ON THE REDUNDANCY OF MONOTONY ASSUMPTIONS

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Introduction.

It is well known that all the results in integral inequalities of Bellman-Gronwall-Reid type demand an assumption of monotony on the functions involved. Since the corresponding theory of differential inequalities does not require the monotonic assumption, it is believed that this extra condition is due to the technics employed rather than the necessity.

It is also well known that in proving the convergence of successive approximations, it becomes crucial to suppose an additional restriction of monotony on the functions satisfying uniqueness criteria. The question whether this additional assumption is really needed has been open for many years. This problem was discussed by the author in [5] where a partial answer was given, namely, it is sufficient if the function $g(t,u)$ satisfying, for example, Kamke's uniqueness condition (without the monotonic assumption) dominates the function

$$g_0(t,r) = \max_{||x-y|| \leq r} ||f(t,x) - f(t,y)||.$$ 

Later Olech and Pliss [3] also gave a partial answer showing that the monotony can be dropped if $g(t,u)$ is of a special type. Very recently Deimling [1] has given the complete answer to the problem by using $g_0$ type function judiciously and working around the critical point. Since
he considers an abstract Cauchy problem and consequently \( g_0 \) may not be continuous in \( r \), his proof needs an equivalent uniqueness criteria.

The known results [4,6,7] that give sufficient conditions for the existence of a limit as \( t \to \infty \) of solutions also assume monotonic condition on the comparison function and it is felt desirable to dispense with this requirement.

In this paper, we develop results that can be applied to show the redundancy of the assumption of monotony in several situations. Employing our auxiliary results, we prove that the monotonicity assumption is superfluous in

(i) the theory of integral inequalities of Bellman-Gronwall-Reid type;

(ii) the convergence of successive approximations including the infinite dimensional situation;

and

(iii) the sufficient conditions for the existence of a limit as \( t \to \infty \) of solutions.

We believe that our method can profitably be used in other situations and in vectorial integral inequalities.
2. Main Results

In this section we shall present the needed results which enable us to remove the assumption of monotony for comparison functions. The idea is as follows: given a comparison function \( g(t,u) \) which is not monotonic in \( u \), we construct, for every \( \beta > 0 \), a comparison function \( G_\beta \) which does possess the monotonic property required and which dominates the given function \( g \). We then utilize this function \( G_\beta \) as a vehicle and show that as \( \beta \to 0 \), we get back the desired results finally. Employing the dominating auxiliary comparison functions to tide over the difficulties is the whole idea of the paper.

We need the following known result [6,p.13] before we proceed further.

**Theorem 2.1.** Let \( g \in C[\mathbb{R}^+ \times \mathbb{R}^+ , \mathbb{R}] \) and let the maximal solution \( r(t,t_0,u_0) \) of

\[
(2.1) \quad u' = g(t,u), \quad u(t_0) = u_0 \geq 0,
\]

exist on \([t_0,\infty)\). Given a compact subinterval \([t_0,T] \subseteq \mathbb{R}^+ \), there exists on \( \varepsilon_0 > 0 \) such that, for \( 0 < \varepsilon < \varepsilon_0 \), the maximal solution \( r_\varepsilon(t,t_0,u_0) \) of

\[
u' = g(t,u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon,
\]

exists on \([t_0,T] \) and

\[
\lim_{\varepsilon \to 0} r_\varepsilon(t,t_0,u_0) = r(t,t_0,u_0)
\]

uniformly on \([t_0,T]\).
Theorem 2.2. Let \( g \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}] \) and let \( r(t, t_0, u_0) \) be the maximal solution of (2.1) existing on \([t_0, \infty)\). Then for each \( \beta > 0 \), there exists a function \( G_\beta \) satisfying the following properties:

(i) \( G_\beta \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}] \), \( G_\beta(t, u, v) \) is increasing in \( v \) for each \( (t, u) \);

(ii) \( G_\beta(t, u, v) \geq g(t, v), \ t \in \mathbb{R}^+, u \geq 0; \)

(iii) the maximal solution \( r_\beta(t, t_0, u_0) \) of

\[
(2.2) \quad u' = G_\beta(t, u, u), \quad u(t_0) = u_0 + \beta, \ u_0 \geq 0,
\]

exists on every compact subset \([t_0, T] \subset \mathbb{R}^+\) and

\[
\lim_{\beta \to 0} r_\beta(t, t_0, u_0) = r(t, t_0, u_0)
\]

uniformly on \([t_0, T]\).

Proof. Define the function

\[
(2.3) \quad g_0(t, r) = \max_{|u-v|=r} |g(t, u) - g(t, v)|,
\]

and note that \( g_0 \) is continuous in \((t, r)\), increasing in \( r \) for each \( t, g_0(t, 0) \equiv 0 \) and \( g_0(t, r) \to 0 \) as \( r \to 0 \) uniformly on every compact subset \([t_0, T] \subset \mathbb{R}^+\). For \( \beta > 0 \), let us now define

\[
(2.4) \quad G_\beta(t, u, r) = g(t, u) + g_0(t, r-u+\beta) + g_0(t, \beta), \ r \geq u-\beta.
\]

Clearly, in view of the properties of \( g \) and \( g_0 \), the conclusion (i) holds. For \( t \geq 0, \ u \geq 0 \), we see that
\[ G_\beta(t,u,r) \geq g(t,u) + g(t,r) - g(t,u-\beta) + g_0(t,\beta) \]
\[ = g(t,r) + [g_0(t,\beta) - (g(t,u-\beta) - g(t,u))] \]
\[ \geq g(t,r), \]

which proves (ii). Since \( G_\beta(t,u,u) = g(t,u) + 2g_0(t,\beta) \), if \( \beta \) is small enough, Theorem 2.1 implies the property (iii) and this proves the theorem.

For convenience, we list below some classes of functions satisfying uniqueness criteria:

\[ U_1 = \left\{ g \in C[[t_0,t_0+a] \times [0,2b],R^+] \mid g(t,0) \equiv 0 \right\} \]

the maximal solution \( r(t,t_0,0) \) of

\[ (2.5) \quad u' = g(t,u), \quad u(t_0) = 0, \]

exists on \( t_0 \leq t \leq t_0 + a \) and is identically zero];

\[ U_2 = \left\{ g \in C[[t_0,t_0+a] \times [0,2b], R^+] \mid g(t,0) \equiv 0 \right\} \]

\[ u(t) \equiv 0 \] is the only solution of \( (2.5) \) existing on \( t_0 \leq t \leq t_0 + a \) such that \( \lim_{t \to t_0^-} u(t) = 0 \];

and

\[ U_3 = \left\{ g \in C[[t_0,t_0+a] \times [0,2b], R^+] \right\} \]

given any \( \epsilon > 0 \) there exist a \( \delta > 0 \), sequences \( \{t_i\}, \{\delta_i\} \) with \( t_i \to 0, \delta_i \to 0 \) as \( i \to \infty \), a sequence of functions \( \{u_i(t)\} \), continuous on \([t_i,t_0+a]\)
such that \( u_i(t_i) \geq \delta t_i \),

\[
D^- u_i(t) \geq g(t, u_i(t)) + \delta_i \quad \text{and} \quad 0 \leq u_i(t) \leq \epsilon, \quad t \in [t_i, t_{0+a}]
\]

If \( g \in U_1(U_2) \), \( g \) is said to satisfy Perron's (Kamke's) uniqueness condition. The class \( U_3 \) is used in \([1]\).

Theorem 2.3. Let \( g \in U_3 \). Then, for each sufficiently small \( \beta > 0 \), there exists a \( G_\beta \) satisfying the following properties:

(i) \( G_\beta \in C[(t_0, t_{0+a}) \times [0, 2b] \times [0, 2b], R^+] \), \( G_\beta(t, u, v) \) is increasing in \( v \) for each \( t \in (t_0, t_{0+a}) \), \( 0 \leq u \leq 2b \);

(ii) \( G_\beta(t, u, v) \geq g(t, v), \quad t \in (t_0, t_{0+a}), \quad 0 \leq u \leq 2b \);

(iii) the sequence of maximal solutions \( r_\beta(t, t_i, \delta t_i) \) of

\[
(2.6) \quad u' = G_\beta(t, u, u), \quad u(t_i) = \delta t_i,
\]

exists on \( [t_i, t + a] \) and \( 0 < r_\beta(t, t_i, \delta t_i) < \epsilon \) on \( [t_i, t_{0+a}] \), where, given \( \epsilon > 0 \), the numbers \( \delta, t_i, \delta_i \) are as in \( U_3 \).

Proof. We construct \( G_\beta \) as in Theorem 2.2 with appropriate modifications. The properties (i) and (ii) are immediate. To prove (iii), we choose \( \beta > 0 \) such that \( 2g_0(t, \beta) \leq \delta_i, \quad t \in [t_i, t_{0+a}] \), and consider the maximal solutions \( r_\beta(t, t_i, \delta t_i) \) of (2.6). Here \( \delta, \{\delta_i\} \) and \( \{t_i\} \) are the same occurring in the definition of \( U_3 \). We have
\[ D^+ u_i(t) \geq g(t, u_i(t)) + \delta_i, \quad [t_i, t_0 + a], \]

\[ u_i(t_1) \geq r_{\beta_i}(t_1), \]

\[ r_{\beta_i}(t, t_1, \delta t_1) = g(t, r_{\beta_i}(t, t_1, \delta t_1)) + 2g_0(t, \beta_i) \]

\[ < g(t, r_{\beta_i}(t, t_1, \delta t_1)) + \delta_i, \quad t \in [t_i, t_0 + a]. \]

Hence by the theory of differential inequalities, [6]

\[ r_{\beta_i}(t, t_1, \delta t_1) \leq u_i(t) \leq \epsilon, \quad t \in [t_i, t_0 + a]. \]

Clearly \( r_{\beta_i}(t, t_1, \delta t_1) > 0 \) and the proof is complete.

3. Bellman-Gronwall-Reid type inequalities.

As an application of Theorem 2.2, we shall now prove a general theorem of Bellman-Gronwall-Reid type without the additional restriction of monotony.

Theorem 3.1. Let \( g \in C[R^+ \times R^+, R], \) \( m \in C[R^+, R^+] \) and

\[ m(t) \leq m(t_0) + \int_{t_0}^{t} g(s, m(s))ds, \quad t \geq t_0. \]

Suppose that the maximal solution \( r(t, t_0, u_0) \) of (2.1) exists on \( [t_0, \infty). \)
Then \( m(t_0) \leq u_0 \) implies

\[
m(t) \leq r(t, t_0, u_0), \quad t \geq t_0.
\]

**Proof.** By the property (ii) of \( G_\beta \) in Theorem 2.2 we have

\[
(3.1) \quad m(t) \leq m(t_0) + \int_{t_0}^{t} G_\beta(s, r_\beta(s, t_0, u_0), m(s)) ds, \quad t \geq t_0,
\]

where \( r_\beta(t, t_0, u_0) \) is the maximal solution of (2.2) which exists on every compact interval \([t_0, T] = \mathbb{R}^+\) by the property (iii). We shall show that

\[
(3.2) \quad m(t) < r_\beta(t, t_0, u_0), \quad t \geq t_0.
\]

Choose \( u_0 = m(t_0) \) so that \( m(t_0) < u_0 + \beta \). If (3.2) is not true, then there exists a \( t_1 > t_0 \), such that

\[
m(t_1) = r_\beta(t_1, t_0, u_0)
\]

and

\[
m(t) \leq r_\beta(t, t_0, u_0), \quad t_0 \leq t \leq t_1.
\]

Hence, using the monotonic character of \( G_\beta \), and the fact \( m(t_0) < u_0 + \beta \), we get from (3.1) the inequality

\[
m(t) < u_0 + \beta + \int_{t_0}^{t} G_\beta(s, r_\beta(s, t_0, u_0), r_\beta(s, t_0, u_0)) ds
\]

\[
= r_\beta(t, t_0, u_0), \quad t_0 \leq t \leq t_1.
\]

At \( t = t_1 \), we have

\[
m(t_1) < r_\beta(t_1, t_0, u_0) = m(t_1),
\]

As another application of Theorem 2.2, we shall prove the following theorem on the convergence of successive approximations omitting the usual restriction of monotony on the uniqueness function. Perron's, as well as Kamke's, uniqueness criteria are treated together.

Theorem 4.1. Let \( f \in C[R_0, R^n] \), where \( R_0 = \{(t,x): t_0 \leq t \leq t_0 + a, ||x-x_0|| \leq b\} \) and for \((t,x),(t,y) \in R_0\),

\[
\|f(t,x) - f(t,y)\| \leq g(t, ||x-y||).
\]

Suppose that \( g \in U_1 \) or \( U_2 \). Then the successive approximations defined by \( x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s,x_n(s))ds \) exist on \( t_0 \leq t \leq t_0 + a \), where \( a = \min(a, \frac{b}{M}) \), \( ||f(t,x)|| \leq M \) on \( R_0 \), as continuous functions and converge uniformly on this interval to the solution \( x(t) \) of

\[
x' = f(t,x), \quad x(t_0) = x_0.
\]

Proof. Let \( g \in U_2 \). Since uniqueness of solutions of (4.2) follows (see Th.2.2.1 in [6]), it is enough to prove that \( m(t) = 0 \). Where

\[ m(t) = \lim_{n \to \infty} \sup_{n} ||x_{n+1}(t) - x_n(t)||. \]

We now proceed with \( G_\beta(t,r_\beta(t),u) \) of Theorem 2.2 instead of \( g(t,u) \), where \( r_\beta(t) = r_\beta(t,t_0,0) \) is the maximal solution of \( u' = G_\beta(t,u,u), \quad u(t_0) = \beta \). Because of the fact
$G_\beta$ is monotone, following the standard proof [6, p.63], we arrive at the inequality

$$m(t) \leq \int_{t_0}^{t} G_\beta(s, r_\beta(s), m(s))ds, \ t \in [t_0, t_0 + \alpha].$$

Consequently, arguing as in the proof of Theorem 3.1, we obtain

$$m(t) \leq r_\beta(t, t_0, 0), \ t \in [t_0, t_0 + \alpha].$$

By Theorem 2.2, $\lim_{\beta \to 0} r_\beta(t, t_0, 0) = r(t, t_0, 0)$ and by $U_1$, $r(t, t_0, 0) \equiv 0$. This implies $m(t) \equiv 0$ and the proof is complete.

If $g \in U_2$, following Olech [2], define

$$g_f(t, r) = \max_{||x-y||=r} ||f(t, x) - f(t, y)||.$$

Noting that $g_f \in C[[t_0, t_0 + \alpha] \times [0, 2b], \mathbb{R}^+]$, $g_f(t, 0) \equiv 0$ and

$$g_f(t, u) \leq g(t, u), \ t \not\equiv t_0,$$

we see by Theorem 2.2.3 in [6] that $g_f \in U_1$. Consequently, with $g_f$ in place of $g$ proves the theorem. The proof of the theorem is therefore complete.

Remark. Similar conclusions can be drawn when $g$ satisfies other general uniqueness criteria (for example, see Theorem 2.2.4 and the comments on page 60 in [6]) as long as $f(t, x)$ is continuous. If, on the other hand, we wish to consider the differential equation (4.1) where
$f \in C[R_0, E]$, $E$ being a Banach space, the function $g_f$ defined above, may not in general be continuous in $r$, and consequently the foregoing trick of replacing $g$ by $g_f$ does not work. This case will be treated separately as an application of Theorem 2.3.

Theorem 4.2. Let $E$ be a Banach space, $f \in C[R_0, E]$, and $||f(t,x)|| \leq M$ on $R_0$. Let the condition (4.1) hold on $R_0$ except at $t = t_0$ and $g \in U_3$. Then the conclusion of Theorem 4.1 is true.

Proof. We recall that there is exactly one solution $x(t)$ of (4.1) on $t_0 \leq t \leq t_0 + \alpha$ and that $m(t) = 0(t-t_0)$ as $t \to t_0$, where

$$m(t) = \lim_{n \to \infty} \sup_{t \in [t_0, t_0 + \alpha]} ||x_n(t) - x(t)||, \quad t \in [t_0, t_0 + \alpha].$$

Hence, given $\varepsilon > 0$ there is $\tau = \tau(\delta) > 0$ such that $m(t) \leq \varepsilon(t-t_0)$, $t_0 \leq t \leq \tau$. Let us choose $\delta, t_0 + t_1 < \tau(\frac{\delta}{2})$, $\delta_1$ and $r_{\beta_i}(t, t_1, \delta t_1)$ according to the definition $U_3$ and Theorem 2.3. Then it follows that

$$m(t) \leq \frac{\delta}{2} \left( t_1 - t_0 \right) < \frac{\delta}{2} \tau \left( \frac{\delta}{2} \right) \leq \varepsilon, \quad t \in [t_0, t_1].$$

It is therefore enough to show that $m(t) \leq \varepsilon$, $t \in [t_1, t_0 + \alpha]$.

Proceeding as in Theorem 4.1, one gets for $t \in [t_1, t_0 + \alpha]$

$$m(t) \leq \frac{\delta}{2} t_1 + \int_{t_1}^{t} G_{\beta_i}(s, r_{\beta_i}(s, t_1, \delta t_1), m(s))ds.$$ 

We have, by Theorem 2.3,

$$r_{\beta_i}(t, t_1, t_1) > \frac{\delta}{2} t_1 + \int_{t_1}^{t} G_{\beta_i}(s, r_{\beta_i}(s, t_1, \delta t_1), r_{\beta_i}(s, t_1, \delta t_1))ds.$$ 

Consequently, by the theory of integral inequalities (see Th.5.1.1 in [6])
we get $m(t) < r_*(t, t', \delta t) \leq \varepsilon$, $[t_i, t_0 + \alpha]$. The proof is therefore complete.

Theorem 4.2 is essentially the result of Deimling [1]. Our proof is short and throws much light on the problem.

5. Asymptotic behaviour.

We consider the differential equation

\[(5.1) \quad x' = f(t, x), \quad x(t_0) = x_0,\]

where $f \in C[R^+ \times E, E]$, $E$ being a Banach space. Assume that $f$ is smooth enough to ensure local existence of solutions of (5.1) for $(t_0, x_0) \in R^+ \times E$. The known results [4,6,7] that offer sufficient conditions for the existence of a limit as $t \to \infty$ of solutions of (5.1) assume either monotony or some additional condition on the comparison function. Using Theorem 2.2, we shall prove this result without the monotonic assumption.

Theorem 5.1. Suppose that

\[||f(t, x)|| \leq g(t, ||x||), \quad (t, x) \in R^+ \times E,\]

where $g \in C[R^+ \times R^+, R^+]$ and the maximal solution $r(t) = r(t, t_0, u_0)$ of

\[u' = g(t, u), \quad u(t_0) = u_0 \geq 0,\]
exists and is bounded on \([t_0, \infty)\). Then every solution \(x(t) = x(t, t_0, x_0)\) such that \(||x_0|| \leq u_0\) exists on \([t_0, \infty)\) and satisfies \(\lim_{t \to \infty} x(t) = y \in E\).

**Proof.** Let \(x(t)\) be a solution of (5.1) such that \(||x_0|| \leq u_0\) which exists on \(t_0 \leq t < \alpha < \infty\) and the value of \(\alpha\) cannot be increased. Let \(T > 0\) be such that \([t_0, \alpha] \subset [t_0, T]\). Defining \(m(t) = ||x(t)||\), we obtain

\[
m(t) \leq m(t_0) + \int_{t_0}^{t} g(s, m(s))ds
\]

\[
\leq m(t_0) + \int_{t_0}^{t} G_\beta(s, r_\beta(s), m(s))ds, \quad t \in [t_0, \alpha),
\]

where \(G_\beta\) and \(r_\beta\) are as given in Th. 2.2. It follows by Theorem 3.1 that

\[
(5.2) \quad m(t) \leq r_\beta(t), \quad t_0 \leq t < \alpha,
\]

where \(r_\beta(t) = r_\beta(t, t_0, u_0)\) is the maximal solution of (2.2). Also, for any \(t_0 < t_1 < t_2 < \alpha\), we have, using monotony of \(G_\beta\),

\[
(5.3) \quad ||x(t_1) - x(t_2)|| \leq \int_{t_1}^{t_2} g(s, ||x(s)||)ds
\]

\[
\leq \int_{t_1}^{t_2} G_\beta(s, r_\beta(s), r_\beta(s))ds
\]

\[
= r_\beta(t_2) - r_\beta(t_1).
\]

Since \(\lim_{t \to \alpha} r_\beta(t)\) exists and is finite, it follows from (5.3) that \(\lim_{t \to \alpha} x(t)\) exists. Defining \(x(\alpha) = \lim_{t \to \alpha} x(t)\) and considering (5.1)
with $x(\alpha)$ as the initial condition at $t = \alpha$, leads to a contradiction because of the assumed local existence. Hence (5.2), (5.3) hold with $\alpha = \infty$.

Since $r(t)$ is bounded and nondecreasing on $[t_0, \infty)$, it follows that $\lim_{t \to \infty} r(t)$ exists. Thus (5.3) gives, together with the properties of $r_\beta(t)$, for $t_0 < t_1 < t_2 < \infty$,

$$ ||x(t_1) - x(t_2)|| \leq r_\beta(t_1) - r_\beta(t_2) $$

$$ \leq |r_\beta(t_1) - r(t_1)| + |r(t_1) - r(t_2)| + |r(t_2) - r_\beta(t_2)| $$

$$ < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon $$

This yields the desired result and the proof is complete.
REFERENCES


