The Truncated Circular Normal Distribution
with Applications in Ballistics and Meteorology

by

Danny Dyer

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1. INTRODUCTION

The use of the circular normal distribution (CND) to describe the behavior of random phenomena of a geophysical nature has been discussed by Crutcher [7, p. 9]. Based on the Mauchly [17] - Hsu [12] test, the harmonic dial points (the Fourier coefficients obtained from harmonic analysis of observations of periodic phenomena) representing i.) lunar semidiurnal atmospheric tides (Chapman and Lindzen [5, p. 66]), ii.) the westerly component of wind in the study of tidal oscillations in the upper atmosphere (Haurwitz [10]), and iii.) terrestrial-magnetic activity relative to solar activity (Bartels [2]) may be treated as observations from a CND. However, the "cloud" of points is often restricted to circular regions in the plane of the harmonic dial. Perhaps then, in this situation, a CND truncated outside a circular region is a more adequate distribution for the dial points.

A CND truncated outside a circular region may also be used to describe in a probabilistic sense gunfire pattern restricted to a circular target. Gunfire pattern, in general, is often approximated by a CND; however, in many situations, e.g., for initial firings of the weapon, the parameters of the distribution are unknown. Estimation of the parameters or functions of the
parameters has received the attention of many authors, for example, Eckler [8], Moranda [18], Kamat [15], Inselmann and Granville [14]. Generally, estimates are based on a sample of $n$ impact-points obtained by firing the weapon at some bounded two-dimensional target—quite often a circular region. Furthermore, the ballistic test is usually such that all shots impact within the circular region, otherwise a censored sample is obtained creating difficult statistical estimation problems. Since sampling is restricted to the circular region, a truncated CND might be a more reasonable distribution for the impact-points.

In this paper recurrence relations for the population moments of a CND truncated outside a circular region are derived. From these recurrence relations one may obtain the maximum likelihood equations for estimating the unknown parameters. These equations are nonlinear and an iterative scheme is given to obtain a solution. The recurrence relations may also be used to obtain an alternate set of estimators which are consistent asymptotically jointly normal. These estimators are easier to compute than the maximum likelihood estimators and may be used as a starting vector in solving the maximum likelihood equations reducing, in general, the number of iterative cycles required to yield a solution. The theory is illustrated with a numerical example based on the harmonic dial points representing determinations of the lunar semidiurnal atmospheric tide.
2. RECURRANCE RELATIONS FOR POPULATION MOMENTS

The density function of a two-dimensional random variable \((U,V)\) having a CND truncated outside \(C = \{(u,v): (u-h_1)^2 + (v-h_2)^2 \leq R^2\}\), where the center \((h_1,h_2)\) and radius \(R\) are known, is

\[
f_{U,V}(u,v) = \frac{p(u,v;\mu_1,\mu_2,\sigma^2)}{P} \quad \text{for} \quad (u,v) \in C, \tag{2.1}
\]

where

\[
p(u,v;\mu_1,\mu_2,\sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{(u-\mu_1)^2 + (v-\mu_2)^2}{2\sigma^2} \right\},
\]

and

\[
P = \iint_C p(u,v;\mu_1,\mu_2,\sigma^2) \, du \, dv.
\]

It will be convenient to translate the origin in the \(uv\)-plane to \((h_1,h_2)\). The density function of \((X,Y) = (U-h_1,V-h_2)\) is

\[
f_{X,Y}(x,y) = \frac{p(x,y;\xi_1,\xi_2,\sigma^2)}{P} \quad \text{for} \quad 0 \leq x^2 + y^2 \leq R^2, \tag{2.2}
\]

where \(\xi_i = \mu_i - h_i, \ i = 1,2\).

Taking the first partial derivative of (2.2) with respect to \(x\) gives

\[
\frac{\partial f_{X,Y}(x,y)}{\partial x} = -(1/\sigma^2)(x-\xi_1)f_{X,Y}(x,y). \tag{2.3}
\]

Upon multiplying both sides of (2.3) by \(x^r y^s\), where \(r\) and \(s\) are non-negative integers, and integrating over \(C\), we have

\[
A_x = -(1/\sigma^2)\mu'_{r+1,s} + (\xi_1/\sigma^2)\mu'_{r,s}, \tag{2.4}
\]
where
\[ \mu_{r,s}^i = \mathbb{E}(X_{r,s}^2) \]

and
\[
A_x = \iint_{C} x y \gamma \left( x, y / \lambda x \right) \, dx \, dy
\]
\[
= \int_{-R}^{R} \left( R^2 - y^2 \right)^{r/2} y^{s} f_{X,Y}(x, y) \left( (R^2 - y^2)^{1/2}, y \right) \, dy
\]
\[
+ (-1)^{r+1} \int_{-R}^{R} \left( R^2 - y^2 \right)^{r/2} y^{s} f_{X,Y}(x, y) \left( -(R^2 - y^2)^{1/2}, y \right) \, dy - \mu_{r-1,s}^i
\]
\[
= (-1)^{s} R^{r+s+1} \int_{-\pi}^{\pi} \sin^{r+1} \theta \cos^{s} \theta f_{X,Y}(R \sin \theta, -R \cos \theta) \, d\theta - \mu_{r-1,s}^i, (2.5)
\]

with the last term on the right-hand side of (2.5) vanishing when \( r = 0 \).

Thus we have the following recurrence relation
\[
(-1)^{r+1} R^{r+s+1} \int_{0}^{2\pi} \sin^{r+1} \theta \cos^{s} \theta f_{X,Y}(-R \sin \theta, R \cos \theta) \, d\theta
\]
\[
= \mu_{r-1,s}^i - \left( 1/\sigma^2 \right) \mu_{r+1,s}^i + \left( \xi_1/\sigma^2 \right) \mu_{r,s}^i \quad \text{for} \quad r > 0, \ s > 0, \quad (2.6)
\]

where the first term on the right-hand side of (2.6) vanishes when \( r = 0 \).

Using a similar method on the first partial derivative of (2.2) with respect to \( y \), we have
\[
(-1)^{s+1} R^{r+s+1} \int_{0}^{2\pi} \sin^{s+1} \theta \cos^{r} \theta f_{X,Y}(R \cos \theta, -R \sin \theta) \, d\theta
\]
\[
= s \mu_{r,s-1}^i - \left( 1/\sigma^2 \right) \mu_{r,s+1}^i + \left( \xi_2/\sigma^2 \right) \mu_{r,s}^i \quad \text{for} \quad r > 0, \ s > 0, \quad (2.7)
\]
where the first term on the right-hand side of (2.7) vanishes when \( s = 0 \).

The integrals in (2.6) and (2.7) may be written in terms of modified Bessel functions (see Gröbner and Hofreiter [9, p. 144]); the values of \( P \) are tabled in Owen [20, p. 172].

3. ESTIMATION OF PARAMETERS

The recurrence relations from the previous section may be used to estimate the parameters \( \xi_1, \xi_2, \sigma^2 \). Let

\[
Q_1(r,s) = \int_0^{2\pi} \sin^{r+1} \theta \cos^s \theta f_{X,Y}(-R \sin \theta, R \cos \theta) \, d\theta
\]

\[
Q_2(r,s) = \int_0^{2\pi} \sin^{s+1} \theta \cos^r \theta f_{X,Y}(R \cos \theta, -R \sin \theta) \, d\theta
\]

3.1 METHOD OF MAXIMUM LIKELIHOOD

Using the method of moments (equate population moments to corresponding sample moments), estimators are found by solving the following system of equations for \( \xi_1, \xi_2, \sigma^2 \):

\[
\begin{align*}
\xi_1 + R\sigma^2 Q_1(0,0) &= m_{1,0}' \\
\xi_2 + R\sigma^2 Q_2(0,0) &= m_{0,1}' \\
2\sigma^2 + \xi_1 m_{1,0}' + \xi_2 m_{0,1}' - R^2\sigma^2 Q_1(1,0) - R^2\sigma^2 Q_2(0,1) &= m_{2,0}'+ m_{0,2}'
\end{align*}
\]

where \( m_{r,s}' = (1/n) \sum X_i Y_i^s \), and \( n \) is the number of paired observations \((X_i, Y_i)\) on which the estimators are based. The above system of equations,
which is equivalent to the system of maximum likelihood equations by a
result of Hotelling [11], may be written as

\[
\begin{align*}
\xi_1 &= m_{1,0}'/(1-V_1) \\
\xi_2 &= m_{0,1}'/(1-V_1) \\
\sigma^2 &= (m_{2,0}'+ m_{0,2}' + R^2 V_0 - \xi_1 m_{1,0}' - \xi_2 m_{0,1}')/2,
\end{align*}
\]  

(3.1)

where

\[
V_n = \left[ R(\xi_1^2 + \xi_2^2)^{-\frac{1}{2}} \right]^n \exp \left[ -(R^2 + \xi_1^2 + \xi_2^2)/2\sigma^2 \right] I_n \left[ (R/\sigma^2)(\xi_1^2 + \xi_2^2)^{1/2} \right]/\pi,
\]

and \( I_n(x) \) is a modified Bessel function of order \( n \). The equations in (3.1) may be solved iteratively. British Association Mathematical Tables [4, p. 213] gives values of \( I_n(x) \), \( n = 0, 1 \), while National Bureau of Standards [19, p. 416] gives values of \( e^{-x} I_n(x) \), \( n = 0, 1 \).

Let \( (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\sigma}^2) \) represent the maximum likelihood estimators of \( (\xi_1, \xi_2, \sigma^2) \). It is well known (Kendall and Stuart [16, p. 54]) that under certain regularity conditions, \( (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\sigma}^2) \) are asymptotically jointly normal with mean vector \( (\xi_1, \xi_2, \sigma^2) \) and variance-covariance matrix whose inverse in Fisher's information matrix \( -E(\partial^2 \ln L/\partial \theta_i \partial \theta_j) \), where \( \theta = (\theta_1, \theta_2, \theta_3) = (\xi_1, \xi_2, \sigma^2) \). Since (2.2) is a member of the exponential class of distributions,

\[
-E(\partial^2 \ln L/\partial \theta_i \partial \theta_j) = -(\partial^2 \ln L/\partial \theta_i \partial \theta_j)_{\theta = \theta}.
\]
where the right-hand side is to be interpreted as the second partial derivative of the logarithm of the likelihood function with the maximum likelihood estimators replaced by the corresponding parameters (Huzurbazar [13]). We thus have as elements in the information matrix:

\[-E(\partial^2 \ln L / \partial \xi_i \partial \xi_j) = (n/\sigma^2)[\delta_{ij}(1-V_1) + \xi_1 \xi_j G/(\xi_1^2 + \xi_2^2)]\]

where

\[\delta_{ij} = \begin{cases} 1 & , \ i = j \\ 0 & , \ i \neq j , \ i,j = 1,2 \end{cases}\]

\[-E(\partial^2 \ln L / \partial \xi_i \partial \sigma^2) = (n/2\sigma^4) \xi_i (H - G) , \ i = 1,2\]

\[-E(\partial^2 \ln L / \partial (\sigma^2)^2) = (n/2\sigma^4) \left\{ 2 - (\xi_1^2 + \xi_2^2) [2H - (R^2/\sigma^2) V_0]/2\sigma^2 \right. \]

\[+ (R^4/2\sigma^4) V_0 (1 + V_0) \right\}, \]

and

\[H = (R^2/\sigma^2)[(1-V_1)(1+V_0) - 1]\]

\[G = (\xi_1^2 + \xi_2^2) \left\{ V_1 (1-V_1)/\sigma^2 + [2V_1 - (R^2/\sigma^2) V_0]/(\xi_1^2 + \xi_2^2) \right\}.\]
3.2 MODIFIED METHOD OF MOMENTS

Using a modified method of moments (replace population moments by corresponding sample moments), we shall obtain estimators of $\xi_1$, $\xi_2$, $\sigma^2$ which are much easier to compute than those from (3.1). Since

$$Q_1(0,0) - Q_1(0,2) - Q_1(2,0) = 0,$$

$$Q_2(0,0) - Q_2(2,0) - Q_2(0,2) = 0,$$

$$Q_1(1,2) + Q_1(3,0) - Q_1(1,0) + Q_2(2,1) + Q_2(0,3) - Q_2(0,1) = 0,$$

we have the following system of equations in matrix form

$$
\begin{bmatrix}
\mu_{1,0}^i + \mu_{0,2}^i - R^2 & 2\mu_{1,0}^i \\
0 & 2\mu_{0,1}^i & \mu_{2,0}^i + \mu_{0,2}^i - R^2 \\
\mu_{1,2}^i + \mu_{3,0}^i - R^2\mu_{1,0}^i & 2(2\mu_{2,0}^i + 2\mu_{0,2}^i - R^2) & \mu_{2,1}^i + \mu_{0,3}^i - R^2\mu_{0,1}^i \\
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\sigma^2 \\
\xi_2 \\
\end{bmatrix}
$$

$$= 
\begin{bmatrix}
\mu_{3,0}^i + \mu_{1,2}^i - R^2\mu_{1,0}^i \\
\mu_{0,3}^i + \mu_{2,1}^i - R^2\mu_{0,1}^i \\
\mu_{4,0}^i + \mu_{0,4}^i + 2\mu_{2,2}^i - R^2(\mu_{2,0}^i + \mu_{0,2}^i) \\
\end{bmatrix}.
$$

(3.2)

Upon replacing $\mu_{r,s}^i$ in (3.2) by $m_{r,s}^i$, and denoting the estimators of $\xi_1$, $\xi_2$, $\sigma^2$ by $\hat{\xi}_1$, $\hat{\xi}_2$, $\hat{\sigma}^2$, we have
\[ \xi_1 = \frac{a_4(a_3a_6 - a_1a_3) + a_2(a_1a_7 - a_6^2)}{a_1(a_3a_6 - a_1a_5 + a_2a_4)} \]
\[ \xi_2 = \frac{a_6(a_2a_4 - a_1a_3) + a_3(a_1a_7 - a_4^2)}{a_1(a_3a_6 - a_1a_5 + a_2a_4)} \]
\[ \gamma^2 = \frac{(a_6^2 - a_1a_7 + a_4^2)}{2(a_3a_6 - a_1a_5 + a_2a_4)} \]

where

\[ a_1 = m_{2,0}^1 + m_{0,2}^1 - R^2, \quad a_2 = m_{1,0}^1, \quad a_3 = m_{0,1}^1, \quad a_4 = m_{1,2}^1 + m_{3,0}^1 - R^2m_{1,0}^1, \]
\[ a_5 = 2m_{2,0}^1 + 2m_{0,2}^1 - R^2, \quad a_6 = m_{2,1}^1 + m_{0,3}^1 - R^2m_{0,1}^1, \]
\[ a_7 = m_{4,0}^1 + m_{0,4}^1 + 2m_{2,2}^1 - R^2(m_{2,0}^1 + m_{0,2}^1). \]

The estimators given by (3.3) are consistent asymptotically (jointly) normal. Consistency follows from the fact that a rational function of sample moments converges in probability to the same rational function of corresponding population moments (Cramér [6, p. 358]). Since the estimators are continuous functions of sample moments with continuous first and second order partial derivatives with respect to the sample moments, their asymptotic joint distribution is normal (Cramér [6, p. 366]). Finally, it should be noted that when \( R \to \infty \), the estimators (3.3) are equivalent to the method of moments (and maximum likelihood) estimators of the parameters of an untruncated CND.

4. APPLICATIONS

The following data are from Bartels [1] and represent 40 determinations of \( L_2 \), the lunar semidiurnal atmospheric tide, at Batavia (now Djakarta).
in Indonesia for each of the years 1866–1905. We shall assume the dial points are from a CND truncated outside the circular region

\[ C = \{(u,v):(u - .030)^2 + (v - .052)^2 \leq (.030)^2\} . \]

The restriction of the dial points to \( C \) is due to the fact that, at Batavia, the amplitude corresponding to a determination of \( L_2 \) for one calendar year is invariably within the range 0.03 – 0.09 millimeters of mercury while the time of maximum barometric pressure is within two hours after upper transit. This describes a sector of an annular region in the harmonic dial as shown in Figure A, and the boundary of \( C \) is the inscribed circle. The data are plotted in Figure A. The sample moments are

\[
\begin{align*}
  m'_{1,0} & = -0.0033 & m'_{0,1} & = 0.0045 & m'_{1,1} & = -0.38225 \times 10^{-4} \\
  m'_{2,0} & = 1.211 \times 10^{-4} & m'_{0,2} & = 1.269 \times 10^{-4} & m'_{1,2} & = -0.141525 \times 10^{-6} \\
  m'_{3,0} & = -1.21605 \times 10^{-6} & m'_{0,3} & = 0.6177 \times 10^{-6} & m'_{1,3} & = -1.0805375 \times 10^{-8} \\
  m'_{4,0} & = 3.43724 \times 10^{-8} & m'_{0,4} & = 3.92037 \times 10^{-8} & m'_{2,1} & = 0.790275 \times 10^{-6} \\
  & & & & m'_{2,2} & = 1.3358775 \times 10^{-8} \\
  & & & & m'_{3,1} & = -1.1025275 \times 10^{-8} \\
\end{align*}
\]

From (3.3),

\[ \hat{\varepsilon}_1 = -0.00375, \quad \hat{\varepsilon}_2 = 0.00579, \quad \hat{\sigma}^2 = 1.25713 \times 10^{-4} . \]

By way of comparison, \((\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\sigma}^2)\) may be used as a starting vector in the maximum likelihood equations (3.1). After a few iterative cycles, the maximum likelihood estimates are
\[\xi_1 = -0.00372, \quad \xi_2 = 0.00508, \quad \sigma^2 = 1.24323 \times 10^{-4}.\]

Since \(\hat{\mu}_1 = \xi_1 + h_1 = 0.02625\) and \(\hat{\mu}_2 = \xi_2 + h_2 = 0.05779\), the 40-year mean determination of \(L_2\) at Batavia has amplitude \((\hat{\mu}_1^2 + \hat{\mu}_2^2)^{1/2} = 0.06347\) millimeters of mercury with phase angle \(\tan^{-1}(\hat{\mu}_2/\hat{\mu}_1) = 65^\circ 34'.\) It is common practice to include an estimate of the probable-error circle \(C_0\) with probable-error \(r_o\), i.e.,

\[C_0 = \{(u,v): (u-\mu_1)^2 + (v-\mu_2)^2 \leq r_o^2\}\]

such that \(P[(U,V) \in C_0] = 1/2\), where \((U,V)\) has density function (2.1). It can be shown that \(r_o = [-2\sigma^2\ln(1 - P/2)]^{1/2}\). Based on the data,

\[\hat{r}_o = 0.01275\] millimeters of mercury.

Chapman and Lindzen [5, p. 9] have suggested that a determination of \(L_2\) may be considered reasonably good if the amplitude of its dial vector is at least three times its probable-error. Each yearly determination in this example satisfies this criterion. Finally, from Figure A we see that 21 of the 40 points lie inside \(\hat{C}_0\), indicating that the assumed underlying distribution appears to be satisfactory (Bartels [3]).
REFERENCES


