REMARKS ON NONLINEAR CONTRACTION AND
COMPARISON PRINCIPLE IN ABSTRACT CONES*

by

J. Eisenfeld and V. Lakshmikantham

Technical Report No. 25

June 1975
Remarks on nonlinear contractions and comparison principle in abstract cones

Introduction. The contraction mapping principle and the Schauder principle can both be viewed as a comparison of maps. For the former one has a condition of the type

$$\rho(Tx, Ty) \leq \psi(x, y)$$  \hspace{1cm} (1.1)

and for the latter one has a condition of the type

$$\gamma(T(S)) \leq \psi(\gamma(S))$$  \hspace{1cm} (1.2)

where $\rho$ is the metric and $\gamma$ is the Kuratowski measure of noncompactness. If $\psi$ is a linear map $\psi x = \kappa x$ from the nonnegative reals $\mathbb{R}^+$ into itself then the map $T$ satisfying (1.1) is said to be $\kappa$-contractive and the map satisfying (1.2) is said to be $\kappa$-set contractive. It is also usually assumed that $\kappa < 1$ in which case the adjective "strict" is used to describe the contractive property.

Instead of taking $\psi$ to be a linear map on the cone $\mathbb{R}^+ \to \mathbb{R}^+$, $\psi$ can be chosen as a nonlinear map from a cone of a Banach space into itself [1], [4]. This innovation provides for greater flexibility in the choice of $\psi$ and it also has the advantage of stronger convergence properties and more accurate estimates. The comparison map $\psi$ is: positive (in the sense that it takes values in a cone), monotone (nondecreasing) and has a unique fixed point which is the zero element of the cone. For a regular cone (such as cones in $L_p^p$, $1 \leq p < \infty$) $\psi$ needs only satisfy the weak continuity.
condition: upper semi-continuous from above (or from the right). However, in the case of a normal cone which is not regular (such as $C[0,1]$) it is assumed in [1], [4] that $\psi$ is completely continuous. The complete continuity condition which is also used by Krasnoselskii [7, p.127] may be replaced by a weaker compactness-type condition in terms of measure of noncompactness along with upper semi-continuity from above. We also manage to avoid strict contractive conditions.

The paper is organized as follows. In Section 2 we state definitions regarding the theory of cones and some propositions which are used as lemmas or to amplify results proved later on. In Section 3 we present some results dealing with maximal fixed points of monotone maps. As a consequence we obtain a generalized Bellman-Gronwall-Reid inequality. In Section 4 we present a generalization of the contraction mapping principle.


2. Cones. Let $E$ be a real Banach space. A set $k \subset E$ is called a cone if: (i) $k$ is closed; (ii) if $u, v \in k$, then $au + bv \in k$ for all $a, b \geq 0$; (iii) of each pair of vectors $u, v$ at least one does not belong to $k$, provided $u \neq \theta$, where $\theta$ is the zero of the space $E$. We say that $u \geq v$ if and only if $u - v \in k$. A cone is called normal if a $\delta > 0$ exists such that $||e_1 + e_2|| > \delta$ for $e_1, e_2 \in k$ and $||e_1|| = ||e_2|| = 1$. The norm in $E$ is called semi-monotone if for arbitrary $x, y \in k$ it follows from $x \leq y$ that $||x|| \leq N||y||$, where the constant $N$ does not depend on $x$ and $y$. 
Proposition 2.1 ([7]). A necessary and sufficient condition for the cone $K$ to be normal is that the norm be semi-monotone.

Proposition 2.2. A decreasing sequence $u_0 \geq u_1 \geq \ldots$ in a space with a normal cone is convergent if it has a convergent subsequence.

Proof. Let $u_n \to u_\infty$, as $n \to \infty$. Then for $m \geq m_K$, $u_m - u_\infty \leq u_n - u_\infty$. By Proposition 2.1, $|u_m - u_\infty| \leq N|u_n - u_\infty|$ as $n \to \infty$. Thus $u_n$ converges to $u_\infty$.

A cone is said to be regular if every decreasing sequence $u_0 \geq u_1 \geq \ldots$ which is bounded from below, i.e., there is a vector $v$ such that $u_n \geq v$, $n = 0, 1, \ldots$, is convergent.

The conical segment $<x_0, u_0>$ is the subset of $E$ of vectors $x$ satisfying $x_0 \leq x \leq u_0$.

A map $\psi$ from a subset of $E$ into $E$ is said to be monotone if $\psi u \geq \psi v$ when $u \geq v$.

If $(A, \rho)$ is a bounded metric space, we define $\gamma(A)$, the measure of noncompactness of $A$, to be $\inf \{d > 0 | A$ can be covered by a finite number of sets of diameter less than or equal to $d\}.$

Proposition 2.3 (Kuratowski [8]). Let $(X, \rho)$ be a complete metric space and let $A_1 \supseteq A_2 \supseteq \ldots$ be a decreasing sequence of nonempty, closed subsets of $X$. Assume that $\gamma(A_n)$ converges to zero. Then if we write $A_\infty = \bigcap_{n \geq 1} A_n$, $A_\infty$ is a nonempty compact set and $A_n$ approaches $A_\infty$ in the Hausdorff metric.

With regard to Kuratowski's theorem we say that a map $\psi$ on a complete metric space $(A_0, \rho)$ into itself is quasi-compact if the sequence of measures of noncompactness $\gamma(A_n)$ of the closed sets $A_{n+1} = \text{cl}(\psi(A_n))$, $n \geq 0$, approaches zero.
A mapping $\psi$ on a partially ordered set into itself is said to be upper semi-continuous from above if whenever $u_0 \geq u_1 \geq \ldots$ and $\psi u_0 \geq \psi u_1 \geq \ldots$ are both monotonic, convergent sequences and $\omega = \lim n u_n$ is in the domain of $\psi$, then $\psi \omega \geq \lim \psi u_n$.

**Proposition 2.4.** Let $\psi$ be monotone and upper semi-continuous from above and suppose that the sequence of iterates $u_n = \psi^n u_0$, of a vector $u_0$, is decreasing and convergent to a vector $u_\infty$, which is in the domain of $\psi$. Then $u_\infty$ is a fixed point of $\psi$, i.e., $\psi u_\infty = u_\infty$.

**Proof.** Clearly $\psi u_n = u_{n+1}$ is also decreasing and convergent to $u_\infty$. From $u_n \geq u_\infty$ and the monotone property we deduce that $\psi u_n \geq \psi u_\infty$ and hence $u_\infty \geq \psi u_\infty$. The reverse inequality follows from the upper semi-continuous from above property.

**Proposition 2.5.** Let $f$ be monotone and upper semi-continuous from above from an interval $[0, a]$ of real numbers into itself such that $f(x) = x$ if and only if $x = 0$. Let $\psi$ be a map from a complete metric space $(A_0, \rho)$ into itself such that

$$\gamma(\psi A) \leq f(\gamma(A))$$

for any subset $A$ of $A_0$. Then $\psi$ is quasi-compact.

**Proof.** Let $A_{n+1} = \text{cl}(\psi A_n)$, $n \geq 0$. Put $r_n = \gamma(A_n)$, $n \geq 0$. One then has from condition (2.1) that $r_{n+1} \leq f(r_n)$, $n \geq 0$. From the monotone property of $f$, it follows that $r_n \leq t_n$, $n \geq 0$ where sequence $t_n$ is defined by $t_0 = r_0$, $t_{n+1} = f(t_n)$. By the monotone property of $f$, $t_n$ is a decreasing sequence. Let $t_\infty = \lim t_n$. From Proposition 2.4, $t_\infty$ is a
fixed point \( f \) and hence \( t_\infty = 0 \). Clearly, \( r_n \) converges to zero.

**Remark:** The map \( \psi \) in Proposition 2.5 is called \( \alpha \)-set contractive, \( \alpha > 0 \), if it is continuous and \( \gamma(\psi(A)) \leq \alpha \gamma(A) \) for any bounded subset \( A \). If \( \psi \) is \( \alpha \)-set contractive with \( \alpha < 1 \) then \( \psi \) is quasi-compact since we may take \( f(x) = \alpha x \) in this case.

3. Fixed points in spaces with cones.

**Theorem 3.1.** Let \( A \) be a closed bounded subset of a Banach space which is partially ordered with respect to a normal cone. Let \( \psi \) be a monotone, quasi-compact, upper semi-continuous from above map from \( A \) into itself. Let

\[
U = \{ u \in A \mid \psi u \leq u \},
\]

\[
L = \{ u \in A \mid \psi u \geq u \}
\]

\[
F = \{ u \in A \mid \psi u = u \}
\]

Then (i) \( U, L, F \) are invariant under \( \psi \). (ii) Let \( \psi_n \) denote the restriction of \( \psi^n \) to \( U, n \geq 0 \). Then the sequence \( \psi_n \) is decreasing, i.e., \( \psi_n u \geq \psi_{n+1} u \) for \( u \in U \), and pointwise convergent to a map \( \phi \), i.e., \( \psi_n u \to \phi u \) for \( u \in U \). (iii) The range of \( \phi \) is \( F \) which is precompact. (iv) The map \( \phi \) is monotone. (v) If \( v \in L \) and \( v \leq u \in U \), then \( v \leq \phi u \leq u \).
Proof. Statement (i) is obvious from the monotone property of \( \psi \) which also implies that \( \psi_n u \geq \psi_{n+1} u \) when \( u \in U \). Let \( u \in A \), then by Proposition 2.3, with \( A_n = \text{cl}(\psi^n(A)) \), and from the quasi-compact property, there is a compact set \( C \) and sequence \( \sigma_n \in C \) such that \( ||\psi_n u - \sigma_n|| \to 0 \) as \( n \to \infty \). Since the sequence \( \sigma_n \) has a convergent subsequence, so does the sequence \( u_n = \psi^n u \). If \( u \in U \), we deduce from Proposition 2.2 that \( u_n \) is convergent. This establishes statement (ii). The fact that \( \phi u \in F \) follows from Proposition 2.4. Now if \( u \in F \) then \( \phi u = u \) so that \( \phi \) maps \( U \) onto \( F \). Also \( \text{cl}(F) \) is precompact because \( F \subset C \).

This completes the proof of statement (iii). Statement (iv) follows because each of the maps \( \psi_n \) are monotone. For statement (v), note that \( v \leq u \) implies \( v \leq \psi v \leq \psi u + \phi u \). Thus \( v \leq \phi u \). This completes the proof.

Theorem 3.2. Let \( \psi \) be a monotone, upper semi-continuous map from a conical segment \( <\theta, u_0> \) into itself and let the following condition be satisfied.

Condition (H): either \( \psi \) is quasi-compact and the cone is normal or the cone is regular (or both).

Then the sequence of iterates \( \psi^n u_0 \) is decreasing and convergent to fixed point \( w \) of \( \psi \). Moreover, \( v \leq \psi u \) implies \( v \leq w \). In particular, \( w \) is the maximal fixed point of \( \psi \) in the segment.

Proof. If \( \psi \) is quasi-compact and the cone is normal then the result is a corollary of Theorem 3.1. If the cone is regular the result follows from Theorem 3.1 of [1].
The following result is a generalization of the Bellman-Gronwall-Reid inequality.

**Corollary 3.1.** Let the hypothesis of Theorem 3.1 be satisfied and let $p$ be a mapping of a set $X$ into the segment $<0,u_0>$. Suppose $T$ is a mapping of $X$ into itself such that $pTa \leq pTa$, $a \in X$. Then if $b$ is a fixed point of $T$, $pb \leq \omega$ where $\omega$ is the maximal fixed point of $\psi$.

**Proof.** Set $u = pb$. Then $u = pTb \leq \psi pb = \psi u$. By Theorem 3.2, $u \leq \omega$.

4. **Contraction mapping principle.** Let $X$ be a set and let $\rho$ be a mapping from $X \times X$ into a cone $k$ of a Banach space. The map $\rho$ is said to be a $k$-metric on $X$ if it satisfies the properties:

$$
\rho(x,y) = \rho(y,x), \quad \rho(x,y) = 0 \text{ iff } x = y,
$$

$$
\rho(x,y) \leq \rho(x,a) + \rho(a,y).
$$

A sequence $x_n$ in the $k$-metric space $(X,\rho)$ is said to be **Cauchy** if

$$
\lim_{m \to \infty} \rho(x_n,x_m) = 0. \text{ The sequence } x_n \text{ is said to be } \text{convergent} \text{ if there is a } y \in X \text{ such that } \lim_{n \to \infty} \rho(x_n,y) = 0. \text{ A } k\text{-metric space is complete if every Cauchy sequence is convergent.} \text{ A convergent sequence } x_n \text{ is said to be } k\text{-convergent to } y \text{ if there is a sequence } u_n \to \theta \text{ in } k \text{ such that } \rho(x_n,y) \leq u_n.
$$

**Theorem 4.1.** Let $(X,\rho)$ be a complete $k$-metric space. Let $\psi$ be a monotone, upper semi-continuous from above map from the conical segment $<0,u_0>$ into itself such that condition (H) is satisfied and such that $\theta$ is the unique fixed point of $\psi$. Let $T$ be a map from $X$ into itself such that
Then for arbitrary \( x_0 \in X \), the sequence of iterates \( x_n = T^n x_0 \) \( k \)-converges to a fixed point \( y \) of \( T \) and \( y \) is the unique fixed point of \( T \).

**Proof.** For any pair of integers \( m > n > 1 \) we have \( \rho(x_n, x_m) = \rho(Tx_{n-1}, Tx_{m-1}) \leq u_0 \). Hence, \( \rho(x_{n+1}, x_{m+1}) \leq \psi \rho(Tx_0, Tx_m) \). Repeating this argument we find that \( \rho(x_{n+1}, x_{m+1}) \leq \psi^n \rho(Tx_0, Tx_m) \). Since \( \rho(Tx_0, Tx_m) \leq u_0 \) and since \( \psi \) is monotone, \( \rho(x_{n+1}, x_{m+1}) \leq \psi^n u_0 \). But by Theorem 3.2 and since \( \theta \) is the maximal fixed point of \( \psi \), \( \psi^n u_0 \) decreases to \( \theta \). Thus \( x_n \) is a Cauchy sequence. Let \( y = \lim x_n \). Then by letting \( m \to \infty \) in the above inequality we have \( \rho(x_{n+1}, y) \leq \psi^n u_0 \). Thus the sequence \( x_n \) \( k \)-converges to \( y \). Since \( \rho(x_{n+1}, y) \leq u_0 \), \( \rho(Tx_{n+1}, Ty) \leq \psi \rho(x_{n+1}, y) \leq \psi^n u_0 \). Therefore \( \rho(x_{n+1}, Ty) \to \theta \) so that \( y = Ty \), i.e., \( y \) is a fixed point of \( T \). Suppose also \( z = Tz \) is also a fixed point. Then \( \rho(y, z) = \rho(Ty, Tz) \leq u_0 \). Hence \( \rho(y, z) = \rho(Ty, Tz) \leq \psi \rho(y, z) \). By Theorem 3.2, \( \rho(y, z) \leq 0 \). This means that \( \rho(y, z) = 0 \) or \( y = z \). The proof is now complete.

The above result weakens assumptions made by the authors in [1] regarding the complete continuity of \( \psi \) and the strict inequality \( \psi u_0 \leq u_0 \). Still it assumes too much. The following theorem represents an economization of Theorem 4.1.

Recall that a map \( T \) is **closed** if whenever \( x_n \in \text{Domain}(T) \), \( x_n \to u \), \( Tx_n \to v \) then \( v \in \text{Domain}(T) \) and \( Tu = v \).
Theorem 4.2. Let \((X, \rho)\) be a k-metric space and let \(\psi\) be a monotone map from a segment \(<\theta, u_0>\) into itself such that

\[
\lim \psi^n u_0 \to \theta \quad (4.3)
\]

Suppose \(T\) is a closed map from a subset \(D\) of \(X\) into \(X\) such that

\[
\rho(Tx, Ty) \leq \psi \rho(x, y), \quad x, y \in D, \quad \rho(x, y) < u_0 \quad (4.4)
\]

Suppose further that \(x \in D\) and \(x_n \in T^n x \in D, \quad n = 1, 2, \ldots\) and that

\[
\rho(x_n, x_0) < u_0 \quad (4.5)
\]

Then \(x_n\) k-converges to a fixed point of \(T\).

\textbf{Proof.} As in the proof of Theorem 4.1, \(\rho(x_n, x_m) \leq \psi^n \rho(Tx_0, Tx_n) \leq \psi^{n+1} \rho(x_0, x_n) \leq \psi^{n+1} u_0\) for \(m > n > 0\). Whence by (4.3), \(x_n\) is Cauchy.

Let \(x_n \to w\). Then \(Tx_n = x_{n+1} \to w\). Since \(T\) is closed, \(Tw = w\). Moreover from \(\rho(x_n, w) \leq \psi^{n+1} u_0\) we conclude that \(x_n\) k-converges to \(w\).

See [9] for further conditions under which an iterative process converges to a fixed point.
REFERENCES


5. S. Keikkila and S. Seikkala, On the estimation of successive approximations in abstract spaces, University of Oulu, Department of Mathematics, Oulu, Finland, 90100.


