LYAPUNOV-LIKE VECTOR FUNCTIONS
USING POINTWISE DEGENERATE SYSTEMS
AS COMPARISON FUNCTIONS

by

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1. The use of Lyapunov-like vector functions is recognized as an important tool for estimating the behavior of a dynamical system. In applications, one needs to determine a suitable comparison function which contains information or properties that can be used to obtain some information about the behavior of the dynamical system.

In recent years, the remarkable property of pointwise degeneracy has been discovered for linear, autonomous, delay-differential equations. For this class of equations one knows that all solutions will reach a subspace in finite time and remain on the subspace thereafter. As this class of equations is particularly simple, it becomes an obvious candidate for use as a comparison function.

In this paper (Section 3), we show how one may effectively use pointwise degenerate systems as comparison functions. Section 4 treats functional differential equations which possess certain monotonic features. These monotonic properties do not appear to allow us to select pointwise degenerate systems as natural comparison functions.

2. Using the notation in Lakshmikantham and Leela [4], we write the functional differential equation with initial function $\phi_0$ as

\[ x'(t) = f(t, x(t), x_t) ; \quad t > t_0, \quad x_{t_0} = \phi_0 \]  \hspace{1cm} (1)

where $t \in \mathbb{R}, x(t) \in \mathbb{R}^n, x_t$ is a translation of $x \in C([-\tau, \infty), \mathbb{R}^n]$.
defined by \( x_t(s) = x(t+s), -\tau \leq s \leq 0 \). If \( C^n = C([-\tau, 0], R^n) \), we define the norm of \( \phi \in C^n \) by

\[
||\phi||_0 = \max_{-\tau \leq s \leq 0} ||\phi(s)||
\]

where \( ||\cdot|| \) denotes any convenient norm in \( R^n \), and for \( \rho > 0 \) we let

\[
C_\rho = \{ \phi \in C^n : ||\phi||_0 < \rho \}.
\]

Although redundant, we sometimes find it convenient to explicitly display in (1) the second argument in \( f \) as \( x(t) \).

Normally, the comparison function will be the linear, autonomous, differential-delay equation

\[
x'(t) = Ax(t) + Bx(t-\tau), \quad t > t_0
\]  

(2)

with initial function \( x_{t_0} = \psi_0 \). Here \( A, B \) are real \( n \times n \) matrices.

Equation (2) is said to pointwise degenerate (p.d.) at \( t_d \) with respect to the nonzero vector \( q \in R^n \) if for all initial continuous functions \( \psi_0 \) the corresponding solution \( x(t) \equiv x(t_0, \psi_0)(t) \) satisfies \( q^*x(t_d) = 0 \).

It is then easy to verify that

\[
q^*x(t) \equiv 0 \quad \text{for} \quad t \geq t_d.
\]
Recognizing that Eqn. (2) is pointwise degenerate at \( t \geq t_d \), motivates the following:

**Definition 1.** The *functional differential system*

\[
y'(t) = g(t, y_t), \quad t > t_0
\]

is said to be degenerate with respect to the nonzero vector \( q \in \mathbb{R}^n \) if there exists a real number \( t_d > t_0 \) such that

\[
q^* y(t_0, \psi_0)(t) \equiv 0, \quad t \geq t_d
\]

for all \( \psi_0 \in C^n \).

For more details on p.d. systems we refer the reader to Popov [5], Asner and Halanay [1], Charrier [2], Choudhury [3], and Zmood and McClamroch [6].

We use the following notation for the Dini derivatives:

\[
D^+ x(t) = \lim_{h \to 0^+} \sup h^{-1}[x(t+h) - x(t)]
\]

\[
D^- x(t) = \lim_{h \to 0^-} \inf h^{-1}[x(t+h) - x(t)]
\]

and normally let \( J \) denote a \( t \)-interval containing \( t_0 \).

Denote \( S_\rho = \{ x \in \mathbb{R}^n : ||x|| < \rho \} \). Then using Lyapunov-like vector functions we define, using a Dini derivative, the derivative of \( V \) along solutions of (1) in
Definition 2. Let $V \in C([\tau, \infty) \times \mathbb{S}_\rho, \mathbb{R}^n_+)$ and $V(t, x)$ be locally Lipschitzian in $x$. For $\phi \in C_\rho$ we define $D^+V_f(t, \phi(0), \phi)$ with respect to the functional differential system $x' = f(t, x_t)$ as follows:

$$D^+V_f(t, \phi(0), \phi) = \lim_{h \to 0^+} \sup h^{-1} [V(t+h, \phi(0) + hf(t, \phi)) - V(t, \phi(0))].$$  \hspace{1cm} (4)

3. Using Eqn. (3) as a comparison function, we can prove the following:

Theorem 1. Let $V \in C([\tau, \infty) \times \mathbb{S}_\rho, \mathbb{R}^n_+)$ and $V(t, x)$ be locally Lipschitzian in $x$. Assume for $q \in \mathbb{R}^n$

$$q^*D^+V_f(t, \phi(0), \phi) \leq q^*g(t, \phi)$$  \hspace{1cm} (5)

for $t > -\tau, \phi \in C_\rho$. Then for any solution $x(t_0, \phi_0)(t)$ of Eqn. (1) we have

$$q^*[V(t, x(t_0, \phi_0)(t)) - \int_{t_0}^{t} g(s, x_s(t_0, \phi_0))ds] \leq q^*V(t_0, \phi_0(0)).$$  \hspace{1cm} (6)

**Proof.** Define $\phi = x_\tau(t_0, \phi_0)$ so that $\phi(0) = x(t_0, \phi_0)(t)$. Let $m(t) = q^*V(t, \phi(0))$ and note that $m(t_0) = q^*V(t_0, \phi_0(0))$. Using the Lipschitzian character of $V$ we obtain

$$D^+m(t) \leq q^*D^+V_f(t, \phi(0), \phi).$$
From Eqn. (5)

$$D^+m(t) \leq q^*g(t,\phi).$$

Hence

$$m(t) \leq m(t_0) + \int_{t_0}^{t} q^*g(s,x_s(t_0,\phi_0))ds$$

and Eqn. (6) follows using the definition of $m(t)$.

We would like to estimate the integral appearing in Eqn. (6). For this purpose we need the following Lemma which is an obvious modification of Theorem 6.3.3 in Lakshmikantham and Leela [4].

**Lemma 1.** Let $f, g \in C[JxR^N\times C_p,R^N], G \in C[Jx[0,\phi),R_+]$, and

$$a(t) = \liminf_{h \to 0^-} h^{-1} \{||\phi(0) - \psi(0) + h[f(t,\phi(0),\phi) - g(t,\psi(0),\psi)]||$$

$$- ||\phi(0) - \psi(0)||\} + ||\phi(0) - \psi(0)||, D_a(t) \leq G(t,||\phi(0) - \psi(0)||a(t)),$$

for $t > t_0$ and $\phi, \psi \in C_p$ satisfying

$$||\phi - \psi||a_t|_0 = ||\phi(0) - \psi(0)||a(t)$$

(8)

where $a(t)$ is continuous on $[-\tau,\infty)$. Let $x(t_0,\phi_0)$ be a solution of (1) and $y(t_0,\psi_0)$ be a solution of (3) with initial condition $y_{t_0} = \psi_0$ such that

$$||\phi_0 - \psi_0||a_{t_0}|_0 \leq u_0$$
Corollary 1. Let \( x(t_0, \phi_0), y(t_0, \psi_0) \) be any solutions of Eqn. (9)
and Eqn. (2), respectively, such that

\[
||\phi_0(s) - \psi_0(s)|| e^{-\mu(A)(t_0+s)} \leq u_0 \quad \text{for} \quad -\tau \leq s \leq 0.
\]

Define

\[
\sigma(t) = ||F(t, \phi)|| \quad \text{for} \quad t \in J, \quad \phi \in C^\rho, \quad \text{and}
\]

\[
b_0 = ||B|| e^{-\mu(A)\tau}.
\]

Then, as far as \( x(t_0, \phi_0)(t) \) exists to the right of \( t_0 \)

\[
||x(t_0, \phi_0)(t) - y(t_0, \psi_0)(t)|| \leq \eta(t)
\]

where

\[
\eta(t) = e^{(\mu(A)+b_0)t} \left[ e^{-b_0 t u_0} + \int_{t_0}^{t} e^{-(b_0+\mu(A))\alpha} \sigma(\alpha) d\alpha \right].
\]

Proof. In Lemma 1 let

\[
G(t, u) = b_0 u + \sigma(t)e^{-\mu(A)t}
\]

and

\[
a(t) = e^{-\mu(A)t}.
\]
If for $t > t_0$, $\phi, \psi \in C_p$ satisfy Eqn. (8), then in particular at $s = -\tau$ we have

$$||\phi(-\tau) - \psi(-\tau)|| e^{-\mu(A)(-\tau + t)} \leq ||\phi(0) - \psi(0)|| e^{-\mu(A)t}. \quad (10)$$

Then using (10) we have for $h$ sufficiently small

$$e^{-\mu(A)t} h^{-1}[||\phi(0) - \psi(0) + h[A(\phi(0) - \psi(0)) + B(\phi(-\tau) - \psi(-\tau)) + F(t, \phi)]||$$

$$- ||\phi(0) - \psi(0)||] - \mu(A)e^{-\mu(A)t}||\phi(0) - \psi(0)||$$

$$\leq e^{-\mu(A)t}\{h^{-1}[||I + hA|| - 1 - h\mu(A)]||\phi(0) - \psi(0)|| + ||B|| ||\phi(-\tau) - \psi(-\tau)|| + \sigma(t)\}$$

$$\leq e^{-\mu(A)t}[\varepsilon(h) + ||B|| e^{-\mu(A)t}||\phi(0) - \psi(0)|| + \sigma(t)]$$

$$\leq e^{-\mu(A)t}\varepsilon(h) + G(t, ||\phi(0) - \psi(0)|| e^{-\mu(A)t})$$

and Eqn. (7) is satisfied as $\varepsilon(h) \to 0$ when $h \to 0^-$. 

**Theorem 2.** Let $V$ be defined as in Theorem 1. Assume

$$q^*D^+V_f(t, \phi(0), \phi) \leq q^*[A\phi(0) + B\phi(-\tau)] \quad (11)$$

for $t \geq -\tau, \phi \in C_p$, and Eqn. (2) is degenerate with respect to starting $q$ at $t_d$. Then for any solution $x(t_0, \phi_0)(t)$ of Eqn. (9), $y(t_0, \psi_0)(t)$
of Eqn. (2) implies for \( t \geq t_0 \)

\[
q^*V(t, x(t_0, \phi_0)(t)) \leq q^*[V(t_0, \phi_0(0)) - \psi_0(0)]
\]
\[
+ L_1 \int_{t_0}^{t} \eta(\alpha)d\alpha + L_2 \int_{t_0}^{t} \eta(\alpha-\tau)d\alpha
\]

where

\[
L_1 = ||q^*A||, \quad L_2 = ||q^*B||,
\]

\[
\eta(t) = e^{(\mu(A) + b_0)t}[e^{-b_0t_0u_0} + \int_{t_0}^{t} \frac{e^{-[b_0 + \mu(A)]\sigma(\alpha)}d\alpha}{e^{-\mu(A)\sigma(\alpha)}}],
\]

and

\[
b_0 = ||B||e^{-\mu(A)\tau}, \quad \sigma(t) = ||F(t, \phi)||, \quad u_0 = \max_{-\tau \leq s \leq 0} \{|\phi_0(s) - \psi_0(s)| e^{-\mu(A)(s + t_0)}\}
\]

Proof. With \( g(t, \phi) = A\phi(0) + B\phi(-\tau) \) the hypothesis of Theorem 1 is satisfied. Now

\[
q^*g(t, x(t_0, \phi_0)) = q^*[g(t, x_\tau(t_0, \phi_0)) - g(t, y_\tau(t_0, \psi_0))] + q^*g(t, y_\tau(t_0, \psi_0))
\]
\[
= q^*A(\phi(0) - \psi(0)) + q^*B(\phi(-\tau) - \psi(-\tau)) + q^*g(t, y_\tau(t_0, \psi_0)).
\]
From the definitions in Eqn. (13)

\[ q^* g(t, x_t(t_0, \phi_0)) \leq L_1 ||\phi(0) - \psi(0)|| + L_2 ||\phi(-\tau) - \psi(-\tau)|| + q^* g(t, y_t(t_0, \psi_0)). \]  \(16\)

From Corollary (1) with

\[ -\tau \leq s \leq 0 \{ ||\phi_0(s) - \psi_0(s)|| - \mu(A)(s+t_0) \} = u_0 \]  \(17\)

we have

\[ ||\phi(0) - \psi(0)|| \leq \eta(t) \ and \ ||\phi(-\tau) - \psi(-\tau)|| \leq \eta(t-\tau). \]  \(18\)

Rewriting Eqn. (2) in integral form

\[ y(t_0, \psi_0)(t) = \psi_0(0) + \int_{t_0}^{t} g(s, y_s(t_0, \psi_0))ds \]

and using the degeneracy property of \(g\) shows that

\[ 0 = q^* \psi_0(0) + \int_{t_0}^{d} q^* g(s, y_s(t_0, \psi_0))ds. \]  \(19\)

From Eqns. (16), (18) and (19) we have for \(t \geq t_d\)

\[ \int_{t_0}^{t} q^* g(s, x_s(t_0, \phi_0))ds \leq L_1 \int_{t_0}^{t} \eta(\alpha)d\alpha + L_2 \int_{t_0}^{t} \eta(\alpha-\tau)d\alpha - q^* \psi_0(0), \]

and the conclusion of the theorem follows using Eqn. (6).
4. We now impose some monotonic properties on the functions.

**Definition 3.** Let \( g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \). Then \( g(t, u, \phi) \) is said to possess a quasimonotone property if

(i) for fixed \((t, \phi)\), \( g_p(t, u, \phi) \) is nondecreasing in \( u_j \),
\[ j = 1, \ldots, n, j \neq p, \]
(ii) for fixed \((t, u)\), \( g_p(t, u, \phi) \) is nondecreasing in \( \phi_j \),
\[ j = 1, \ldots, n. \]

**Remark.** \( u_j \) and \( \phi_j \) are components of the vectors \( u \) and \( \phi \) and should not be interpreted as elements of \( \mathbb{R}^n \).

**Lemma 2.** Let \( g \in C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n) \) and possess a quasimonotone property. Let \( v, w \in C([t_0, \tau), \mathbb{R}^n) \) and
\[ v_{t_0} < w_{t_0}. \] (20)

Assume that

\[ D_- v(t) \leq g(t, v(t), v_t) \] (21)
\[ D_- w(t) > g(t, w(t), w_t) \]

are satisfied for \( t \in (t_0, \tau) \). Then
\[ v(t) < w(t), \ t \in [t_0, \tau). \] (22)
Proof. Define $m(t) = w(t) - v(t)$. Because of (14) $m_{t_0} > 0$.

Suppose assertion (22) is not true. Then, the set

$$ Z = \bigcup_{i=1}^{n} \{ t \in [t_0, \infty) : m_i(t) \leq 0 \} $$

is nonempty. Let $t_1 = \inf Z$. Consequently, there exists $j$ such that

$$ m_j(t_1) = 0 $$

while

$$ m_{t_1} \geq 0. $$

Thus, we obtain for small $h < 0$

$$ h^{-1}[m_j(t_1+h) - m_j(t_1)] \leq 0 $$

which, in turn, implies that

$$ D_- m_j(t_1) \leq 0. $$

From (21) and the definition of $m(t)$ we deduce at $t = t_1$

$$ g_j(t,w(t),w_{t_1}) \leq D_- w_j(t) \leq D_- v_j(t) \leq g_j(t,v(t),v_{t_1}). \quad (23) $$

But $m_{t_1} \geq 0$ implies $w_{t_1} \geq v_{t_1}$. Since $g$ possesses a quasimonotone property, we deduce that

$$ g_j(t_1,v(t_1),v_{t_1}) \leq g_j(t_1,w(t_1),v_{t_1}) \leq g_j(t_1,w(t_1),w_{t_1}) $$

which together with (23) is a contradiction. Thus, $Z$ is empty and (22) is proved.
For functional differential inequalities we prove the following comparison lemma. Let \( C^n_+ \) denote the set of all nonnegative functions belonging to \( C^n \).

**Lemma 3.** Let \( m \in C[[t_0 - \tau, \infty), R^n_+] \) and satisfy the inequality

\[
D_m(t) \leq g(t, m(t), m_\tau), \quad t > t_0
\]

where \( g \in C[\mathbb{J} \times R^n_+ \times C^n_+, R^n] \). Assume that \( g \) possesses a quasimonotone property and let \( r(t_0, \phi_0), \quad \phi_0 \in C^n_+ \) be the maximal solution of

\[
x' = g(t, x(t), x_\tau)
\]

existing for \( t \geq t_0 \). Then

\[
m_\tau \leq \phi_0
\]

imply that

\[
m(t) \leq r(t_0, \phi_0)(t), \quad t \geq t_0.
\]

**Proof.** Let \( x(t_0, \phi_0, \epsilon) \) be a solution of

\[
x'(t) = g(t, x(t), x_\tau) + \epsilon, \quad x_{t_0} = \phi_0 + \epsilon, \quad \epsilon \in R^n_+.
\]

Then,

\[
m_{t_0} < \phi_0 + \epsilon, \quad x'(t_0, \phi_0, \epsilon)(t) > g(t, x(t_0, \phi_0, \epsilon)(t), x_\tau(t_0, \phi_0, \epsilon)),
\]

and \( D_m(t) \leq g(t, m(t), m_\tau) \). From the previous lemma
\[ m(t) < x(t_0, \phi_0, \epsilon)(t), \quad t \geq t_0. \]

But

\[
\lim_{\epsilon \to 0} x(t_0, \phi_0, \epsilon)(t) = r(t_0, \phi_0)(t) \text{ uniformly for } t \geq t_0.
\]

We now prove the fundamental comparison theorem for Lyapunov-like vector functions.

**Theorem 3.** Let \( V \in C([-\tau, \infty) \times S_\rho, R^n_+], \) and \( V(t, x) \) be locally Lipschitzian in \( x. \) Assume that, for \( t \in J, \phi \in C_\rho, \)

\[
D^+ V_f(t, \phi(0), \phi) \leq g(t, V(t, \phi(0)), V_t)
\]

where

\[
V_t = V(t+s, \phi(s)), \quad -\tau \leq s \leq 0, \quad g \in C(J \times R^n_+ \times C^n_+, R^n),
\]

\( g \) possesses a quasimonotone property, and \( r(t_0, \sigma_0) \) is the maximal solution of the functional differential system

\[
u' = g(t, u, u_t), \quad u_{t_0} = \sigma_0 \in C^n_+.
\]

existing for \( t \geq t_0. \) If \( x(t_0, \phi_0) \) is any solution of (1) defined in the future such that

\[
V_{t_0} = V(t_0 + s, \phi_0(s)) \leq \sigma_0
\]
then we have

\[ V(t, x(t_0, \phi_0)(t)) \leq r(t_0, \sigma_0)(t), \quad t \geq t_0. \]

**Proof.** Let \( x(t_0, \phi_0) \) be any solution of (1) such that

\[ ||x(t_0, \phi_0)|| < \rho. \]

Let \( \phi = x(t, \phi_0) \) and define

\[ m(t) = V(t, x(t_0, \phi_0)(t)) \]

so that

\[ m_t = V(t+s, \phi(s)) = V_t. \]

By hypothesis \( m_{t_0} \leq \sigma_0 \). Arguing as in Theorem 1

\[ D^+ m(t) = D^+ V_f(t, \phi(O), \phi) \leq g(t, V(t, \phi(O)), V_t) \]

or

\[ D^+ m(t) \leq g(t, m(t), m_t). \]

The monotone property of \( g \) implies

\[ D_- m(t) \leq g(t, m(t), m_t). \]
Then by Lemma 3

\[ m(t) \leq r(t_0, \sigma_0)(t), \quad t \geq t_0. \]

In application, it does not appear that pointwise degenerate systems will be useful as comparison functions. Indeed if \( g(t, x(t), x_t) = Ax(t) + Bx(t - \tau) \) then the quasimonotone property of \( g \) requires that \( a_{ij} \geq 0, \quad i \neq j \), and \( b_{ij} \geq 0 \) where \( A = (a_{ij}), B = (b_{ij}) \). In this case \( e^{At} \geq 0, \quad t \geq 0. \)

**Theorem 4.** Let \( g(t, x(t), x_t) = Ax(t) + Bx(t - \tau) \) possess the quasimonotone property. If \( q \geq 0 \) then (2) cannot be pointwise degenerate at \( t_d = 2\tau. \)

**Proof.** Let \( X(t)e^{R^{nxn}} \) be the solution of the matrix equation

\[ \dot{X}(t) = AX(t) + BX(t - \tau), \quad X(t) = 0, \quad t < 0, \quad X(0) = I. \]

The necessary and sufficient conditions for (2) to be pointwise degenerate at \( 2\tau \) (see Zmoood and McClamroch [6]) are

\[ q^*X(2\tau) = 0 \quad \text{and} \quad q^*X(t)B = 0 \quad \text{for} \quad \tau \leq t \leq 2\tau. \]

Now

\[ X(t) = e^{At} + \int_{\tau}^{t} e^{A(t - \alpha)}B e^{A(\alpha - \tau)}d\alpha, \quad \tau \leq t \leq 2\tau. \]
and assuming degeneracy at $2\tau$

$$0 = q^*X(2\tau) = q^*A2\tau + \int_0^{2\tau} q^* e^{A(2\tau-\alpha)}B e^{A(\alpha-\tau)}d\alpha.$$ 

Since each term is a combination of positive elements we have

$$q^* e^{A2\tau} = 0,$$ thus $q^* = 0$, a contradiction.

**Remark.** It is well known (see any of the references on p.d. systems) that $t_d = 2\tau$ (with $t_0 = 0$) is the minimal time for degeneracy to occur. Examples do exist where $t_d > 2\tau$ (see Asner and Halanay [1]), but these do not possess the quasimonotone property. Even if the matrices $A$ and $B$ satisfy the mixed quasimonotone property of Lakshmikantham and Leela [4], then one can always transform to the matrices of Theorem 4. The $q$ vector will also transform with a sign regular patters to the nonnegative $q$ of Theorem 4.
REFERENCES


