NON-TRADITIONAL SOCIO-MATHEMATICAL NORMS

IN UNDERGRADUATE REAL ANALYSIS

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

December 2009

ACKNOWLEDGMENTS

I give all praise, honor, and glory to God through Jesus Christ for the opportunities I have enjoyed to do this work and the abilities He has provided that made it possible.

I am overwhelmingly grateful to my family for their support and love, especially my Father, Dr. Ross Clinton Dawkins, Jr. who has raised me to be a scientist in heart and mind as he is.

My most heartfelt thanks go out to my mentor and friend Dr. James A. Mendoza Epperson for his leadership, guidance, patience, and faith in me. I cannot count the times I have told people what a blessing you have been on every level of my graduate experience. You are a superb mentor in every way.

Thanks also to my committee members for their participation in this process. I give special acknowledgment to Dr. Christopher Kribs-Zaleta and Dr. Theresa Jorgensen for helping me create a meaningful mathematics education doctorate experience where none was previously defined. Thanks also to Dr. D.L. Hawkins for his academic support and statistical help with previous projects.

Thanks to the Coffee Haus Downtown for fueling a large majority of the hours

that went into this work. I also am thankful for all of the wonderful music that

provided the soundtrack to my working hours, but I will not name artists here.

<p align="right">November 13, 2009</p>

ABSTRACT

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This study builds upon the framework of classroom norms (Cobb, Wood, &

Yackel, 1993) and socio-mathematical norms (Cobb & Yackel, 1996) to

understand how non-traditional socio-mathematical norms influence student

reasoning and transitions to advanced mathematical thinking in undergraduate

real analysis. The research involves a qualitative investigation of classroom

instruction and interactions, student and instructor interviews, and class exams.

The study explores the roles of each norm as the students constructed

understanding of advanced mathematics and transitioned to advanced

mathematical thinking. I define "non-traditional" according to research accounts

of traditional instruction in proof-based mathematics courses and considerations on the culture of the mathematics community. Evidence from this study indicates that classrooms in which students participate in constructing mathematics and act as mathematical validators strongly facilitates the transition to advanced modes of mathematical thinking and promotes students' mathematical autonomy. Students moved toward mathematical mindsets common to mathematicians by practicing the creative and constructive processes similar to research mathematicians. These norms led the classes to operate as microcosms of the greater mathematical community and institutionalize meaning as a classroom community (Cobb, 1989).

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure   Page

LIST OF TABLES

CHAPTER 1

INTRODUCTION

When one speaks of "advanced mathematical thinking" transcription becomes non-trivial insomuch as there exists a distinction between "advanced mathematical thinking" and "advanced, mathematical thinking." Is the mathematics advanced or the thinking advanced? In the classical work on the matter, Tall's (1991b) *Advanced Mathematical Thinking*, he described the goal of advanced mathematical teaching as students becoming "mature mathematicians at an advanced level" (p. 7). "Mature mathematicians" implies that students engage in advanced thinking about mathematics while "at an advanced level" implies that students think about advanced mathematics, thus Tall indicates advanced mathematical thinking represents the composition of advanced mathematics and advanced reasoning.

Tall (1991b) indicates that the distinction between advanced mathematical thought from elementary mathematical thought lies in the actions of defining and deducing. Only in university mathematics do students usually encounter the practice of rigorous defining laying the foundation for rigorous validation of mathematical statements, validation established by deductive argument about

those definitions. Certainly these actions constitute a large part of what the mathematical community does and thus student participation in these activities represents some form of students becoming mathematicians.

However, much university mathematics teaching fails to directly guide students toward participating in such actions. Quoting Skemp (1971), Tall (1992) points out that far too much mathematics instruction "teaches the product of advanced mathematical thought, not the process of advanced mathematical thinking" (p. 509). Tall (1991a) spoke about this distinction saying:

> There is a huge gulf between the way in which ideas are built cognitively and the way in which they are arranged and presented in a deductive order. This warns us that simply presenting a mathematical theory as a sequence of definitions, theorems and proofs (as happens in a typical university course) may show the logical structure of the mathematics, but it fails to allow for the psychological growth of the developing human mind (p. xiv).

In other words, logical presentation puts mathematical thought on display without necessarily promoting mathematical thinking among students or even giving them footholds by which to move toward such practice.

Mathematics education research at the elementary level also identifies the importance of teaching both mathematical content and mathematical ways of thinking. Cobb and Yackel (1996) observed how the emergence of a set of normative structures of mathematical interaction helped guide students toward more independent mathematical meaning making. In a classroom characterized by certain social norms of mathematical interaction, these researchers observed the classroom acting as a microcosm of the mathematical community and therein

2

observed the students' transition to advanced thinking about elementary mathematics.

In some very important ways, Cobb's and Tall's viewpoints harmonize across the curricular gaps between their areas of research focus. Cobb (1989, 1994) argues that mathematics teaching bears responsibility to both promote individual student cognitive development and to enculturate students into the mathematical community. As students make meaning, the teacher must guide them to do so in a way that sufficiently matches standard meanings adopted by society. This parallels perfectly Tall's (1991a) previous quote addressing the classroom's need for dual attendance to the logic of mathematics as well as the psychological development thereof. So while Cobb observed the role socio-mathematical norms played in promoting student cognitive development at the elementary school mathematics level, Tall affirmed the need for university teaching that promotes similar advanced cognitive constructions about advanced topics.

The present study builds upon the framework of classroom norms by Cobb, Wood, and Yackel (1993) and socio-mathematical norms by Cobb and Yackel (1996) to understand the following questions at the advanced mathematical level:

1) How does the establishment of non-traditional socio-mathematical norms in an introductory real analysis course affect students' understanding of analysis and proof-based mathematics and influence students' mathematical autonomy?

2) How does the establishment of specific socio-mathematical norms affect students' meta-mathematical understanding of real analysis and proof-based mathematics alongside students' understanding of the content of real analysis?

The term "non-traditional" will be clarified by the literature's description of traditional proof-based instruction. In this work, I describe the nature of an undergraduate real analysis classroom that I argue is analogous to those observed by Cobb et al. (1993) at the elementary level and I investigate students' transition to advanced mathematical thinking in the context of their experience in the classroom. This study assumes that the purpose of advanced mathematics courses is to promote advanced mathematical thinking. By this I mean that they should facilitate students' transition toward ways of thinking that emulate those of mathematicians as well as promote student comprehension of advanced mathematical topics in a way compatible with the shared meanings of the mathematical community. The research literature provides a description both of mathematicians' way of thinking and the community's shared values by which I make this general goal more explicit.

An interesting question arises amid the translation of Cobb's (1989, 1994) work to the advanced mathematical level. He deems "enculturation" (Cobb, 1994)

to mathematics as one goal of mathematical instruction, which assumes that mathematics is part of the dominant culture in which elementary school children live. Since most of our society is relatively conversant in arithmetic, this assumption appears valid. However, it is not so obvious whether training upper-level undergraduate math students in mathematical culture should also be called "enculturation" or whether the term "acculturation" is more appropriate, which would mean students are being initiated into a different or foreign culture. Though this question is philosophically interesting, we shall throughout this study maintain Cobb's (1994) language of "enculturation."

CHAPTER 2

LITERATURE REVIEW

Several key strains from the research literature inform and situate the

present investigation. Parallel to Tall's (1991b) overview of advanced

mathematical thinking I review the literature regarding visualizing, defining, and

proposing and proving. Though visualizing is not inherent to all mathematical

thinking, the visualizing/arithmetizing dynamic holds a vital place within the

history of mathematical understanding and progress (Alcock & Simpson, 2004;

Dreyfus, 1991; Eisenberg & Dreyfus, 1991). The processes of defining and then

proposing and proving theorems based upon those definitions constitute the key

processes that set advanced mathematical thinking apart from its curricular

predecessors (Alcock & Simpson, 2002; Edwards & Ward, 2008; Tall, 1991b).

Finally, I consider some more global issues of mathematics instruction discussed

in the literature.

2.1.   Visualizing

An oft-heard layman's response to a mathematician asks whether the

latter is an "algebra person" or a "geometry person." This pervasively perceived

dividing line between how people think about mathematics has grounding in

analyses of mathematical history. Poincare (1913, as cited in Tall, 1991b) pointed to a clear division of the mathematical community into analytical thinkers like Weierstrass or graphical thinkers like Riemann. Krutetskii (1976) described three categories of preferred reasoning among different mathematics students: visual reasoning, verbal-logical reasoning, and harmonic reasoning which blends the two. Eisenberg (1991) quoted Hilbert to highlight how, in the last one and a half centuries, many mathematicians have valued graphical reasoning while still considering it a second-class citizen in proof-based mathematics only acceptable when coupled with (the possibility of) proper analytical argument (Dreyfus, 1991; Stylianou & Silver, 2004).

Intuitive power represents one of the prime strengths attributed to visual reasoning above analytical reasoning. What people perceive visualization lacks in rigor they claim it makes up in immediacy and ability to compel the mind. Poincare (1913, as cited in Tall, 1991b) pointed to this distinction when he compared the analytical approach to trench warfare and the visual to a cavalry charge. He also said of Riemann's visual arguments, "each of his conceptions is an image that no one can forget, once he has caught its meaning" (Poincare, 1913, as cited in Tall, 1991b, p. 4). Fischbein's (1987) classical work on mathematical intuition also identified the connection between visual image or argument and intuition. Fischbein also indicated that visual arguments often prove so compelling to some students that the images provide disincentives to formal proof.

According to Hogarth (2001), cognitive science defines intuition in terms of its immediacy in that it is pre-deductive knowledge or reasoning. He indicates that there exists a connection between the mind's non-sequential experience of visual input and the intuitive mode of cognition. Analytical modes of processing by comparison are more sequential and intentional and thereby more deductive. Visual reasoning by itself is not intuitive reasoning, but intuitive notions are often disproportionately controlled by visual images. This strong association between visual reasoning and intuition, which tends to press itself upon the mind, explains some mathematicians' distrust of visual modes of argument because any assumptions the image pre-supposes may be all the more likely to stay implicit since conclusions drawn from visual inputs approach having pre-deductive nature. This connection also helps explain the convincing power of visualization and its connection with affect (Presmeg & Balderas-Canas, 2001).

Despite the way in which mathematicians such as Hilbert (1862-1943) and the Bourbaki school sought to ignore or subsume both intuition and visualization through the application of axiomatic-deductive and analytical methods, visualization stands firmly established in much mathematical thinking and, others would argue, indispensible in the pursuit of mathematical knowing (Eisenberg & Dreyfus, 1991; Vinner, 1989). Tall (1991b) argued for the synergistic benefits of attendance to both logical-analytic and visual reasoning saying:

> In general it may be possible to use the complementary power of
> visualization to give a global gestalt for a mathematical concept, to show
> its strengths and weaknesses, its properties and non-properties, in a way

that makes it a logical necessity to formulate the theory clearly. Visual ideas without links to the sequential processes of computation and proof are insights which lack mathematical fulfillment. On the other hand, logical sequential processes, without a vision of the total picture, are blinkered and limiting. It is therefore a worthy goal to seek the fruitful interaction of these very different modes of thought (p. 18).

Eisenberg (1991) went on to argue that when the classroom ignores the role visualization plays for mathematicians, the result goes beyond historically dishonest to damaging. "The Hilbert-Bourbakian view of mathematics has produced generations of semi-literates, in part because the pictures which motivate the proofs and which are behind the big ideas are seldom emphasized in the classroom" (p. 148).

Mathematics education research on K-12 instruction identifies systemic issues undermining visualization in the classroom and among students. At the grade school and calculus level, research finds that non-visualizers generally outperform visualizers (Eisenberg & Dreyfus, 1991; Vinner, 1989) despite many research-based calls for instruction that uses and values multiple representations (Hughes-Hallet, 1991; NCTM, 2000). Some argue that this classroom disparity stems from a curricular and historical favoritism toward analytical and algebraic methods (Vinner, 1989). Others argue that the absence of instruction which equips students to use visual reasoning properly and overcome the "one-case concreteness of an image or a diagram" leads students to avoid visual reasoning (Eisenberg & Dreyfus, 1994, p. 47). Although some question the claim that students avoid visualization, most agree that visualization deserves both a more prominent place in instruction which would require more direct training in visual

9

reasoning in order to be used properly (Aspinwall, Shaw, & Presmeg, 1997; Eisenberg & Dreyfus, 1991).

Aspinwall et al. (1997) criticize the way that many traditional uses of visual representations in the classroom treat graphs or diagrams as something to which students must acquiesce, another something to learn, rather than something students produce and use as a pathway to learning other ideas. They indicate the existence of great potential in visual modes of exploration saying:

> Traditionally, graphic representations have been treated as a desired end in mathematics curricula; students' progress has been measured by how closely they are able to express their mathematical understanding as accurate manifestations of the instructional representations. A concrete proposal for managing graphs and other diagrams is to treat them as instructional activities that constitute a starting point for students' mathematical constructions. Such diagrams can make it possible for teachers to guide students into novel experiences by drawing on students' prior knowledge and experience. Students interpreting mathematical meaning in these activities would form increasingly sophisticated mathematical conceptions (p. 314).

When teachers use visualization as a means of mathematical exploration rather than a curricular goal, it can become a tool with which students construct mathematical understanding. Presmeg and her colleagues' (Aspinwall et al., 1997; Presmeg, 1997; Presmeg & Balderas-Canas, 2001) extensive work on visualization mostly observed students at or below the calculus level, but others have examined issues of visualization in proof-based settings.

Research on student thinking about sequences and series affirmed both a clear division of students as visual thinkers or verbal-algebraic thinkers and also some attitudinal differences towards proof that stem from visualization's intuitive

effects. Alcock and Simpson (2004, 2005a) investigated student reasoning about sequences and series. They describe how students could be neatly classified according to their tendency to visualize or their tendency toward verbal and algebraic modes. Whereas the latter group displayed more propensity toward seeking proof and verification, the former group more often appeared convinced of truth or falsehood according to their visual explorations and felt little impetus to algebratize or further establish the validity of their conclusions. Interactions between visualization and defining or visualization and proof are further discussed below in the exploration of the defining and proof research literature.

2.2. Defining

The centrality of definitions in proof-based mathematics, and thereby much of advanced mathematical thinking, descends from the structure of mathematical proving dating back to Euclid (c.371 - c.285 BC). The tradition of proof in mathematics uses formal or precise rather than informal or inductive definitions to establish truth deductively. For the purposes of proof, a definition means nothing more and nothing less than the logical entailments of its definition and as Selden and Selden (2008) noted:

> While understanding the logical structure of a definition or a theorem is certainly not sufficient for constructing a proof, it is definitely necessary. In other words, if you do not understand what something really says, you certainly can't prove it (p. 102).

However, research has found that students might not understand this statement the way it is meant because they do not necessarily think that "what something

11

really says" depends more on the formal definitions of the terms therein rather than their intuitive definition (Edwards & Ward, 2008).

Mathematics education research explores mathematical definition in three primary ways: identifying how students construct understanding of definitions, contrasting student perception of the role of mathematical defining, and proposing alternate ways for students to engage definitions.

2.2.1 Students Constructing Understanding of Definitions

The research literature presents several accounts, both theoretical and empirical, of how students come to understand proofs. Dubinsky and his colleagues (Asiala, Brown, Devries, Dubinsky, Mathews, & Thomas, 1996; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997) in the APOS school of mathematics education research argue that students progress through the Action-Process-Object trajectory in their understanding of quantifiers and their relationships within formal analysis definitions (Dubinsky, 1991). As students come to coordinate the relationships between quantifiers and quantities, these structures are encapsulated into units through reflexive abstraction. They call the specific order and nature of the stages and this process and the transitions in between a genetic decomposition of the definition, "genetic" implying this parsing to be inherent in the make-up of definitions in the mind of the knower.

Others have questioned the universality or inherence of this approach to definition understanding. Pinto and Tall (2002) describe the successful and

sophisticated development of one student's concept of sequence convergence that was inextricably tied to two-dimensional graphical images of sequences. This student was able to overcome the "one-case concreteness of an image or a diagram" (Eisenberg & Dreyfus, 1994, p. 47) through creating what Pinto and Tall called generic images rather than specific images. Generic images portray the key aspects of a set of examples without establishing any extra visual or behavioral coherence to the images' form. For instance, the student drew a convergent sequence avoiding any familiar examples or strong behavioral properties like monotonicity. Pinto and Tall (2002) contrast this student's constructed understanding of the definition with Dubinsky, Elterman, & Gong's (1988, cited in Pinto & Tall, 2002) account by defining the processes of *giving meaning* to a definition and *extracting meaning* from a definition. The student in their study learned to interpret the graphical images with which he was already familiar via the definition thus giving meaning to the definition through the image. Dubinsky et al.'s genetic decomposition account describes how students might extract meaning from the formal definition itself by reflection upon the quantities and quantifiers therein.

Alcock and Simpson (2002) identify three distinct modes of reasoning that students use to answer questions about categories, which are sets of examples that definitions are constructed to describe. Some students use a prototypical example to represent a whole class of examples. The authors call this use of a prototypical example *generalizing*. Other students use sets of examples to

consider different aspects' range of variation within a category, but still reason about the whole category in terms of exemplars. The authors refer to this second approach as *property abstraction* because students using this method of reasoning observe properties that are common amongst their chosen set of examples and abstract them to the entire class. Students using the third approach to classification reason about the definition itself to understand the set of examples it entails. This approach makes no reference to prototypes and is called *working with definitions*.

Though Alcock & Simpson (2002) acknowledge that none of these mental activities are foreign to mathematical thought or practice, they distinguish the first two modes from the latter by distinguishing between extracting the definition or defining property from the set of examples, where the set establishes the definition, versus those definitions whose constitution establishes the set itself, the definition truly defines the set. They explain:

> The result is a fundamental difference in the nature of the category students work with. For both Wendy [example generalizer] and Cary [example property abstracter], the category is pre-existing (and non-classical) and for Cary the properties of the category *follow* from it. Greg's [example definition user] approach to property use, however, goes beyond Cary's by inverting the property/category relationship: the defining property *determines* the category (Alcock & Simpson, 2002, p. 32)

This distinction matches the one that Edwards and Ward (2008) observe between *extracted* definitions and *stipulated* definitions. Alcock and Simpson (2002) discuss the cognitive limitations to the two modes dependent upon examples, pointing out that the transition to advanced thinking involves moving

14

from the particular to the general, which the shift from prototype to definition enables. As they point out, "appropriate use of the definition means that any correct deductions he makes will be valid for all members of the mathematical category of convergent sequences" (p. 32).

According to memory researchers from the cognitive sciences students tend to use examples because this approach pervades common thinking in non-technical contexts (Bower, 2000). Alcock and Simpson (2002) argue that students must be trained to overcome their dependence on these habits of mind in favor of mathematical modes of thought such as reasoning from definitions. However, they assert that since many mathematics professors take these structural aspects of definitions and their usage for granted, students often receive no direct instruction in such matters.

2.2.2. *Differences Between Student and Mathematician Concepts of Definition*

Alcock and Simpson (2002), Edwards and Ward (2008), and Vinner (1991) all point to an important distinction between student and advanced mathematical use of definitions. Alcock and Simpson (2002) presented accounts of both types of reasoning. They emphasized the logical differences between the two viewpoints. Edwards and Ward (2008) specifically investigated student conception both of specific definitions and mathematical defining itself. They found that many students think of mathematical definitions in the same way as they would lexical definitions insomuch as their meaning is extracted from

15

intuition or the sets they intend to describe. In this way, if there is a conflict between students' idea of a definition and the logical entailments of the formal definition, they give preference to the idea because in their thinking it is primary. Alcock and Simpson (2002) and Edwards and Ward (2008) both identify that though mathematical definitions do have a history and meanings have shifted over time, in the practice of mathematics definitions are stipulated to mean only what the definition entails and are not contextually dependent.

Vinner (1991) discussed a similar distinction between approaches to definitions in terms of his previously coined constructs of concept image and concept definition (Tall and Vinner, 1981). The concept image contains all of the images, notions, applications, connections, etc. associated with a given concept, while the concept definition represents only the form of words used to delineate that concept. Intuitive notions and sets of examples belong to the concept image, and thus Vinner frames any possible conflict between formal definitions and prototypes or intuition as an issue of whether a student's concept image or concept definition holds dominance. Mathematicians give the concept definition dominance, but reason using the interplay of the two. Students often tend to give dominance to the concept image.

One of the starkest consequences of the distinction between student use and advanced mathematical use of definition appears in the case of formal theories of infinity. In line with the historical story, no other advanced

16

mathematical topic receives stronger resistance from students than Cantorian definitions of infinite cardinalities and the implications thereof. Research has found that students' intuitive and informal notions of infinity prove resistant to instruction (MacDonald & Brown, 2008; Tirosh, 1991). Cantor argued that mathematical theories should be accepted based on their self-consistency rather than on the beliefs of mathematicians, like the more limiting beliefs of his persistently antagonistic mentor Kronecker (Maor, 1987). The mathematical community subsequently accepted Cantor's viewpoint and affirmed his theories based upon the logical rigor and consistency, suppressing the intuitive difficulties induced. However, when students still operate from a prototype-based viewpoint of definitions, relying more heavily on intuition and their concept image, they find little impetus to transition to thinking in terms of the accepted theories of infinity based on the bijection definition of cardinality (MacDonald & Brown, 2008).

However, Alcock and Simpson (2002) and Edwards and Ward (2008) also acknowledge that the concept of stipulation does not capture the full complexity of mathematical defining. There can be disagreement about which property of a set of examples best captures the essence of that category (Alcock & Simpson, 2002). Freudenthal (1973, as cited in DeVilliers, 1998) points out that there are mathematical definitions of both *descriptive* and *constructive* natures. Descriptive definitions take a given concept and choose a subset of its properties that are taken as foundational, and then the rest of the properties are extracted from this subset. Constructive definitions define a new concept by its demarcation in a

17

definition. Freudenthal (1973, as cited in DeVilliers, 1998) complained about instruction that treats definitions as completely arbitrary, because he argued the historical reality is that most definitions were not produced in final form, but rather appeared during the final stages of the organizing activity.

Tall (1992) points out that differentiability can either be defined in terms of secant lines approaching a unique tangent line or in terms of what he calls *local straightness*. The distinction matters both in conceptual accessibility for students and also in generalization since the notion of secant and tangent lines or the limit of a difference quotient are all absent from the multi-dimensional extension of the derivative. Local straightness conceptually extends quite easily to the multi-dimensional case. This case reveals that, even though in the practice of mathematics the definition determines the concept, there are cases in which a shift in the concept or concept image (like from limits of difference quotients to approximation by a linear transformation) precipitates a shift in the formal definition. Moreover, the context of application (single variable functions or multi-variable functions) determines the formulation of the definition.

Edwards and Ward (2008) also point out that mathematical definitions come laden with a set of community values rather than solely logical entailments. Van Dormolen and Zaslavsky (2003, as cited in Edwards & Ward, 2008) provide a list of common criterion required of a mathematical definition including: that concepts be defined as subgroups of larger sets, that an exemplar exist, that

multiple definitions be logically equivalent, and that definitions fit into a deductive system. However there are several more aesthetic criteria also commonly employed such as minimality, elegance, and treatment of degenerate cases (examples which satisfy the definition but not the concept). Most mathematicians assume and operate according to such criteria, but few teach directly about such matters (Vinner, 1991).

In summary, the research literature indicates that mathematical definitions:

- are often extracted originally from observed patterns (DeVilliers, 1998),

- change over time (Edwards & Ward, 2008),

- can be context dependent (Tall, 1992), and

- are designed to satisfy value-driven as well as logical demands (Edwards & Ward, 2008).

Thus, mathematical defining represents an inherently human activity.

The question then arises as to what constitutes standard meaning of mathematical concepts. Alcock and Simpson (2002) indicate that standard meaning for definitions depends upon communal acceptance by mathematicians:

> This process of choosing [defining] properties is institutionalized within the mathematics community. While definitions… can often be traced to properties abstracted from an individual's prototype, one function of the community is to debate which of these properties best capture what is common to those objects in the category under discussion. This is a worthwhile enterprise, because making such definitions facilitates communication on a large scale by making reasoning in the subject systematic (p. 32).

The process of defining is a communal activity mutually negotiated for the purpose of meaning making and communication. This activity constitutes part of the practice of advanced mathematics.

1.2.3 Defining in the Classroom

This negotiative and human aspect of mathematical defining stands in particular contrast to common proof-based classroom practice in which definitions are often presented as complete and as the beginning or means to classroom mathematical exploration rather than the object or goal thereof. As DeVilliers (1998) puts it,

> The construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching (p. 249).

Many have called for change in this regard, often suggesting that students might need to be engaged in the act of defining to understand mathematical definitions properly. Mason and Watson (2008) conjecture that almost all definitions represent "an important shift in the way of perceiving and thinking that someone made in the past, and has to be re-experienced by each learner" (p. 200).

Vinner (1991) argued that students should learn to construct definitions from sets of examples of the objects being defined, thereby learning the importance of the correspondences and differences between concept images and concept definitions. He said:

> Our belief is that mathematical concepts, if their nature allows it, should be acquired in the everyday life mode of concept formation and not in the

technical mode. One should start with various examples and non-examples by means of which the concept image will be formed…
[Students] should be trained to use the definition as an ultimate criterion in various mathematical tasks. But in order to achieve this goal, one should do more than introducing the definition. One should point at the conflicts between the concept image and the formal definition and deeply discuss the weird examples (p. 80).

DeVilliers (1998) echoed this recommendation when he referenced mathematician Felix Klein's (1849-1925) bio-genetic principle of teaching.

Essentially, the genetic approach departs from the standpoint that the learner should either retrace (at least in part) the path followed by the original discoverers or inventors, or to retrace a path by which it could have been discovered or invented. In other words, learners should be exposed to or engaged with the typical mathematical processes by which new content in mathematics is discovered, invented and organized (p. 248).

Branford (1908) made a very direct attack on presenting definitions rather than constructing them:

To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational (p. 216-7).

These mathematicians all argue that because definitions represent the final product from a process of mental organizing and reorganizing, it appears dishonest to expect students to simply acquire the definition without engaging in a similar mental process. Stated more positively, one way to guide students to construct concept definitions that match those adopted by the mathematical community is to guide them to emulate the processes of the definitions' original

conception. This hearkens back to Tall's definition of advanced mathematical thinking in terms of students participating in mental processes and activities indicative of those practiced by mathematicians.

2.3. Proposing and Proving

Since the time of Euclid's Elements, proof has been a centerpiece of the mathematical endeavor (Mariotti, 2000). Thus mathematics educators have often asked the question: To what extent, at what levels, and in what form should proof appear in the mathematics classroom? However, the present intersections between proof and the classroom receive much criticism in mathematics education literature. The converse question then arises, which is: to what extent, at which points, and in what way should the classroom context affect the treatment of mathematical proof?

On one hand, Cobb (1994) points out that mathematics education is as much a matter of enculturating students into standard practice as it is of individual cognitive development. Deductive proof has been a part of mathematics culture for thousands of years and most programs for undergraduate mathematics majors reflect this expectation that students assimilate to the culture of proofs. In fact, the standard form of instruction in most proof-based courses participates in the tradition of minimal, rigorous, analytical proof: what Alibert and Thomas (1991) call the "usual 'linear code' type" (p. 220) of proof or "standard linear proof style" (p. 223) and Hanna (2000) called "sterile

formalism" (p. 15). Though this tradition on some level predates them, Hilbert and

the Bourbaki school are often given credit for their heavy contribution thereto

(Alibert & Thomas, 1991; Eisenberg, 1991; Hanna, 1991, 2000).

On the other hand, many in the mathematics education community have

questioned the appropriateness and sufficiency of proof instruction characterized

by the minimal, rigorous style. For instance, Alibert and Thomas (1991) say:

> The linear formalism of traditional proof may be described as the minimal
> code necessary for the transmitting of the mathematical knowledge. It
> appears, however, that in several important respects, it is a sub-minimal
> code, resulting in an irretrievable loss of information vital for understanding
> (p. 220).

Alibert & Thomas argue that if minimal, linear proof constitutes students' only

exposure to proof-based mathematics, then they will automatically begin lacking

essential elements of proof's content and meaning. Hersh (1993) echoed this

assertion saying, "the passage from an informal, intuitive theory to a formalized

theory inevitably entails some loss of meaning or change of meaning. The

informal by its nature has connotations and alternative interpretations that are not

in the formalized theory" (p. 390).

All three of these criticisms are founded upon the premise that since

instruction intends to teach students how to prove in addition to some portion of

the proof canon, the linear form of argument fails because it does not provide all

of the information they would need to understand the construction of proof or to

construct proofs themselves. Mariotti (2006) points out the pedagogical danger

of teaching only the rigorous, logical aspect of proof:

Proof clearly has the purpose of validations- confirming the truth of an assertion by checking the logical correctness of mathematical arguments- however, at the same time, proof has to contribute more widely to knowledge construction. If this is not the case, proof is likely to remain meaningless and purposeless in the eyes of students (p. 198).

The proposition thus arises to reformulate formal mathematics for the purposes of education. Hanna (1989, 1991), Hersh (1993), Hoffman (cited in Dreyfus, 1991), and Tomoczko (cited in Dossey, 1992) have called for a new philosophy of mathematics that reflects both the rigorous demands of the research mathematician and the communication and comprehension demands of the mathematics educator. As Alibert and Thomas (1991) stated:

Whilst most of the work in mathematics education rightly seeks to improve the learning and communication of mathematics by supplementing the formalism, it is also important to look at the formalism itself and consider how it too might be improved, leading to better communication and understanding (p. 220).

Thus we now turn our focus to re-evaluations of the nature of mathematical proving in the research literature and some of the curricular corollaries thereof.

2.3.1. The Dual Nature of Proving

It has been widely observed that proof entails two different aspects. A series of names have been associated to these two aspects of proof. Hersh (1993) described the roles of proof as explaining and convincing. Alibert and Thomas (1991) discuss the differences between the goal of providing understanding and connections and the goal of convincing. Mariotti (2006) refers to the parallel notions of acceptability and validation. She also describes Duval's (1991, cited in Mariotti, 2006) distinction between "the semantic level, where the

epistemic value of a statement is fundamental, and the theoretical level where, in principle, only the validity of a statement is concerned" (p. 182). Harel and Sowder (1998) said, "The process of proving includes two sub-processes: *ascertaining* and *persuading*" (p. 241). Proof thus relates both to:

- the need for understanding and insight into the validity of a given conjecture which is a dialectical purpose and

- the need for logical verification of the validity of a given conjecture which is a logical/theoretical purpose.

Alibert and Thomas (1991) argue that a proof's sufficiency in the logical verification aspect does not necessarily imply it meets the comprehension criterion by quoting Fields Medal winner Pierre Deligne regarding a proof he produced, "I would be grateful if anyone who has understood this demonstration would explain it to me" (p. 220).

At least three main resolutions have been proposed for the apparent tension between these dual roles. Mariotti (2000, 2006) references Duval's (1992/93) argument that the rift between the two in some cases "may be irretrievable" (p. 182) such that he divides "proof" into two distinct parts: *argumentation* for the dialectical goals and *proof* for the theoretical. Duval (1992/93, cited in Mariotti, 2000, 2006) resolves the tension by proposing that the duality of roles reveals a duality of concept.

This distinction provides a constructive viewpoint for assessing the instructional benefits and difficulties of various mathematical statements. Boero, Garuti, Lemut, and Mariotti (1996) refer to the compatibility of convincing argument and rigorous proof for a given statement as *cognitive unity*. Statements with strong cognitive unity facilitate intuitive explorations as to why the statement is true that simultaneously pave the way for constructing formal proof. Statements that lack cognitive unity must be dealt with differently in the classroom. Later, we examine more closely Boero et al.'s (1995, 1996) work on cognitive unity.

A second way of addressing the dual roles of proof calls into question the traditional perception of formal proof as pure logical deduction. In the wake of the formalization of non-Euclidean geometries, three major philosophies of mathematics and proof developed (logicism, formalism, and intuitionism) that differ in their view of what mathematics is and thus what forms of proof and theory are acceptable (Dossey, 1992). None of these theories have proven robust enough, however, to warrant universal adoption or absolute prominence. In this way, defining mathematical proof remains quite difficult.

Hersh (1993) argues that defining mathematical proof becomes increasingly difficult with the introduction of computer-based proving and certain probabilistic rather than deterministic results. He points out that proofs come in many forms and that the premium mathematicians place upon elegance and

aesthetics in proof reveal that more than logical validation is at work. More substantially, he points out that predicate calculus never appears in the presentation of most mathematical proof. He argues that a proof's validity rests more on the acceptance of the mathematical community than upon pure logical rigor: "The proof of the pudding is in the eating; the proof of the theorem, in the refereeing" (p. 392).

Hanna (1989, 1991) extends the assertion that mathematical proof and the validation thereof represent social constructs over and above logical ones. Hanna lists five major criteria for a proof's acceptance that correspond to: result believability, theoretical value, theoretical harmony (not conflicting with previous results), author reputation, and how convincing the argument is. "If there is a rank order of criteria for admissibility, then these five criteria all rank higher than rigorous proof" (Hanna, 1991, p. 58). She argues that even professional proving relates more to understandability and insight than to rigor:

> [Russian logician Manin says] the truth of a theorem in the eyes of the mathematical community becomes established indirectly, that is, not because the proof has been verified as error-free, but because the results are compatible with other accepted results and the arguments used in the proof are similar to ones used in other accepted proofs. (p. 59)

Later she continues:

> The role of proof in the process of acceptance is similar to its role in discovery. Mathematical ideas are discovered through an act of creation in which formal logic is not directly involved... While a proof is considered a prerequisite for the publication of a theorem, it need be neither rigorous nor complete. Indeed the surveyability of a proof, the holistic conveyance of its ideas in a way that makes them intelligible and convincing, is of much more importance than its formal adequacy. (p. 59)

Ultimately both Hersh (1993) and Hanna (1989, 1991) argue that the ideal of mathematical proof based solely on logical validity and rigor is a myth and does not reflect the actual practice and culture of research mathematicians. Although there is a distinction between the roles of proof as communicating and convincing, the practice of proof carries a largely social, and thus contextually dependent, aspect and should be treated thus in the classroom. Proof then, like defining, is an inherently human activity.

The third major clarification on the dual nature of proof references the history of mathematics to establish the sequential interplay between informal exploration of mathematics and formal elaboration. Kitcher (1984, as cited in Hanna, 1991) references Euler, Cauchy, Weierstrass, and Newton while Mariotti (2006) cites an example from Arabic mathematics involving Thabit ibn Qurra (836-901) and qu'al-Kawarizmi (780-850) to argue that mathematical theory advances in a succession of two stages: one stage which introduces and explores new ideas without dependence upon a foundational, formal theory and a second, latter stage which develops more rigorous theory to undergird the previously formulated discoveries. Mariotti (2000) describes the dual process as "the intuitive construction of knowledge and its formal systematization" (p. 28). Polya (1944, as cited in Recio & Godino) reflected this two-stage aspect of mathematical formation when he said, "Mathematics presented with rigor is a systematic and deductive science, but mathematics in gestation is an empirical and inductive science" (p. 98).

Kitcher (1984, as cited in Hanna, 1991) and Mariotti (2000, 2006) argue that the roles of convincing and then rigorously establishing correspond to the two historical stages of the mathematical endeavor. In the same way that individual problem solving involves moments of insight and periods of more analytical exploration, periods of solving and checking (Carlson & Bloom, 2005), historical mathematical discovery experienced periods of insight followed by periods of establishment. These same processes then both hold sway in the classroom as the need for convincing and the need for validation.

2.3.2 Classifying Proof Instruction and Production

As a counterpoint to the afore-mentioned criticisms of traditional proof-based instruction, commonly called the *definition-theorem-proof* (DTP) format, Weber (2004) describes both the instruction and reasoning of one "traditional" advanced mathematics teacher. Most of the aforementioned complaints about instruction in proofs and definitions respond to the standards of DTP teaching; however, Weber's study stands alone in its careful account and analysis of a DTP professor and classroom and the reasoning behind it.

Built around the acknowledgement that teaching flows out of a complex mixture of an instructor's knowledge, skills, goals, and beliefs, Weber's (2004) study described the style(s) of instruction used in the classroom, the intentions and views of the teacher that drove these instructional choices, and an account

of the learning linked to these style(s) of instruction. The study did not set out to espouse or decry such instruction, but to understand and describe it.

Weber (2004) classifies the teaching he observed as "traditional" according to the following set of characteristics:

> The instruction largely consists of the professor lecturing and the students passively taking notes, the material I presenting in a strictly logical sequence, the logical nature (e.g., formal definitions, rigorous proofs) of the covered material is given precedent over its intuitive nature, and the main goal of the course is for the students to [be] capable of producing rigorous proofs about the covered mathematical topics (p. 116).

Under this overarching description, Weber observed three distinct styles of instruction in the professor's (Dr. T) practice: logico-structural style, procedural style, and semantic style. Table 2.1 presents the characteristics of each style and the primary analysis topics upon which the professor employed each style of instruction.

Weber (2004) identified several beliefs Dr. T held about the teaching and learning of introductory real analysis that strongly motivated his diversity of teaching styles and his progression through the semester. These beliefs grew out of his previous experiences with students and his perception of the various topics and proofs, especially their increasing difficulty across the semester.

One of the beliefs Weber (2004) identified was, "The ideas in these proofs [about sets and functions] are divorced from other intuitive ideas in mathematics... one can go from place to place in these proofs just by following

Table 2.1: Styles of instruction in a traditional analysis course (Weber, 2004).

Instructional Style	Characteristics	Primary Topics
logico-structural	• intended to give students confidence in proving • identifying hypothesis and conclusions • translating both according to definitions until they intersect • linear validation of argument • almost no diagrams	• set theory • axioms of real numbers • basic properties of functions
procedural	• intended to provide students with heuristics and techniques for proofs about limits • writing an incomplete argument to illustrate proof structure • doing "scratch work" with inequalities to fill in the proof's framework • no semantic or intuitive discussion of proof meaning and very few diagrams	• limits
semantic	• intended to associate images with concepts over and above emphasis of definitions and theorems • intuitive descriptions of the idea a concept tries to capture often with a two-dimensional diagram • both definitions and examples thereof discussed in terms of relationship to intuitive diagram • some proofs presented in complete form and analyzed for comprehension • other proofs preceded by intuitive discussion using the diagram • rigorous proof always provided in a handout if not in the lecture	• topological topics

his nose" (direct quote from Dr. T, p. 128). The professor also expressed the understanding that:

- If students find analysis too difficult, they will become frustrated and give up on the course.
- Students need to have an elementary understanding of logic to follow an advanced mathematics course. An understanding of logic and advanced mathematical concepts cannot co-emerge.
- There are basic symbolic skills (e.g., proof techniques, working with inequalities) that students need to master before tackling tougher problems.
- Students cannot intuitively understand advanced mathematical concepts without sufficient experience working with these concepts at a symbolic level. (Weber, 2004, p. 128)

Weber notes that neither the beliefs that guided Dr. T's instruction nor his instruction itself reflect any of the three most cynical reasons past researchers provided for the prevalence of DTP instruction. Kline (1977, as cited in Weber, 2004) attribute DTP instruction to professors' lack of time or desire to teach advanced courses well due to research demands. Davis and Hersh (1981, as cited in Weber 2004) indicate that professors act upon vain desire to appear brilliant rather than a desire to help students learn. Leron and Dubinsky (1995, as cited in Weber 2004) claim that professors surrender to a belief that students simply cannot ascertain advanced topics in one semester-long course. None of these three claims appeared relevant or valid in Dr. T's case. In addition, Weber points out that because a stable set of thoughtful beliefs about the teaching and learning of analysis founded Dr. T's instruction, change in his teaching practice will only follow a significant set of motivating experiences and belief alterations.

Weber (2004) also interviewed Dr. T's students asking them to prove and explain a set of three results, one representing each of three topics taught in the three instructional styles. He classified their responses according to three styles of reasoning identified in the literature: *natural learners* reason using their intuitive understanding, *formal learners* reason using logical entailments of definitions, and *procedural learners* master techniques modeled in instruction pushing sense-making to later reflection. He found that though one student approached every problem as a natural learner, the majority of students responded to the different tasks using different approaches. Most students exhibited formal learning behavior on a question regarding functions evaluated over a set, half exhibited procedural learning behavior on a question about limits of sequences, and all of the students exhibited natural learning behavior on a question about topological closure. Weber summarized his observations saying "the lecture styles of Dr. T appeared to have a direct effect on the way some students attempted to learn the material" (p. 131).

Alcock and Simpson (2004, 2005a, 2005b) have worked extensively to classify student thinking and proving at the advanced mathematical level. In addition to their previously cited classification of student thought according to visual and analytic modes, Alcock and Simpson (2005b) distinguishes between students proving using *syntactic* and *referential* modes of reasoning. Writing and unpacking definitions and assumptions using logical entailments or standard proof approaches guide syntactic proof construction. Conceptual and intuitive

notions as well as visual explorations guide referential proof construction.

Weber's (2004) categories observed in Dr. T's class correspond strongly with

Alcock and Simpson's. He divides syntactic proofs into the *logico-structural* style

and *procedural* style with the former focusing on logical entailments and the latter

upon applying standard proof approaches. He refers to the referential approach

as *semantic* proof construction.

2.3.3. Recommendations for Proof in the Classroom

Alibert and Thomas (1991), Hanna (1991), Hersh (1993), Mariotti (2006),

and Tall (1991b) all advocate for classroom instruction that attends to both the

convincing and the validating aspects of proof. They also indicate that classroom

proof should reflect more of the thought and insight that produced the proof

rather than the bare logical necessities. Multiple proposals exist for how these

changes can be accomplished.

Before students can prove statements, they must understand meaningfully

what those statements say. The referential or semantic proof approach, as

described by Weber (2004) and Alcock and Simpson (2005b), attends to the

mathematical objects to which a mathematical statement refers for the purposes

of proving. Selden and Selden (2008) indicate many students might not have this

approach accessible to them due to weak concept images with which to make

sense of mathematical statements. They say, "In order to use a concept flexibly,

it is important to have a rich concept image, that is, a lot of examples, facts,

properties, relationships, diagrams, and visualizations, that one associates with that concept" (p. 103). They advocate that instruction attend to students' need to construct a more robust experience with concepts of interest.

Another recommendation focuses on the origins of statements that are being proven. Whereas many proof-based classrooms present those statements that will be proven, researchers indicate that students could benefit from the process of proposing mathematical statements to be verified or disproven as a group. Alibert and Thomas (1991) described the novel teaching environment the French Grenoble group developed in which theorems arose in an environment of scientific debate. Jahnke (2005) proposes a proving classroom entailing a "culture of why questions" (p. 435). Along similar lines, Mason and Watson (2008) said:

> We have also found that learners respond well to be being called upon
> and expected to use their own powers to specialize and generalize, to
> imagine and express, to conjecture and to convince, to organise and to
> characterise... Thus the challenge is to promote a movement from merely
> *assenting* to what they are told or to do, to taking the initiative and
> *asserting* (in the form of making, testing and validating conjectures,
> constructing examples which illustrate conditions, and generalizing
> particular tasks to a class of 'types' of tasks) through using and developing
> their natural powers (author's italics, p. 193).

The theme common to an environment of scientific debate, a culture of why questions, and a classroom of asserting is that students become directly involved in asking the questions and proposing the statements that they then test and alter in pursuit of valid theorems. Mariotti (2006) suggests that there is a connection between students working together and their engagement in the

asserting and testing processes. "Collaborative work, amongst peers or in small groups, seems to be a favourable social context in which to make cognitive conflicts arise as they are naturally brought to students' consciousness in confronting answers and arguments" (p. 190-191).

Regarding the need for proofs that are accessible and informative to students, Leron (1985) advocated using structural proofs. Structural proofs (or the top-down approach) start with an outline of the argument without details or full justification that allows students to perceive the general approach. The level of detail and justification then increases as the class expounds upon aspects of this overall structure. Rather than sequential presentations whose ordering derives from logic, importance determines the order of presentation of structural proofs.

Raman (2003) offers the "key idea" of a proof as another approach to proof that provides more global insight and constructive information than linear presentation. Studies in novice understandings of proof show that whereas mathematicians differentiate easily between routine and significant aspects of a proof, many mathematics students' only approach to comprehending a proof is to examine it line-by-line (Selden & Selden, 2003). The key idea of a proof, as Raman (2003) defines it, represents the aspect of the proof that was the most significant hurdle to the construction of the argument. It thereby provides a link between the public and private aspects of proof because it connects both to the

thought process by which the proof was made and the logical structure of the proof that makes it valid. Centering student engagement with a proof on a key idea thus provides insight into the validity and the history or construction of the proof.

Mariotti (2000, 2006) argues that proof should not be thought of in isolation, but as one member of a triad consisting of (1) the statement to be proved, (2) the argument, and (3) the greater theory within which the argument validates the statement. This quality of proof parallels the mathematical criteria for definitions requiring they fit into a larger deductive system (Edwards & Ward, 2008). Mariotti (2000) states:

> The fact that the reference theory often remains implicit leads one to forget or at least to underevaluate its role in the construction of the meaning of proof. For this reason is seems useful to refer to a 'mathematical theorem' as a system consisting of a statement, a proof and a reference theory (p. 29).

For the classroom, she proposes that rather than teaching students theorems and proofs as distinct units, teachers must simultaneously develop understanding of conjectures and justifications as well as an overall system of theory in which those conjectures are meaningful and justifications necessitated. Mathematical ideas and theorems must be what she calls "theoretically situated" (Mariotti, 2006, p. 184). So whereas Alibert and Thomas (1991) called linear formalism a sub-minimal code of proof information, Mariotti argues that individual theorems and proofs constitute a sub-minimal body of justification and meaning to which students are expected to acclimate.

37

2.3.4. Unifying the Empirical and Logical Aspects of Proof

The final recommendation for proof in the classroom builds upon Fischbein's (1982) distinction between the empirical and logical modes of proof. He found that most students desired to see a validating example of a statement (an empirical form of verification) even when presented with sufficient logical proof (the logical form of verification). He pointed out that mathematicians use examples to inform their intuition alongside the construction of formal proof and identified that students quite often desire similar verification of proof (Fischbein, 1982). Jahnke (2005), based on similar observations in the operations of the empirical sciences, argued for a classroom form of proof that incorporated more empirical justifications.

This distinction between the two modes of proof parallels the distinction between Vinner's (1991, p. 80) two modes of defining: "the everyday life mode" and "the technical mode." The empirical and logical modes of proof also correspond strongly to the modes of defining: extracting definitions that depend upon prototypes or sets of prototypes and stipulating definitions that depend only upon the logical entailments of the defining statement (Alcock & Simpson, 2002; Edwards & Ward, 2008). Fischbein's (1982) notions of the empirical and logical forms of proof match well with Mariotti's (2006) description of Duval's characterizations of argument and proof.

I already briefly described Boero and colleagues' idea of cognitive unity (Boero et al., 1995, 1996; Mariotti, 2006), which describes the level of correspondence between the informal argument and the logical proof for a given statement. In order to better understand the interplay between these two constructs, they engaged students in the processes of proposing, arguing, and proving in an empirical environment rather than a mathematical one. The class explored different phenomenon relating to shadows trying to explain or model their observations systematically. They chose this context because of the rich foundations of experience that students already possessed with shadows and because it lent itself to empirical exploration and verification (Boero et al., 1995, 1996; Mariotti, 2006).

During these discussions, the researchers observed students producing sets of different conjectures and arguments for those conjectures. As the class discussed these different arguments, the ideas being expressed transitioned from informal observations toward more formal propositions and students began to link the arguments together in logical chains that approximated formal proofs rather than arguments. Thus, they identified continuity between argument and proof in that the students were able to transform and construct one into the other during the course of their conversations. These findings introduced the notion of cognitive unity since not every statement lent itself to this semi-unification of argument and proof. Shifting the starting point of the conversation from proving based on principles to proposing based on empirical observations facilitated

these students' successful transition from argument to proof, from the empirical to the logical (Boero et al., 1995, 1996; Mariotti, 2006).

Although Boero et al. (1995, 1996) stretched beyond the boundaries of mathematics to physical phenomenon to engage their students in an empirical context of proving, Vinner's (1991) recommendations and Pinto and Tall's (2002) findings indicate that mathematics can also provide fertile soil for quasi-empirical explorations that use examples as a starting point to transition into more logical approaches. However, it would appear that such an approach might be contingent upon students first expanding their set of founding experiences with various mathematical objects to supplement the role experience played in Boero et al.'s (1995, 1996) findings. In the context of defining, this is why Selden and Selden (2008) noted the importance of students' development of robust concept images.

2.4   General Curricular Considerations

The previous sections about visualization, definitions, and proof presented some critiques of traditional teaching often centered upon its attention to logical detail and relative ignorance of psychological phenomena and issues related to the content being taught. Such issues arise at all levels of instruction, however, and so we now examine relevant literature taken from various contexts of mathematics education research on teaching that considers both logical and psychological issues in the classroom.

Each of these topics helps clarify the difference between characterizations of "traditional" instruction and formative alternatives, which have many names such as "reform-oriented" or "student-centered." Despite the variety of sources calling for change, a unified voice arises constructing a relatively well-defined image of one research-based alternative to the "traditional." This distinction serves the present study both to classify the form of instruction observed and define the "non-traditional" label adopted to describe it.

2.4.1. Teaching Teachers to Attend to Student Thinking

"It appears that the more use that teachers make of their knowledge of student thinking while teaching, the more mathematics their students will learn" (Speer & Hald, 2008, p. 309). The research on Cognitively Guided Instruction (CGI) highlights this connection (Carpenter, Fennema, Franke, Levi, & Empson, 2000; Fennema et al., 1996; Franke & Kazemi, 2001). The CGI research group integrated the understanding they had developed of student conceptions of arithmetic with their teacher preparation in such a way as to promote teacher awareness of student thinking. The key ideas behind the successful teaching model they promoted among the teachers in their program include teachers:

1) identifying student thinking,

2) understanding and valuing different solution methods common among students,

3) allowing students to solve problems and express their solutions as much as possible,

4) valuing and making use of student solutions for instruction, and

5) catering instruction to their students' thinking even on an individual level by attending to the lines of reasoning they display.

Shifting instruction toward dependence upon student input and student understanding introduces a level of contingency in the classroom and requires the teacher to continue her/his learning both of mathematics and of student thinking about mathematics (Ball & Bass, 2000). Franke and Kazemi (2001) indicate that this shift has far-reaching effects in instruction:

> Focusing on students' mathematical thinking remains a powerful mechanism for bringing pedagogy, mathematics, and student understanding together. As teachers struggle to make sense of their students' thinking and engage in practical inquiry, they elaborate how problems are posed, questions are asked, interactions occur, mathematical goals are accomplished, and learning develops. Teachers' experimentation around student thinking becomes part of their practice... teachers see a clear relationship between their learning and their students' learning (p. 108).

As they tried to evaluate these changes their program engendered in these teachers' instruction over time, Fennema et al. (1996) developed frameworks to describe various levels of cognitively guided instruction and cognitively guided beliefs about instruction by which they gauged teacher practice and beliefs and identified change over time. Table 2.2 presents the framework of levels of Cognitively Guided Instruction. Higher levels of CGI indicate that a teacher values and attends to student thinking. The highest levels indicate that student understanding guides a teacher's instruction in very direct ways.

Table 2.2: Fennema et al.'s (1996, p. 412) framework of levels of Cognitively Guided Instruction.

Level	Description
1	Provides few, if any, opportunities for children to engage in problem solving or to share their thinking.
2	Provides limited opportunities for children to engage in problem solving or to share their thinking. Elicits or attends to children's thinking or uses what they share in a very limited way.
3	Provides opportunities for children to solve problems and share their thinking. Beginning to elicit and attend to what children share but doesn't use what is shared to make instructional decisions.
4A	Provides opportunities for children to solve a variety of problems, elicits their thinking, and provides time for sharing their thinking. Instructional decisions are usually driven by general knowledge about his or her students' thinking, but not by individual children's thinking.
4B	Provides opportunities for children to be involved in a variety of problem-solving activities. Elicits children's thinking, attends to children sharing their thinking, and adapts instruction according to what is shared. Instruction is driven by teacher's knowledge about individual children in the classroom.

2.4.2 The Mathematics Teaching Cycle

Simon (1995) explored the complex difficulties teachers face in

constructing curriculum insomuch as they must simultaneously attend to

communally defined goals for learning (enculturation) and present mathematical

conceptions of their students (cognitive development). He references

Brousseau's (1981, 1983, 1987, as cited in Simon, 1995) assertion that it is the

job of the teacher to take contextless mathematical ideas accepted by the

community and embed them in a situational context for students to explore.

Brousseau adds that this mathematical situation must be relevant enough to the student for her/him to respond to the "milieu," i.e. the difficulties introduced by the situation and the conceptions necessary to make sense of it, rather than simply the demands of the teacher, i.e. finding the answer being sought. That is, the student must consider the context deeply enough to create a conceptual disequilibrium that they will exert effort to relieve. Brousseau then calls for the teacher to create other activities that guide the student toward "decontextualizing and depersonalizing" the ideas that have been encountered embedded in the situation. In so doing, students can transform the individual conception they constructed to mediate the given situation, which is their individual cognitive development, into something compatible with the shared meanings of the mathematical community, which accomplishes enculturation (Simon, 1995).

This description of the careful construction/ selection of mathematical activities or situations in which to engage the students implies a necessary attention to student thinking and reasoning insomuch as the situation is predicted to induce personal cognitive disequilibria. Simon quotes Lampert's (1990, as cited in Simon, 1995) explanation of how these instructional choices relate to the norms and goals of the general mathematical community:

> The most important criterion in picking a problem was that it be the sort of problem that would have the capacity to engage all of the students in the class in testing and making mathematical hypotheses. These hypotheses are imbedded in the answers students give to the problem, and so comparing answers engaged the class in a discussion of the relative mathematical merits of various hypotheses, setting the stage for the kind of zig-zag between inductive observation and deductive generalization

that Lakatos and Polya see as characteristic of mathematical activity (Lampert, 1990, as cited in Simon 1995, p. 39).

This quote mentions how students propose hypotheses for a problem, which are their attempts to respond to the disequilibria introduced by the task at hand. Teachers must either be aware enough of student thinking to predict the responses students will have or be flexible and attentive enough in the moment to identify the reasoning behind a novel hypothesis. Then the community together must evaluate these hypotheses in a move toward shared resolution, understanding, and meaning. As this communal meaning develops, the teacher must guide both the activities and conversations toward her/his overall learning goals.

To describe the complex interplay between the teacher's learning goals (representing the institutionalized knowledge of the mathematical community), instructional activities or situations, and student conceptions, Simon (1995) developed the Mathematics Teaching Cycle (see Figure 2.1).

The left-hand column represents the teacher's knowledge of mathematics that in the university classroom especially embodies the presence and influence of mathematical culture. However, it is broken up into the teacher's knowledge of mathematical ideas in general (the de-contextualized concepts) and her knowledge of those ideas embedded in various contexts and tasks.

The right-hand column represents the teacher's perception of student thinking. This overall domain of the teacher's thinking subdivides into her general
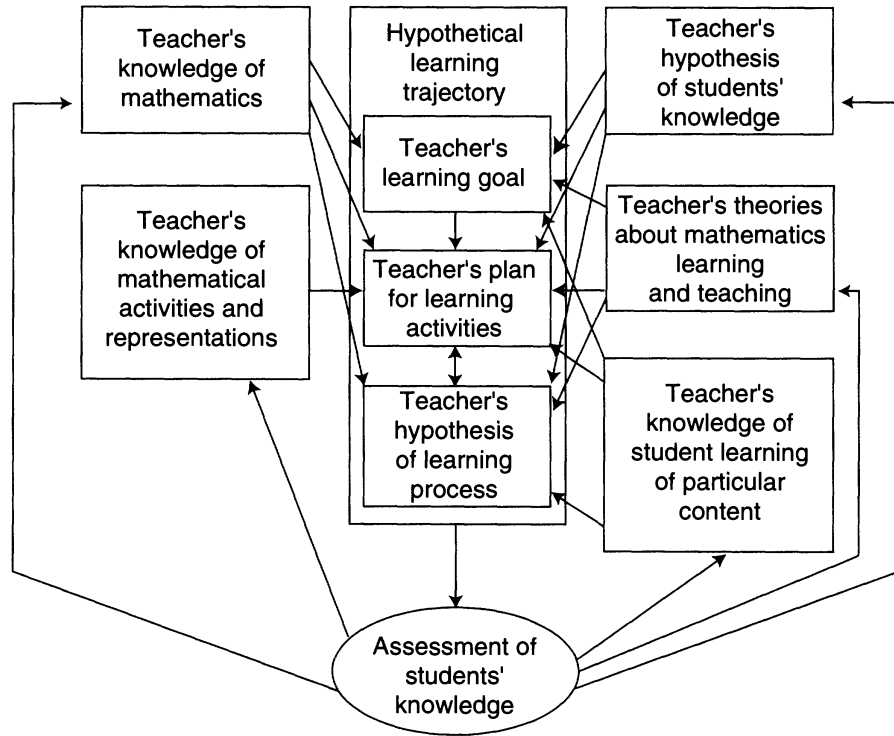
Figure 2.1: Simon's (1995, p. 137) diagram of the Mathematics Teaching Cycle.

perception of student thinking, her conception of how student understanding

grows and changes, and her perception of student thinking about a given

concept.

The middle column represents where the teacher's understanding of the

mathematics and of student thinking intersect to create learning activities as

described by Brousseau (situations) and Lampert (problems). The top entry in

that column represents the ideas the teacher desires for students to interact with,

the second represents the contextualization of that idea into a situation or

problem, and the third represents the teacher's predictions regarding the hypotheses students will produce to mediate the problematic situation.

The circle at the bottom represents the information a teacher receives from observing student thinking in the classroom which feeds back into her conception of the mathematics, her understanding of the task in question, and her perception of the three levels of student thinking.

According to Simon (1995), this cycle occurs in the mind and classroom of any teacher. However, each classroom gives more emphasis or weight to certain portions of the cycle. The selection of particular pathways as primary will strongly influence the nature of the mathematical activities that appear in a class and the interactions that ensue from those activities. For instance, the different classroom structures provide teachers with differing amounts and types of information about student knowledge. Even when this information is provided, teachers must choose how and to what extent to use this information to reevaluate their conceptions of the mathematics and their students' thinking (Simon, 1995).

A given teacher's theories of student learning will affect the weight given to the left and right sides of the diagram, which corresponds to the extent to which the classroom activities are built around attention to student thinking or toward ascent to canonical conceptions and the culture of mathematics into which students are enculturated. Though excess can be found in either direction (Marongelle & Rasmussen, 2008), the overwhelming traditions of university

mathematics education give enculturation almost absolute primacy, similar to

what Cobb (1989) stated regarding traditional elementary education:

> [Teaching in which students do their own truth-making] contrasts sharply
> with traditional instruction in which students are presented with codified,
> academic formalisms that, to the initiated, signify communally-sanctioned
> truths that have been institutionalized by others (p. 38).

In other words, the teacher in traditional classrooms holds absolute authority over

what is acceptable ("to the initiated") because he/she is the representative in that

classroom of the true authority—the mathematical powers-that-be outside the

classroom (Cobb et al., 1993). Ball (1991, as cited in McClain & Cobb, 2001)

eloquently described an alternate form of teaching that attends to both students

and mathematics culture as keeping an ear on student reasoning and an eye on

the mathematical horizon.

2.4.3. Teacher Knowledge that Oils the Mathematics Teaching Cycle

Regarding the intricate interactions Simon's (1995) Mathematics Teaching

Cycle describes, research has found that teachers require a refined and specific

form of mathematics understanding to incorporate student understanding into

their teaching. Ball and Bass (2000) rue the division between subject knowledge

and teaching methods that has prevailed in the minds of policymakers and

educators throughout the 20th century. They explain that conventional wisdom

indicates that the mathematics a teacher needs to know consists only of the

mathematics they teach. The mathematics a teacher presents then stands

unaffected by student conceptions thereof or how students actually experience

the mathematics in the classroom. They conclude, "The gap between subject matter and pedagogy fragments teacher education by fragmenting teaching" (Ball & Bass, 2000, p. 85).

Calls to repair this fragmentation appear throughout much of the research literature at all levels (Cobb, 1989, 1994; Fennema et al., 1996; Recio & Godino, 2001; Tall, 1991a). Tall (1991a) pointed out that advanced mathematical instruction that only attends to logical issues "fails to allow for the psychological growth of the developing human mind." Alibert and Thomas (1991) stated even more directly the need for the classroom to influence teachers' conceptions of mathematics, if not the entire community's, saying, "it is also important to look at the formalism itself and consider how it too might be improved." Mathematicians such as Klein, Freudenthal, and Blandford all called for students to engage the thought behind mathematical definitions rather than just the content of the definitions themselves (Blandford, 1908; De Villiers, 1998). Mathematics Teaching Cycle (figure 2.1) also highlights the complex interactions between curricular goals and student response to classroom activities (Simon, 1995). The concerted voice of these researchers thus harmonizes with Ball and Bass (2000) that teacher content knowledge falls short if it is unaffected by student knowledge and thinking.

The alternative to this fragmented form of teacher knowledge is what Shulman, Wilson, Grossman, and Richert (1986, as cited in Ball & Bass, 2000)

coined "pedagogical content knowledge". Pedagogical content knowledge represents the intersection between a teacher's knowledge of the mathematical content and her/his understanding of student thinking about that mathematics. Ball and Bass (2000) also discuss the contingent nature of such understanding in that teachers must have a form of pedagogical content knowledge that allows them to listen and adapt in the moment, which she calls *pedagogically useful mathematical understanding*. "Knowing mathematics for teaching must take account of both the regularities and the uncertainties of practice, and must equip teachers to know in the contexts of the real problems they have to solve" (p. 90).

2.4.4. Classroom Interaction Patterns

One final aspect of classroom pedagogy that clarifies the differences between a "traditional" classroom and a classroom that incorporates student thinking, deals with student-teacher communication patterns. Nickerson and Bowers (2008) observed a student-centered classroom and identified two powerful and non-traditional interaction patterns that repeated themselves time and again. They define traditional interaction patterns by Mehan's (1979, as cited in Nickerson & Bowers, 2008) description of the initiate-respond-evaluate (IRE) pattern: "a teacher *initiates* a question, next, a student *responds*, and finally there is an *evaluative* interaction" (p. 180). They point out that this pattern can result in students catering their response less to the problem at hand and more to perceived teacher expectations because of students' anticipation of the instructor's impending evaluation (Nickerson & Bowers, 2008).

Nickerson & Bowers (2008) explain that Mehan further defined four types of questions he observed teachers introducing:

1. *Choices*, which are those questions that dictate the student agree or disagree with a statement provided by the teacher;

2. *Products*, which require students to provide factual responses;

3. *Processes*, which call for students' opinions or interpretations; and

4. *Metaprocesses*, which are elicitation questions that ask students to reflect upon the process of making connections between a question and a response to formulate the grounds of their reasoning.

Nickerson and Bowers (2008) distinguish between the first two forms of questioning that are more computationally focused and the latter two that are more conceptually focused. The former two forms of questioning are particularly indicative of the IRE interaction pattern.

Nickerson and Bowers also group Wood's (1994, as cited in Nickerson & Bowers, 2008) characterization of the *focusing* and *funneling* patterns into the traditional category because these patterns describe lines of questioning focused on recall and evaluation rather than reasoning. The researchers conclude that much mathematics classroom discourse is heavy on teacher-generated questions and lacking in student-initiated comments. They indicate that such interaction patterns hold sway in the classroom particularly because they teach students implicit lessons about the kind of learning expected of them. Furthermore, they argue that IRE and funneling/focusing promote procedural and

recall-focused learning while more conceptual modes of classroom discourse engender deeper synthesis and conceptual explanation (Nickerson & Bowers, 2008).

However, Nickerson and Bowers (2008) identified two previously undocumented interaction patterns in their classroom of study:

- elicit-respond-elaborate (ERE) pattern- the teacher *elicited* observations, students *responded*, and the teacher *elaborated* on their comments and

- proposition-discussion (PD) pattern- students make a *proposition* and then others *discuss* it.

Regarding the first pattern, they contrast eliciting and initiating because the former centers the conversation on ideas students introduce. They also contrast elaboration and evaluation because elaboration appeared formative by encouraging further discussion. Regarding the second pattern, they again emphasized the role students played in producing the ideas the class discussed. They concluded from their observations in this classroom that replacing IRE patterns with ERE and PD patterns promoted conceptual understanding by changing the implicit lessons teacher-student interactions taught and thereby changing the type of learning students understood to be expected of them. The ERE and PD patterns affirmed the importance and value of student input and promoted conceptual rather than computational responses (Nickerson & Bowers, 2008).

The authors also emphasized that these interaction patterns are not dictated solely by the professor, but must arise socially and inherently depend upon the students involved.

> Our claim is based on the view that all communication practices are inherently social and therefore follow the rules of all interpersonal communication: the messages that are received are not necessarily those intended by the sender, and meanings are often implicitly negotiated between speakers (in this case, the students and the teacher). For these reasons, classroom communication patterns cannot be explicitly laid out by the teacher alone. Instead, they are negotiated through an implicit process of trial and error by which the students might offer an explanation that serves the implicit function of an opening offer: in essence, the student is asking, *'is this acceptable'* The reaction- from both teacher *and the students*- sends an implicit message back to the student and the rest of the class which, in turn, moves the process of negotiation forward one more round. (p. 188)

The following theoretical framework (Chapter 3) will further explore this theme of classroom interaction and its far-reaching effects upon student learning as well as the inherently social nature of how such patterns arise in a given classroom.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1. Constructivism(s)

I derive my theoretical framework from the constructivist tradition (Piaget, 1964; Vygotsky, 1978), especially as it has been formulated by Cobb (1994). There has often been a division between the individual perspective of constructivism tracing from the work of Piaget (1964) and von Glasersfeld (1984, cited in Cobb, 1994) and the socially oriented perspective tracing its roots to Vygotsky. Constructivism posits that knowing is not the acquisition of objective information or is it the understanding of an external reality, but is rather a personal conceptualization developed to meet the needs of problematic situations.

From the individual constructivist viewpoint, people cannot have full access to one another's conceptions and so teaching does not represent a transfer of knowledge from the mind of a teacher to the student, but rather the teacher provides the student with novel experiences and situations in which and through which the student alters and develops her/his understanding to meet the demands introduced by the information and tasks they face.

Piaget (1964) described the process of individual conception change via an equilibrium metaphor from the physical sciences. Novel tasks and experiences become problematic to individuals when they cannot be properly understood or mediated by present conceptions and thus a cognitive disequilibrium is induced. Piaget (1964) identified a set of mechanisms by which prior conceptions are then altered in light of new experiences to create a new state of cognitive equilibrium; these mechanisms include assimilation, accommodation, and reflexive abstraction.

Von Glasersfeld (1984, as cited in Cobb 1994) used the term "viable" to describe the resulting correspondence between individual conceptions and their external experience. According to his "radical constructivism," student attempts to make sense of their experience are not right or wrong from some objective viewpoint, because he does not assume the existence of such an external standard. The conditions used to define understanding are much more contextual. Students' mental constructions are measured according to whether they are "viable" for making sense of experiences or meeting the demands of tasks.

The social account of constructivism often references Vygotsky's (1978) assertion that:

> Any function in the child's cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category… Social

relations or relations among people genetically underlie all higher (mental) functions and their relationships (p. 57).

The personal phenomenon of learning thus represents initially participating in a communal discourse in which meaning arises and then that meaning is internalized. Internalization then represents a key construct of this theory and this occurs in what is called the zone of proximal development. This zone refers to a period in individual psychological development in which concepts are accessible to the individual within the context of social interaction, but not yet attained by the individual learner (Cobb, 1989; Vygotsky, 1978).

The social viewpoint emphasizes the role community plays in establishing what "viable" might mean in a given situation. In the classroom setting, problematic situations are introduced in the community of teacher and students and thus their resolution is established collectively. In other words, an individual's meaning making can only be validated against the backdrop of a communal pursuit of understanding and thus the emergence of learning has an inherently social nature (Cobb, 1989). Even when an individual pursues understanding in isolation, she/he does so according to rules of validity, which derive from the communal reality of understanding. As Cobb, et al. (1993) stated: "mathematical activity can be viewed as intrinsically social in that what counts as a problem and as a resolution have normative aspects."

Cobb, Wood, and Yackel (Cobb, 1989, 1994; Cobb, et al., 1993; Wood, Cobb, & Yackel, 1995) argue that the individual and social viewpoints of

constructivist theory are merely complementary and not mutually exclusive. They

point out that mathematical learning is simultaneously the process of students

making meaning of their mathematical experiences and the process of students

being enculturated into the "institutionalized ways of knowing" (Cobb, 1989, p.

38) of the mathematical community. The conceptions of a given student are

unique to that student and constructed by that student, but they were

established, evaluated, and affirmed according to communal standards and

goals. Cobb (1989) cited other scholars' general assertion of the same duality:

> Theorists such as Comarof (1982) and Lave (1988) propose that the
> relation between the mutual construction of cultural knowledge and
> individual experience of the lived-in world is dialectical. In this formulation,
> it can be argued that cultural knowledge (including mathematics) is
> continually recreated through the coordinated actions of the members of a
> community… Each child can be viewed as an active reorganizer of his or
> her personal mathematical experiences and as a member of a community
> or group who actively contributes to the group's continual regeneration of
> the taken-for-granted ways of doing mathematics (p. 34).

Cobb paralleled these researchers' synthesis to physicists' dual treatment of

matter as wave and particle. The two views are complementary and neither can

fully account for the full complexity of classroom phenomena.

This synthetic view of individual and social constructivism affirms the

complementary roles these two halves of Simon's (1995) Mathematical Teaching

Cycle play. They are not independent or mutually exclusive, but instead should

be synergistic. To understand how the Mathematics Teaching Cycle leads to the

communal development of mathematical meaning, I now turn to Yackel and

Cobb's (1996) construct of socio-mathematical norms.

3.2. Socio-mathematical Norms and the Reform-Based Classroom

Every well-established community has norms of interaction that simultaneously guide and define the activity therein. In the mathematics community, norms underlie the scholarly activity of defining, proposing theorems, and proving those theorems. Cobb et al. (1993) explore how norms develop in the mathematics classroom that dictate if, when, and how students contribute to the discussion. These norms become a backdrop to the activities within such a community and thus represent taken-as-shared notions of the parameters of interaction in that community.

Though in theory every classroom has such norms, the work of Cobb et al. (1993) took place in the context of a reform-oriented classroom with a series of important characteristics:

- the teacher rejected "the assumption that all the students should make certain predetermined mathematical constructions when they completed and discussed their solutions to particular instructional activities" and thus student input was valued and integrated into the classroom discourse (p. 93),

- the teacher did not set herself up as the sole validator of mathematical knowledge, but rather "the teacher and students together constitute a community of validators" (p. 93),

- the teacher mediated attention to student thinking and her responsibilities to enculturate students into more institutionalized notions of mathematics by

reformulating students' "explanations and justifications in terms that were more compatible with the mathematical practices of society at large and yet were accepted by the children as descriptions of what they had actually done" (p. 93).

This mediation implied a certain flexibility to accommodate the statements and responses of the students. The teacher alternated between conversations about mathematics and conversations about talking about mathematics. In the former case, the teacher avoided the traditional IRE interaction pattern (Mehan, 1979, as cited in Nickerson & Bowers, 2008) in which the teacher merely evaluates the student's response against the pre-determined and desired response, but instead asked "information-seeking questions" (Cobb et al., 1993, p. 111) whose answers are valid insomuch as they provide the teacher with insight into the student's thinking.

However, in the latter case of conversations about talking about mathematics, the teacher became more directive in order to establish the norms of interaction she desired for the classroom. They provide an example of such an interaction where a student showed embarrassment at having provided a wrong answer, so the teacher asked the class whether such a mistake was acceptable before strongly affirming the positive answer to her question. The student's response of embarrassment, which does not fit with the norms the teacher desired for her classroom, triggered the teacher to shift from discussing the

mathematical problem at hand to speaking about the mathematical discussion at hand (Cobb et al., 1993).

Teachers possess an unequally powerful role in introducing norms in classrooms, but students play an important role in establishing these norms communally (Cobb et al., 1993). Because of the aforementioned structure of the classroom around student thinking:

> The mathematical meanings and practices institutionalized in the classroom were not immutably decided in advance by the teacher but, instead, emerged during the course of conversations characterized by what Rommetveit (1986) called a genuine commitment to communication (Cobb et al., 1993, p. 93).

The teacher has the power to inhibit such communication in the classroom and because most students have not experienced it in mathematics classrooms, "the teacher had to guide the children's developing abilities to engage in genuine mathematical communication as they worked together" (Cobb et al., 1993, p. 103). However, it was only by the collective participation of the teacher and students together that the collective meaning was shared and "explaining, justifying, and collaborating had become objects of reflection in a consensual domain" (p. 102).

Cobb and Yackel (1995) extended their previous work to point out the existence of not only norms of conversation and argumentation (e.g. explaining, justifying, and collaborating), but also norms establishing what the community accepts as mathematically significant and acceptable (e.g. mathematically valid

explanation, mathematical justification). They deemed these social constructs of mathematical meaning "socio-mathematical norms."

3.3. Constructivism Meets Advanced Mathematical Thinking

The present study seeks to classify the classroom observed according to the models previously described of traditional or reform-oriented instruction in order to understand students' transition to mathematical thinking in this context. We then will identify the socio-mathematical norms that developed in the classroom in order to explain and interpret the student thinking observed. However, I define the overarching goal of proof-based courses to be the promotion of advanced mathematical thinking. By this I mean students thinking about advanced mathematical topics in ways similar to mathematicians. Thus, I assume transitioning students from being mathematical receivers toward being mathematicians as a key role of advanced mathematical courses.

Most of the following aspects of mathematical knowing involve students' structural understanding of mathematics. Student understanding of the advanced mathematics itself appears throughout this investigation, however I assume two factors regarding the relationship between students learning the content of advanced mathematical thought and students learning the process advanced mathematical thinking:

1. Learning the content of advanced mathematical thought proves insufficient and ineffective if students do not simultaneously learn the process of

advanced mathematical thinking. This parallels the dichotomy of procedural

learning and conceptual learning at more elementary levels.

2. Learning advanced mathematical thinking strongly facilitates student's

individual construction of advanced mathematical thought.

According to constructivist thought, students do not acquire mathematical

knowledge from the mind of the teacher, but construct mathematical meaning

themselves in ways, hopefully, compatible with the shared meanings of the

mathematical community in which they are engaged (Cobb, 1994). Thus, if

students and teachers have differing structural understandings of mathematics,

then there exists little cause for hope that they will make similar conclusions

based on their shared mathematical experiences or construct compatible

mathematical meanings.

Students' perceived need for formal proof provides a well-documented

instance of this general principle. Harel & Sowder (1998) found that students

often rely on authoritative or prototype-based proof schemes. Without replacing

these schemes with a more sophisticated structural understanding of

mathematics within which students feel the need for rigorous validation of

mathematical truth, proof comes across as superfluous and arbitrary (Mariotti,

2006). As Mariotti's (2006) previously cited quote stated, "proof has to contribute

more widely to knowledge construction. If this is not the case, proof is likely to

remain meaningless and purposeless in the eyes of students."

I now outline the five aspects that comprise my working definition of the activity and mindset of a mathematician.

3.3.1. Aspect 1: Develop a Sense of Mathematical Autonomy

Cobb, Wood, and Yackel (1990) and McClain and Cobb (2001) reported the emergence of mathematical autonomy in a classroom characterized by sharing and testing ideas. In the same way that mathematical truth is established by mutual acceptance in the mathematical community of validators (Hanna, 1989, 1991), when the classroom community takes on the role of mutual validation of mathematical meaning they are acting as mathematicians by institutionalizing their own meaning (Cobb, 1989). In this sense, the classroom becomes a microcosm of the mathematical community at large.

3.3.2. Aspect 2: Use Visualization for Sense-Making and Problem Solving.

In the same way that some mathematicians almost completely ignore visual arguments, not every student must make extensive use of visualization (Tall, 1991b). However, many mathematicians do use the power of visualization to make sense of situations and even to guide their construction of proof (Eisenberg & Dreyfus, 1994). At least some students, like mathematicians, should use visual images as tools for mathematical exploration and sense-making rather than visual images becoming an end toward which instruction heads or visualization completely being ignored (Aspinwall et al., 1997). In the same way that researchers identified students' need for specific training in visual

63

reasoning, mathematicians observe the correspondences and distinctions between visual intuitions and analytically verified results (Alcock & Simpson, 2002; Eisenberg & Dreyfus, 1994). Thus, the transition to advanced mathematical thinking might also include students reasoning directly about the values and the limitations of visual representations.

3.3.3. Aspect 3: Create Definitions Within a Body of Theory.

Students should understand that mathematical definitions are constructed to describe in a functional way a category of mathematical objects in a functional manner. When mathematicians create a definition, they must choose the property of a category that best captures the essence of that category (Alcock & Simpson, 2002). Students taking part in the thought processes of defining in general and understanding the reasoning behind the form of specific definitions both constitute aspects of advanced mathematical thinking. Once the definition is formulated around a defining property, then students should understand that the definition's logical entailments determine fully the set of elements rather than the examples defining the category; that is, for proof purposes definitions are stipulated rather than extracted. If counter-intuitive examples are included or normal cases are excluded, then the definition must be changed to alter the situation (Alcock & Simpson, 2002).

3.3.4. Aspect 4: Propose, Test, and Validate Statements

The volume *Advanced Mathematical Thinking* (Tall, 1991) proposed several "more cognitively appropriate approaches" to advanced mathematical teaching (p. xiv); the word "more" measured against traditional teaching that for the most part only addresses the enculturation or the logical side of teaching responsibility. One of these suggestions particularly reflects Cobb and Yackel's (1996) set of socio-mathematical norms: that of scientific debate. Cobb and Yackel (1996) reported the salient influence of argumentation and communication for children while Alibert and Thomas (1991) reported the value of mathematical debate for advanced student development. The process of debate engages students in the mathematical activities of making propositions, developing arguments, and assessing arguments: both their own arguments and their peers'. However, like mathematicians, making arguments over time should transform into or give way to developing proof (Boero et al., 1996).

In the same way the mathematical community develops standards of rigor for proof that are collectively "institutionalized" (Cobb, 1989, p. 38), the class should develop a mutual sense of what constitutes valid proof. In addition, students should come to understand the importance of the context of proof in that they are valid in relation to the statement they validate and some body of theory within which they are couched (Mariotti, 2000, 2006).

65

The research indicates that mathematicians understand proofs more globally viewing many proofs to consist of only a few major ideas, while students, on the other hand, assess proofs more on a line-by-line basis with less global sense of the overall structure of the argument (Selden & Selden, 2003). For this reason, Leron (1985) introduced structural proofs and Raman (2003) introduced key ideas as tools to give students more global views of proof. Emulating mathematicians' understanding of proof in this way constitutes one other aspect of how students move toward advanced mathematical thinking.

3.3.5. Aspect 5: The Platonic Experience of Discovery

These observations of the mathematical endeavor together paint a picture of the operations of the mathematical community. In short, students should come to understand mathematics as a constructive human activity because they participate in this activity themselves. I do not imply students must believe in the same philosophy of mathematics here described. Cobb (1989) explored the aspects of the mathematical endeavor, especially the sense of discovery, that promote Platonic philosophies of mathematics among mathematicians. Cobb (1989) points out that if students are to participate in the activities of the mathematics community then they should have similar Platonic experiences, but he argues that strict formalism hinders these experiences. Students should, in the course of transitioning to advanced mathematical thinking, emulate the experience of discovering or constructing mathematics themselves and thereby attain to advanced mathematical thought. This study assesses students'

transition to advanced mathematical thinking according to these five aspects of

mathematical activity.

CHAPTER 4

METHODOLOGY

4.1. The Context

This study was conducted at a mid-sized university (25,000 students) in the Southwest. All data was gathered over three consecutive 15-week semesters. Each semester of study the same professor[1] taught first-semester undergraduate real analysis. At this university, this course generally includes a proof-based development of real numbers, sequences, limits of functions, and continuity.

Data gathered includes field notes of class meetings, biweekly professor interviews, weekly student interviews with a small group of volunteers from the class, copies of students' written class notes, and copies of student exams. Only interview participant exams were gathered during the first and second semester while all exams of consenting students were gathered in the third semester. During the first two semesters, written field notes recorded all written communication on the blackboards, major aspects and key quotes from professor and student verbal communication, and physical gestures employed to

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[1] Dr. Barbara Shipman is the professor observed in this study. At her request, her identity is revealed here.

aid discussion of course material. During the final semester, every class session was video-recorded for further analysis.

All interviews were audio-recorded and any written records of the interactions maintained. The professor answered questions regarding:

- how she prepared her lessons for coming class periods and what her teaching goals were for those meetings,

- her instructional intent in specific explanations or activities in previous class meetings,

- her perception of and response to particular interactions she had with students either during or after class,

- her perception of the students' thinking and understanding and what student actions engendered those perceptions,

- how she constructed her homework and exams and what she expected from the students in their comprehension or performance thereupon, and

- which events seemed to her to be most salient in developing student understanding.

Thus, the professor interviews primarily focused on identifying her expectations and intentions before class meetings, particular activities, homework, and examinations and then her subsequent response to and interpretations of these elements (see Appendix A for more specific examples of professor questions).

During the course of the study, I became aware of the professor's intentional focus on and awareness of specific student misconceptions and understood that this focus would directly impact the students in the study. Thus, subsequent interviews also included inviting her to point out the misconceptions she anticipated in upcoming material, how she had become aware of them, her plans to address them, the misconceptions she observed in student work and thinking, and her plans towards those as well. The interviews also began to investigate her conception of the course material and how she intended for students to understand and reason about course topics.

Each semester, the class was solicited for interview volunteers (Appendix B presents the protocol associated with this solicitation and informed consent). In accordance with the interview methodology in Alcock and Simpson (2004, 2005), volunteers were invited to participate in interviews in pairs if they desired. The student interview participants were selected from among mathematics majors who volunteered so as to represent a variety of final grades in the "Intro to Proofs" course which serves as a pre-requisite to analysis; in this way, each semester's group of 5 or 6 interview participants included one or two students each who made an A, a B, and a C in "Intro to Proofs." Over the three semesters, 10 students were interviewed as individuals and 6 students were interviewed in three pairs; thus 16 total students participated.

Students were not compensated at all for participation, but they were made aware that the interviews would focus on course material and would thereby probably have the effect of guided study or review sessions. Students during each semester reported that the interviews had been beneficial to their learning and course performance, so students in the latter two semesters were made aware of these endorsements upon being invited to participate.

The interviews (at various times) invited students to:

- recall and explain definitions, theorems, and proofs,

- explain and assess mathematical statements and proofs,

- recall and explain aspects of classroom discussion,

- report the extent, content, and focus of their out-of-class studying and test preparation,

- relate the nature, content, extent, and quality of their group interactions outside of class,

- complete homework activities or other novel activities,

- explain their reasoning on written exam questions (after they were returned),

- articulate their confusion about any mathematical topics with which they were uncomfortable, and

- comment about their course experiences both positive and negative (see Appendix B for more specific examples of student interview questions).

Particular attention was paid to the modes (verbal, symbolic, graphical, etc.) in which students chose to communicate and reason about their ideas. During data

analysis, the spontaneous presence of any language, diagrams, lines of reasoning, examples, etc. from the class discussion in student explanations and reasoning were coded. Interviews ascertained students' ability to recall and interpret classroom explanations and discussions as well as their strategies and success on questions and activities presented to them.

The line of questioning with students moved from more general to more specific in order to give them latitude in the way they expressed their ideas. For instance, I began by inviting them to explain to me "What does it mean for a sequence to converge?" In this way, students could choose to explain their understanding verbally, with a graph, with the formal definition, etc. Sometimes they would ask me to clarify the question so I would request them to tell me, "How do you think about it?" Depending on the content of the class discussion on the given topic, I would then invite them to show me how they visualized the concept or the proof or ask them to write and explain the formal definition or proof.

Some questions appeared during interviews because of the unique or notable nature of the classroom discussion surrounding them. However, definitions that received larger amounts of course attention also appeared repeatedly in interviews to observe the evolution of students' understanding of a given topic over the weeks of its development in the course until they were tested over it. For this longitudinal aspect of the investigation, questions fell into three

groups: 1) questions about topics which have been discussed in class but not studied at home, 2) questions about topics that students have discussed in class and done homework about, 3) questions about topics regarding which students have both done homework and engaged in some form of test preparation. Interview questions were chosen over time to represent each of these three groups. Since classes met twice weekly and interviews only weekly, lecture sensitive lines of questioning could not always be uniformly administered to each interview participant.

During the first two semesters of study, the questions focused primarily on assessing student understanding of the course content in general by way of asking them directly or discussing homework questions. In the final semester, a grant[2] to support new curriculum development for this course was received. To support grant objectives, the interviews also included using the activities that the professor developed for the new curriculum to assess student understanding. Typically interviews presented the activities to students after they appeared in class, but occasionally they appeared first in interview to observe students' initial thinking and responses.

4.2.   The Participants

At the time of data gathering, the professor had been teaching for over 10 years and had previously taught undergraduate analysis twice. She has received

multiple teaching awards and is widely regarded as an excellent, if not difficult, instructor. Each class met twice weekly for 80 minutes at a time. Each course began with 20-30 students and ended with at least 13 students.

The professor placed students into groups of three or four during the first week and they did their turn-in homework together as a group. They turned in one homework assignment for each of the four midterms, and she gave cumulative final examinations. Each exam involved 5 to 20 true-false questions, several proofs, and some exams asked students to recall the statements of definitions or theorems.

In addition to her normal office hours, the professor also picked several hours during the week for "study sessions" when anyone from the class who was available could meet her in a spare classroom and do homework and review together. In general, at least a third of the class regularly attended those sessions. She also strongly encouraged students to visit her office and/or send their work to her for feedback, a habit that every semester of study saw some students regularly following.

Mathematics majors represented the largest portion of the class cohort with smaller groups of physics, computer science, and engineering majors. The department requires an "Introduction to Proofs" course as a prerequisite to analysis, which most of the students finished directly prior to taking analysis. Because of the relatively small number of mathematics majors in the department

and many of the group taking the proof course together, it was not uncommon for groups of students to begin analysis with prior personal and academic relationships built in previous classes. The class's mutual familiarity also derived from their activity in the student mathematics organization at the university and/or working in the mathematics tutoring center for lower-level mathematics classes. At least three students from each of the first two semester cohorts either previously or concurrently participated in undergraduate mathematics research programs.

As of the first semester of study, the department set aside one classroom on the first floor of the building for studying such that no official class meetings were held there. The professor held her study sessions in that room throughout the semester. Throughout the rest of the week, many of the students gathered on the third floor in or near the mathematics tutoring center to work together on analysis. She was regularly present in her office on the fourth floor and visits from students were frequent throughout the semesters of study.

During the following two semesters of study the department dedicated another room across the hallway from the mathematics tutoring center to the student mathematics club. During the second semester of study, I observed at least half of the analysis students in that room, with a significant portion spending time there multiple times per week. The conversations in the room naturally oscillated between socializing and mathematics. Most of this group had several

mathematics classes together and so the blackboards in the room were regularly scrawled with work from various courses, analysis being among them. A smaller but non-trivial portion of the class frequented the mathematics club room during the third semester. Other homework groups chose to gather at tables on the first floor of the mathematics building, and I regularly observed them there working together throughout the semester.

During both of the latter two semesters, the professor reported stories of students working together on analysis in the mathematics club room, coming to her office to clarify what they could not resolve, and then returning to work downstairs. Several students also indicated during student interviews how the extensive interactions in the mathematics club room had been very helpful to their learning.

4.3. Analysis

Interviews were transcribed and then coded according to the open coding method described by Strauss and Corbin (1998). Categories varied in nature. Some captured broad sets of interactions such as examples of visualization or comments on the structure of mathematics. Other categories organized every instance of a given question or example appearing during an interview. The most specific categories focused on strands of the data such as one student's particular habits or qualities or the interactions between two individuals.

Categories were developed that reflected each of the overall goals of the interviews.

The interview and classroom data founded the construction of a thorough model of the professor's instruction. This model, once articulated, was validated against the instructional activities that appeared in the classroom in that the model had to describe in some degree every mathematical activity. Elements of this model then guided further exploration of the coded data from student interviews and students' work on exams. Identifying correspondences between the major aspects of the instructional model and the reasoning and understanding students displayed revealed the socio-mathematical norms established in the classes.

CHAPTER 5

RESULTS

I identified three clusters of socio-mathematical norms that developed in the classroom over the course of the study: 1) norms about visualization, 2) norms about meaningful mathematical communication, and 3) norms about constructing mathematics. For each cluster of norms, I will outline how the professor made specific instructional moves to promote these norms. In addition to describing the teacher's facilitation, I provide accounts of student interaction, understanding, and mathematical reasoning which highlight the establishment of these as socio-mathematical norms in the classroom community. For simplicity, I will refer to these clusters of norms simply as visualization, meaningful mathematical communication, and constructing mathematics, though none of these by itself represents a socio-mathematical norm. When possible, I will propose the specific norm I observed at work.

For the purposes of cross-referencing interview and lecture data, each quote is referenced according to the source, the month, and the day. Lecture

references will appear (Lect 7-31)[3] and student quotes will indicate the first two letters of the students name so they appear (Cy 12-25). All dialogue taken from interviews involved only the interviewer and one student, so when necessary a first initial indicates the speaker.

5.1. Visualization

Graphical reasoning cannot be extricated from mathematical history or practice, but that does not mean it is always given due attention in mathematics classrooms (Aspinwall et al., 1997; Eisenberg, 1991; Eisenberg & Dreyfus, 1994; Tall, 1991a). However, graphical and diagrammatic images pervaded the study classroom's conversations. I here adopt the professor's language for all such visual images by referring to them as "pictures" unless greater specificity is warranted. The professor promoted visualization for making mathematical meaning in a number of key ways:

1. A large majority of explanations and discussions during class meetings integrated some form of picture.

2. The introduction to many instructional sections centered upon examining sets of particular examples (of sets, sequences, functions, etc.) which represented most of the important dimensions of possible variation (Mason & Watson, 2008) for the subsequent mathematical conversations, such as sequences

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[3] All lecture quotes come from Shipman, B. (year omitted to preserve student anonymity). Portions of lecture presentations are in preparation for publication under NSF grant DUE #0837810.

which are bounded but not convergent or properly divergent but not monotone.

3. The professor modeled solving problems and constructing proofs using pictures and directly instructed students to do the same on homework and tests.

4. She guided the class in activities specifically designed to hone students' mental images of given analysis situations and concepts, sometimes pointing out common misconceptions induced by graphical images which were not sufficiently general.

5. The teacher engaged the class in activities where students worked in groups to produce pictures that portrayed particular theorems that the class then presented, discussed, and evaluated for relative strengths and weaknesses.

Thus the professor's instruction modeled the use of visualization, helped develop student's visual images, involved discussions about proper use of visualization, and directly elicited visual reasoning from students. The following vignette illustrates some of the various ways visualization was integrated into classroom discussion.

5.1.1. Scaffolding Proof Through Pictures

During a lecture in the second semester of study, the professor introduced the following true-false question to the class for assessment:

T/F If f: D→R is bounded and g: D→R is bounded and $f(x) \leq g(x) \, \forall \, x \in D$, then sup f ≤ sup g. (Lect 9-29)

The teacher asked the students how they would begin if they were in their groups and starting to work on this problem. Three students proffered suggestions: start to prove by picking an element, find a counterexample, or draw a bunch of examples. The professor keyed in upon the final suggestion agreeing that drawing pictures would help them to understand the proof. Figure 5.1 presents the three graphical images she drew on the board pointing out they will draw at least one pair of functions intercepting though they may not.
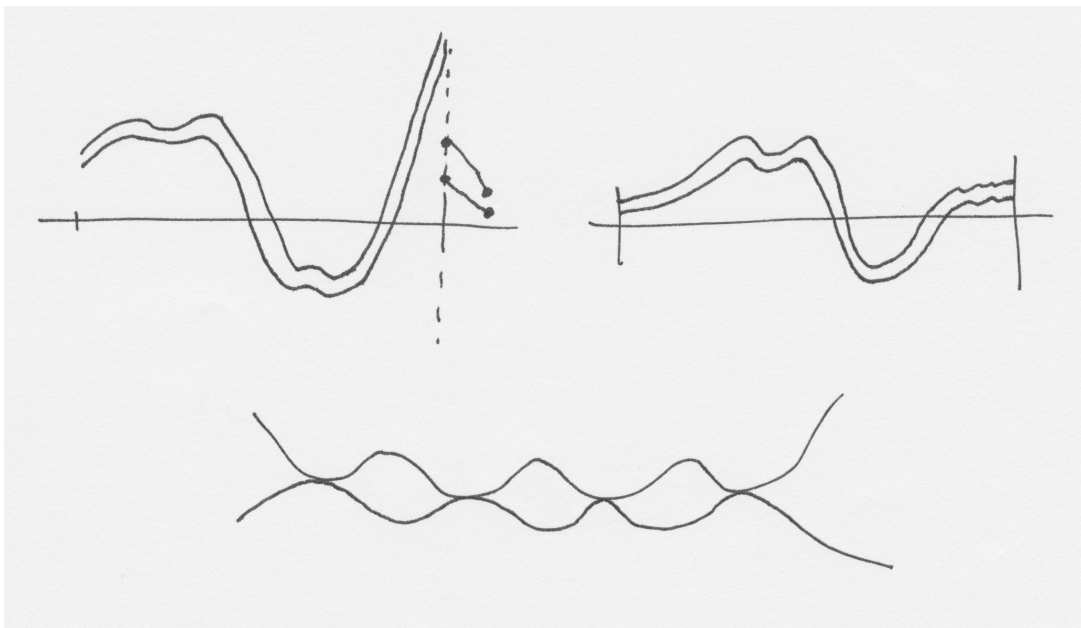
Figure 5.1: Three examples portraying the condition $f(x) \leq g(x) \ \forall \ x \in D$ (Lect 9-29)

It took some time before Locke noticed that neither of the first pair of functions had a supremum and thus they did not fit the assumptions of the statement. However the class agreed that they could not think of a counterexample, so they decided to attempt to prove the statement.

The professor told the class that when trying to prove an inequality, it is often helpful to assume the opposite and "get in trouble." After writing the initial hypothesis on the front blackboard, the professor walked to the blackboard on the side of the room and reminded the class of a previous true-false statement as Figure 5.2 shows.
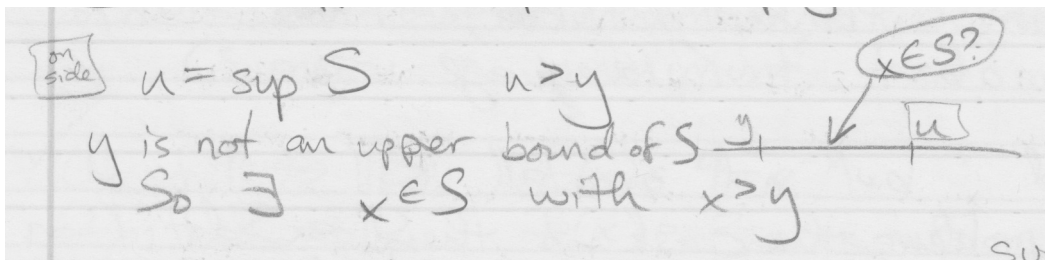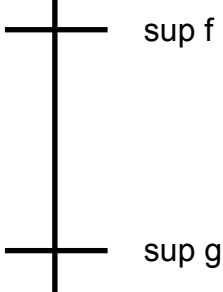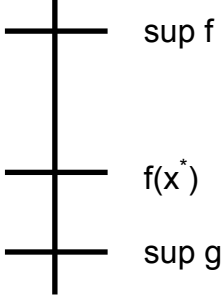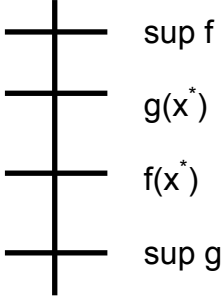
Figure 5.2: Professor's reminder of previously proven statement (Lect 9-29).

She wrote the statement in its original formulation in terms of supremum u, point y, and a set S. To her inquiry regarding whether some element of S is greater than y, Locke said that if no element of S were greater than y, then y would be the supremum. She agreed modifying his statement to say y is an upper bound.

The professor then helped the class apply this statement to the prior assumption that sup f > sup g. For the rest of the proof, the professor introduced a y-axis diagram and used it to motivate/explain each of the next logical steps. Table 5.1 presents the parallel verbal, graphical, and written development of the proof.

Table 5.1: Using a y-axis diagram to produce a proof (Lect 9-29).

Discussion	Diagram	Written Proof
In order to show that something is greater than or equal to something else, it is often good to assume the opposite and use proof by contradiction.	sup f sup g	Assume sup g(D) ≤ sup f(D)
Whenever something is less than the supremum of a set, then what do we know about it? There has to be an element of the set greater than that. What does an element of the set look like in this case? Some function value $f(x^*)$.	sup f $f(x^*)$ sup g	Then sup g(D) is not an upper bound of f(D). So ∃ $x^* \in$D such that $f(x^*) \geq$ sup g
How then does this help us? How can we use our hypothesis? By our hypothesis we know that $f(x^*) \leq g(x^*)$. So what then? This cannot happen. This is a contradiction because an image of g is strictly larger than the sup of the function.	sup f $g(x^*)$ $f(x^*)$ sup g	By hypothesis, $g(x^*) \geq f(x^*)$ So $g(x^*)$ > sup g(D) This contradicts definition of sup g. Thus sup f ≤ sup g.

This vignette shows how analytic proof production was coupled with graphical explorations. Before the proof, the professor took up one student's suggestion to examine several example pairs of functions portraying the assumptions of the statement. She accompanied the recall of the previously proven statement using a line diagram parallel to that used during its initial

exploration. She then used a y-axis diagram to both represent and motivate each step in the proof, which was then translated into an analytical statement.

This vignette also shows two of the three major types of graphical images that the professor used in instruction. The first type includes standard graphical images of commonly defined functions (what Pinto & Tall, 2002 call a *specific* picture). The examples which the professor drew before the proof are examples of the second type of graph in which particular examples are drawn, but they have no visual pattern by which a rule could be produced and are thereby meant to represent an "arbitrary function" (what Pinto & Tall, 2002 call a *generic* picture). The y-axis diagram represents the third category of images where no example is drawn, but aspects of the function are mapped onto the frame of coordinate axes to represent individual properties and relationships for classes of functions.

5.1.2. Students Using Visualization on Tests

To establish the presence of visualization as a socio-mathematical norm, the students had to adopt visual representations and forms of reasoning as viable ways to communicate and make mathematical meaning. For general evidence of students using visual reasoning, I analyzed the available exams from the third semester looking for use of pictures. The tests included three types of questions at different times: true-false questions, always proofs, and occasionally writing definitions. Table 5.2 presents the frequencies with which students used

visualization on tests from the third semesters (there was insufficient data during the first and second semesters of study). The representations used almost always reflected those that the professor introduced during class.

Table 5.2: Numbers of students in the first and third semesters using pictures on exams.

Semester	Test	Total	Pictures (%)	T/F (%)	Non-T/F (%)
3	1	23	18 (78)	5 (22)	18 (78)
	2	21	16 (76)	9 (43)	14 (67)
	3	19	10 (53)	5 (26)	5 (26)
	Final	15	13 (87)	12 (80)	4 (27)

The overall rates of picture use varied with respect to the topic being covered. The first test almost always included lots of diagrams used to construct bijections between infinite sets and heuristic diagrams of functions used to answer questions about function composition, one-to-one, and onto. Students sometimes used the professor's horizontal number line representation for sequences, but others adopted a two-dimensional Cartesian representation.

One strong trend in the results is the increase in use of pictures to answer true-false questions over the course of the semester. There was a simultaneous increase in the frequency of writing out explanations for true-false questions alongside answers, but this still indicates a movement toward using visual means to assess mathematical statements. During that semester, eight students drew

pictures to aid their production of definitions (on two tests which elicited definition recall).

5.1.3. Students Recalling Proof Diagrams

Throughout the student interviews, when asked to recall a proof that was presented with both visual and analytical explanations, students showed a propensity to recall visual representations associated with proof first and more often than aspects of the analytical argument. However, the accuracy of their recall and comprehension of the visual and analytical explanations correlated strongly with other factors in their understanding, namely their concept definition (Vinner & Tall, 1981).

The proof that the composition of continuous functions is continuous was one example where the teacher developed using a picture before writing an analytic proof. Figure 5.3 shows the triple number line diagram she used to discuss the key idea of the proof (Raman, 2003): letting the delta obtained from the continuity of g equal the epsilon to which the continuity of f is applied.

She introduced the fact that the continuity of f and g each introduce an epsilon and a delta in their respective domains and ranges. She also pointed out how the definition of continuity requires them to "work backwards" from the range of g to the domain of f. Each of the first two semesters a student proposed that the epsilon from f should equal the delta from g. During the first two semesters, the class proceeded to use these observations to write an analytic proof.
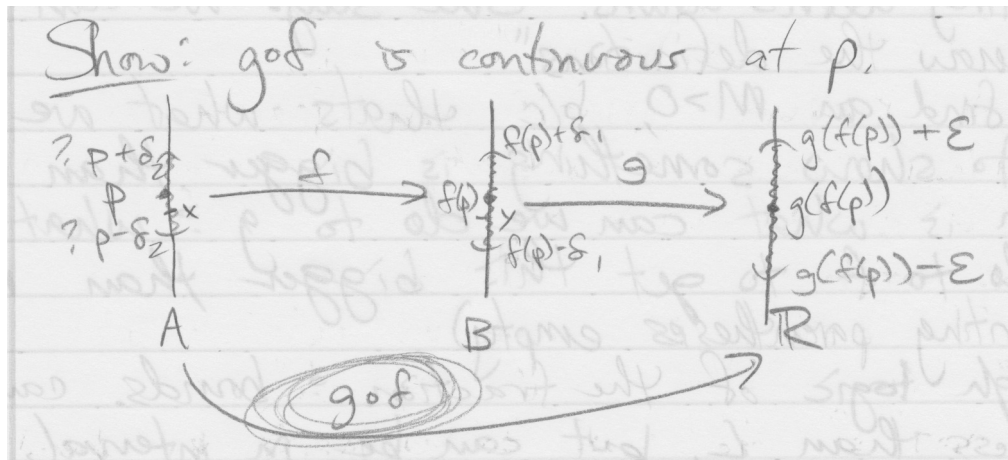
Figure 5.3: Diagram used to prove the composition of continuous functions is continuous.

Each semester, I asked students in interview to tell me about the proof not making particular reference to either the drawing or the analytic aspect. This interview happened soon after the initial lecture so that none of the students had reviewed the proof between class and interview. Only one student out of the 11 that could participate in this series of questions did not begin by drawing some form of the diagram. Table 5.3 displays the correspondences between aspects of students' recall of the proof. Figures 5.4 presents two of the diagrams that students produced during interviews that contained all of the important elements of the proof situation. Figure 5.5 contains two student diagrams that were incomplete.

Several key trends arose among the students' diagrams and explanations. Three of the four students who drew a correct diagram from memory (including neighborhoods on each of the three number lines) gave a correct articulation of

the logical dependencies between the four epsilons and deltas (note: the fifth student who drew a correct diagram had his partner articulate the logic).

Table 5.3: Aspects of student recall of proof diagram and argument.

Student(s) (one pair met together)	Complete picture	Proper account of proof logic	Recalled equality of center ε and δ	Incomplete picture	Tried to use limit def. of continuity
Cyan	X	X	X		
Edgar	X	X	X		
Vincent	X		WN		
Ronso	X	X	X		
Auron	WN	WD	WD		
Tifa | Barrett	X	WD	WD		
Cid				X	X
Celes			X	X	X
Rikku				X	X
Locke	No Picture				X
X: student exhibited behavior; WN: behavior exhibited with class notes available; WD: behavior exhibited with diagram presented					

Students changed the images such that they were not producing a veridical image of her diagram (a graphic memory) but rather a conceptual reconstruction. For example, several students drew circles instead of number lines and one student drew dotted lines between the three number lines that funneled into the next neighborhood rather than being parallel like her diagram. Four of the five students who could not produce a correct articulation of the logical relationships between the neighborhoods tried to apply the definition of continuity that requires that the limit of the function equal the value at the point rather than the epsilon/delta definition that the proof used.

Complete Diagrams for the Proof that the Composition of Continuous Functions is Continuous	
	Cyan (11-25)
	Vincent (4-17)

Figure 5.4: Complete student diagrams of the composition of continuous functions.

Two students did not remember the proof at all initially, but when presented with the diagram they were able to fully explain the logical relationships between the neighborhoods and the key idea of the proof. A third student when presented with the correct diagram was able to recall that the delta from the f function "will work for f of g" and she remembered that, "there exists a delta interval, an epsilon interval, you can just use them to redefine the intervals you want for this function." However, she never did assert the equality of the

neighborhoods in the center set, but only said that they "got some information here" pointing to the center. Several students who were unable to produce the correct diagram or logical argument reported having understood the idea when it was presented during class.
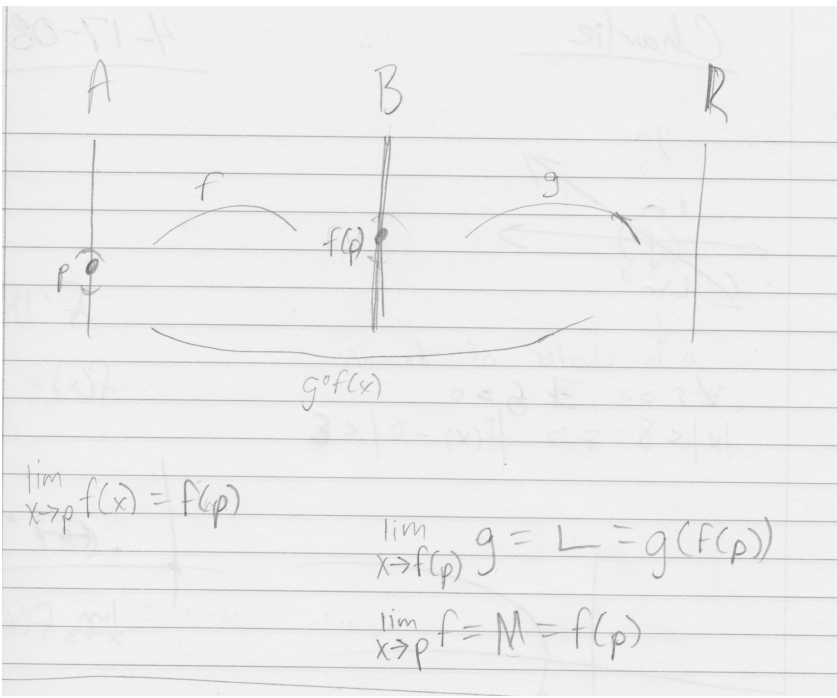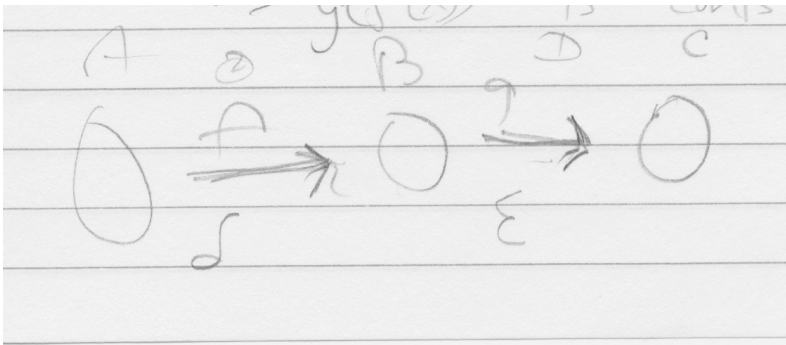
Incomplete Diagrams for the Proof that the Composition of Continuous Functions is Continuous	
	Cid (4-16)
	Celes (4-18)

Figure 5.5: Incomplete student diagrams of the composition of continuous functions

5.1.4. Students Communicating Using Pictures

The second semester of study provided a strong example of how the class communicated via graphical images. The class had proved the monotone convergence theorem (any bounded monotone sequence converges) and was moving toward proving the Bolzano-Weierstrass (BW) theorem (any bounded sequence has a convergent subsequence). The professor began the lecture by reminding the class that a divergent sequence sometimes has a convergent subsequence. The class considered separately unbounded divergent and bounded divergent sequences. The students produced examples that both did and did not have convergent subsequences.

The professor then wrote on the board, "A <u>bounded divergent</u> sequence (always, sometimes, never) has a convergent subsequence." Locke very quickly said that the statement should read "always." The students spent some time trying to find a counterexample, but the professor affirmed that in fact none existed and presented the class with a statement of the BW theorem. Locke suggested that bounded divergent sequences "converge to two different things." Cyan extended that suggestion to say that a sequence is bounded and divergent if and only if it has two different limits of subsequences. The professor rephrased his claim and presented it to the class as a true-false question, "If a sequence is bounded and diverges, then it has subsequences that converge to different limits" (Lect 10-15).

However, to move forward in talking about the Bolzano Weierstrass theorem, she redirected the discussion back to the previously proven monotone convergence theorem (MCT). Locke almost immediately responded to the statement of the MCT that they needed for any bounded sequence to have a monotone subsequence. The professor affirmed this suggestion writing on the board, "To prove B-W using MCT, we will prove [Locke's] lemma: Any bounded sequence has a monotone subsequence" (Lect 10-15). Throughout the rest of the course, they always referred to the monotone subsequence theorem as Locke's Lemma.

As the class moved on to prove Locke's Lemma using the "peaks" argument, the professor exchanged her normal number line representation of sequences for a two-dimensional representation of points connected by lines which gave the appearance of a mountain range as Figure 5.6 displays. She presented the proof case in which a sequence has infinitely many peaks (sequence elements which are greater than or equal to every sequence element with a higher index).
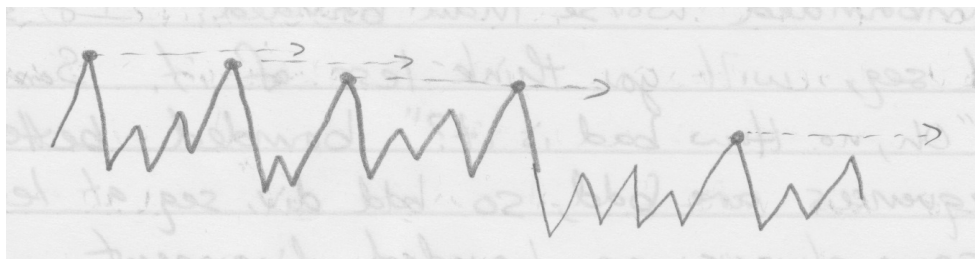
Figure 5.6: How the professor portrayed a sequence with infinitely many peaks. (Lect 10-15)

As the class then considered the case where the sequence had only finitely many peaks, Cyan told the class that he was unable to think of such a case. The professor handed him the chalk and asked him to draw an example of a sequence with exactly three peaks. He draws three peaks, but then is stymied by his inclination to draw the rest of the sequence tending to infinity because the sequence must increase to prevent any further peaks. Locke then got up and filled in the rest of the diagram for Cyan as presented in figure 5.7. The professor went on to use an expanded diagram of such a sequence with finite peaks (figure 5.8) to construct the rest of the proof.
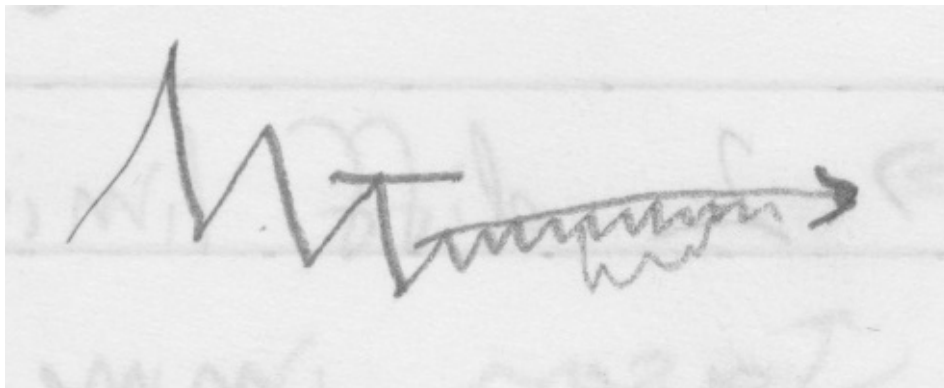
Figure 5.7: Locke's image of a sequence with finitely many peaks (Lect 10-15)

This instance of Locke seeing counterexamples or images that much of the rest of the class missed was not isolated. During a later interview, Cyan praised Locke's ability to see counterexamples and visualize well saying:

> I love discussing things with [Locke] because he often sees the point of view that I don't have. And some people will see my point of view, especially if we have been discussing the material already, because some people don't know it at all. And then they come and we discuss it and then they see my point of view and then whatever flaws I have. Like I said,

when I don't understand some of the implications of the definition, some people carry those flaws into the classroom. They will say, "Oh well I think this," or, "Well intuitively I think that," and [Locke] will have some other intuition altogether. He'll have an example in his head. "No, this is why." It's like, "Wow, I needed the picture." If he shows me the picture, he doesn't have to draw it out, but he can explain the picture and I will visualize it and I will say, "Oh, ok." So I really learn it from that. And I am really learning well in this class. (Cy 10-14)
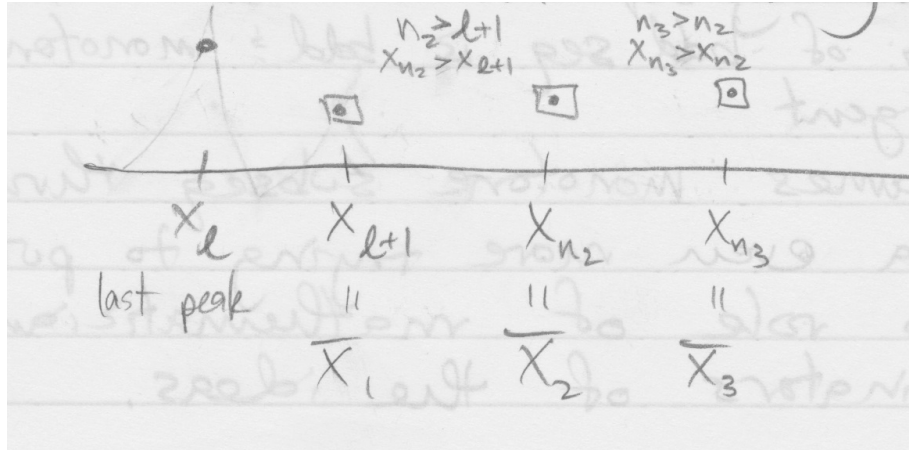
Figure 5.8: The professor's translation of Locke's image (Lect 10-15).

Cyan spent a large amount of time studying and helping his classmates in the mathematics club room provided by the department, and so he cited how his misunderstandings affected the other students whom he tried to help. Not only did Cyan attribute to Locke a unique perspective, but also he particularly emphasized the benefits he felt he gained from Locke's visual images or examples.

In addition to Cyan's report that Locke's mental images helped his visualization and conceptual understanding, Locke spoke during an interview about the ways in which the professor's true-false questioning procedure helped

refine his internal images of definitions. He said, "True-false were probably most helpful for kind of like shaping an overall sort of picture as to what the definition really means. Like, that was definitely the most helpful thing for me" (Lo 12-11). Though formal definitions have analytic form, Locke speaks of the refinement of his "picture" of what the definition means.

5.1.5. Reasoning Graphically About More General Functions

As the professor had done in class, I invited students to explain the sequential criterion for function limits both verbally and graphically. Cyan was explaining his understanding when I felt I needed him to draw the picture he was referencing in his head.

I: Okay so can you draw me a picture, kind of what you see when you are talking about all that?

Cyan: Sure. I can draw a random function and if I have a cluster point in the domain. And I skipped a lot of stuff when I was describing it, cause you have to have all these mandatory things like a sequence that converges to a cluster point in the domain, but the elements of the sequence not equal to the cluster point. So if I have x's or a sequence of x's and they are converging to p [drawing x's on the x-axis near the point labeled p], then the sequence of the image of this or whatever sequence x_n converging to p. Then the sequence of the function, or the image that's in that sequence, if I have this and this happening where this converges to L. So all my y values for this function, like in this little neighborhood, they are all in here [pointing to a neighborhood drawn on the y-axis around the point labeled L]. So like, y, y, y, y and the y is just this sequence right here [writing y's near L]. And so they are converging to some point L. I changed where I was drawing it because before I used to draw it like on here [the graph of the function], like my y's on here, and then I think oh just don't think about that, just think about what their value is.

I: And why is that? Why does that help you or what made you switch?

Cyan: Uh, I think because some functions if we define them not so nicely as this, it is not so clear cut that you can say oh this y this y. Maybe it is, but. Yeah. It might be but we had some functions, I don't know the name of the one, but the one where it had the rationals and irrationals [the function defined as x at every rational number and 0 at every irrational] and that is not as clear, cause you have a line. It's not a continuous line; it's a dotted line. So I wanted to do it this way and now I can think of a range instead of thinking of points on the function. (Cy 11-11)

Cyan observed that a graphical representation of a function defined differently on rational and irrational numbers will be misleading because the line does not represent the function correspondence for every point in the domain. This led him to shift from thinking of the y's or outputs as "on the function," by which he meant on the graph of the function, to drawing them on the y-axis because he wanted to "just think about what their value is."

5.1.6. Pitfalls of Graphical Explanations of Definitions

Throughout the study, the professor discussed definitions in terms of the idea behind the definition and the formal definition itself. One of the final topics that the course covers is uniform continuity. The professor explained uniform continuity in terms of adding to the continuity property the condition that "the same delta work for every point in the domain," in which case one must find a smallest delta to establish the property. During this conversation, she would compare individual points on a function to show that steeper points required smaller deltas, and so she began to use the language of deltas working or not working "if you move in that direction" meaning in a direction where the function grew more shallow or more steep respectively.

She used this kind of explanation to look at many common functions and let the class determine whether they were uniformly continuous or not. She particularly emphasized the contrast between the functions square root of x and natural log of x because though they both "get infinitely steep," the former is uniformly continuous and the latter is not. She articulated this observation saying that even though square root of x has a vertical tangent at zero, it has a steepest point at which we can find a smallest delta. Similarly, tangent inverse has a steepest point and is uniformly continuous. The teacher centered all of these conversations around graphs of these functions drawing epsilon and delta neighborhoods at various points. She could also use this notion of "steepest point" to explain why the square function and $^1/_x$ are not uniformly continuous.

Three first semester students' reasoning during interviews revealed some of the strengths and pitfalls introduced by the informal language the professor used in these graphical explorations of uniform continuity. First, Vincent failed to capture the idea that uniform continuity is a global property of functions and so he would talk about functions being uniformly continuous "if you go this way." He said the following:

Vincent: Well it is uniformly continuous if you go this way… but if we go this way it wouldn't work out for us… If we were to go this way, we couldn't, like, find the small delta because it's only going. Like, it's getting steeper; I remember that tendency. I remember her saying that being steeper, the delta gets smaller and smaller and smaller and as we get closer and closer to zero. We get, the delta gets smaller, but it is kind of hard to pick a point next to zero because of how dense it is between two points. And so finding the smallest delta in this case.

I: So is square root of x uniformly continuous?

Vincent: Well it depends which way we're going

I: What do you mean which way we're going?

Vincent: Like how I always looked at it is if you were to go this way, if you were just continually going this way, if you just picked a delta here, it would work for any point to the right of it. If I pick a delta here it would work for any point to the right of it. But if I pick a delta here, it is not going to work over here. And that's what I mean by which way you are going. (Vi 5-5)

Vincent correctly remembered the professor saying that deltas will work if you move in a direction where the function gets less steep, but he applied this notion to the function being uniformly continuous or not rather than to the delta which is a candidate to fulfill the definition of uniform continuity. He also misapplied her explanation of the ever-increasing slope for the natural log function to the square root function. I decided to try to get him to make a global statement about the uniform continuity of the function.

I: So do you think square root of x on its whole domain is uniformly continuous or not?

Vincent: Hmm, I want to say not because you want to be able to go either way. You won't be able to pick a delta over here that will work over here you won't be able to pick a delta over here that will work over here. And like I said we can keep on picking a smaller delta than the previous one, and so that will give us some trouble because we aren't able to find a smallest delta. But if we were to cut off our domain, then… (Vi 5-5)

Vincent's misunderstanding of the professor's initial explanation of comparing delta values across the domain of functions kept him from either understanding or remembering the professor's claim that square root of x is uniformly continuous. He cited her language of finding a smallest delta and

claimed that removing part of the domain would solve the problem. At this point I reminded him of the uniform continuity theorem and pointed out that it applied to the function square root of x on the interval [0,1]. Upon this reflection he reoriented his thinking:

Vincent: Oh. Hmm. Well it is, that would be closed. Well can we pick delta around zero?

I: Why do you say that?

Vincent: Well, cause it is getting steeper as we go towards zero and zero is the last point. (Vi 5-5)

He trusted the theorem enough to reconsider his previous argument and was able to recreate the professor's argument about the square root function using the notion of steepest point, though he never mentioned recalling her saying it. After this however, he began to reverse his thinking paying attention to shallower points instead of steeper points and was unable to get around this misconception.

A second student from the first semester of study, Celes, focused on the professor's notion of steepest point and thereby claimed that the square root function is not uniformly continuous on the half-open interval (0,1]. She said, "Oh because if this is open, this is getting, this is like basically the limit of this is going to zero. So this is getting smaller and smaller and smaller and smaller. So you can't ever get a smallest one. Cause it will always be smaller" (Ce 5-7). However, Celes went on to correctly produce the formal definition of uniform continuity from memory, but thought it was wrong until I let her look at

99

her notes. She felt like something was missing and that the definition needed an H somewhere (H appeared as an index in other definitions the class used). She never went back to reevaluate her earlier assertion about the square root function.

During her interview Aerith (also from the first semester of study) made the same mistake that Celes did, but was able to use her understanding of the formal definition to correct her misapplication of the notion of steepest point (which she called cutting or cut off point). However this appeared to be a discovery for her because she initially indicated that she did not understand the formal definition.

I: What is your understanding of uniform continuity?

Aerith: Well actually like I don't really understand. I know some of the things like if the slope goes to infinity and there is like no cutting point, then it's not uniform continuity, but like by reading the definition I don't really see what's the difference between continuous function and uniform continuous… The slope cannot go to infinity or else the slope. I guess what I am trying to say, it cannot be very deep [steep]. The graph is very deep then it has to be a closed interval.

I: If it's closed, then is it uniformly continuous or not?

Aerith: Yes, it is.

I: Okay, what if it comes down and it's an open point.

Aerith: Hmm. I guess it's not, because there is always a next point before you hit this point, so you can't find a, like, cut off point. I guess. (Ae 5-2)

She remembered that the square root function is uniformly continuous and the professor's steepest point argument and came to the same conclusions that Celes did. However, she began to evaluate her conclusion in her own terms.

100

I: Why are you hesitant? What do you think?

Aerith: Well, cause I was thinking this one is because for slope you need two points and since there's cut off point. You always can find another point that, that you can find a slope, but in this case there is always next one and next one and next one and it is going to infinity. I am not sure though. I guess it is, it is uniform continuous. (Ae 5-2)

Here she makes a shift from thinking of the steepness of the function at a single point to thinking in terms of pairs of points as in the formal definition. She seemed to shift from tangent thinking to secant thinking between pairs of points saying, "because for slope you need two points." She then shifted to corresponding pairs of points and seeing that the secant slopes are unbounded before making her discovery via the idea of the formal definition.

Aerith: Well cause by definition it says x_1-x_2 will be less than delta and $f(x_1)$-$f(x_2)$ will be less than epsilon and there is like you can find two points from here and that holds the definition. I just think when, it's like, this thing basically is saying when x_1 and x_2 getting closer, the image of, I mean the value of these two points will getting close, too. Like they were getting close, too. So you can find two points that's, they are really, like the function is like this, you can find two points that are really close and the value is really close, too.

I: So you think it is uniformly continuous?

Aerith: Yeah I think so. (Ae 5-2)

Aerith expressed her initial confusion regarding the formal definition, particularly how it differs from the definition of continuity. However, she remembered it well enough that as she tried to reason about the novel question of the deleted point she came up with a personal articulation of what the definition means, namely that as points in the domain get closer, so do their corresponding image values. Once she shifted from thinking of uniform

continuity in terms of the steepest point to her new verbalization of the formal definition, she correctly assessed that removing a point does not change the uniform continuity of a function.

5.2. Meaningful Mathematical Communication

Communicating mathematics through multiple pathways and media became a central aspect of the culture of the classroom of study. The structure of the class promoted sharing ideas, asking questions, and developing language for communicating about mathematics. The class spent time directly discussing mathematical language and conventions as well as adopting some of their own. This local language appeared among several forms of language used to discuss mathematics, namely intuitive, formal, and metaphorical. The classroom dialogue shifted between these various modes or used them in parallel promoting their correspondence.

5.2.1 Structural Promotion of Communication

A number of elements in the course offered chances for students to communicate their ideas about mathematics and thereby gain access to one another's thoughts. First, students completed all homework and many class activities in groups of three or four. The professor regularly took votes about true false statements she presented and invited one or both sides to explain their reasoning. The study sessions and mathematics club room provided extra time

and context for students to work together and for smaller student-teacher discussions.

Vincent expressed some dissatisfaction with the group aspect of the class. He complained about the gap in comprehension he felt between him and his group members:

> I usually try to do what I can. Half of the time I am like over here and they are both answering of it in the right direction. I don't know. It sucks because I can't comprehend it as well as they do. And then I have to understand it by seeing the work that they do, which helps out, but I feel bad because I am not putting out the. I guess I am not doing, I don't feel like I am doing an equal amount of work that they are doing just cause they are always getting the answer correct and I'm not. So, that's never fun. (Vi 4-8)

However, later that month he reported having worked extra hard with the goal of trying to find answers before his fellow group members. By the final interview, he still maintained that he was unable to keep up with his group and that had hurt his class performance because having the answers available hurt his motivation. When asked about what he enjoyed in the class, he responded:

> This is the first math class where I actually tried working with other people. In all the other ones I thought that I could go through it alone, but this one I really felt like I couldn't. I felt that I really needed to work with other people to actually learn this material. I mean, some of it I got on my own, but not all of it. So my favorite aspect was actually working with other people. It was fun for a change. Or something different for a change. I have never done that. (Vi 5-5)

Despite the gap he perceived between the understanding of his group mates and himself, Vincent reported a very positive feeling toward the interactions he had with other students and the new experience of learning mathematics in community.

5.2.2. Developing Mathematical Language(s)

In several key ways, the class attended to the difficulties posed by mathematical language and notation. Some conversations addressed notational issues such as expressing that a sequence tends to infinity using limit notation even though the limit does not exist. In another such activity, the professor provided students with examples of statements past students had written that were incompatible with standard notation such as "$x \in f$" where f is a function or "$y < S$ where $S \subseteq R$." The list of statements also included examples of accepted notation and the class identified any errors rather than the professor. The class discussed the differences between three classes of mathematical objects (points or numbers, sets, and functions), and which relationships (i.e. subset, less than, element of) are appropriate for each.

When the professor introduced a new definition to the class, she would often begin by letting students articulate their understanding of what the definition should be. For instance, students articulated the definition of function limit saying the function value approaches L as x approaches p. Once the class developed some common language articulation of the definition, she would write a more common and intuitive form of the definition on the board to be translated into mathematical language. Through two or three iterations, the class translated the definition from common language into the form of the standard formal definition. In this way, the class practiced expressing their ideas in common language before attempting to formalize their pre-formed idea.

Cyan described the benefits of this process when he spoke of his

interactions with the textbook. He said:

> The book actually the definitions were more difficult, so I understood them, but only after we had already gone over them. So if I just tried to read the book I might have had a lot of trouble trying to get the definitions and understanding what they meant. Mainly because of the way they lead up to writing a definition. I think that it's like it will give you some terms and stuff and if you'd been completely in tune with the book maybe you'd understand what the terms are when they give them to you. Like, ok, I know what they are talking about right now, and then they give me a definition. And it's like ok. I understand it. But instead we have been talking in I guess looser terms in class and then when you go and you have a definition in class in your mind and then you read it out of the book, then you identify what the book is using with what you have been using. And the book could be hard to keep up with some of the things. After we have gone through first, I think it's a lot easier to keep up with, so it has been a lot easier for me to read after the fact. (Cy 12-14)

Cyan contrasts the difficulty of trying to interpret symbolic expressions of

definitions first with the class' practice of using "looser terms" first. He describes

how he is able to correspond the language the class has adopted with the

symbolic formulations of definitions in the book saying, "you identify what the

book has been using with what you have been using." He assessed that this

translation process was much easier than trying to understand symbolic

expressions first.

The professor also guided students to imitate this translation process to

write proofs. An example of this occurred during a study session in the third

semester while a student was attempting to prove that if the function $g \circ f$ is

surjective, then f must be surjective. The professor first guided the student to

draw a picture of the situation. Using a diagram like that in figure 5.9, the student

said, "If f isn't surjective, it misses something. Then there's no way for g to get to

it." The professor affirmed her idea and encouraged her to turn it into a proof.

The student however got stymied trying to write the proof down, so the professor

asked her to say in English what it means for the function to not be surjective.

The student responded, "I didn't get hit by anything." The professor strongly

affirmed this idea and with some guidance the student went on to produce a

complete proof.
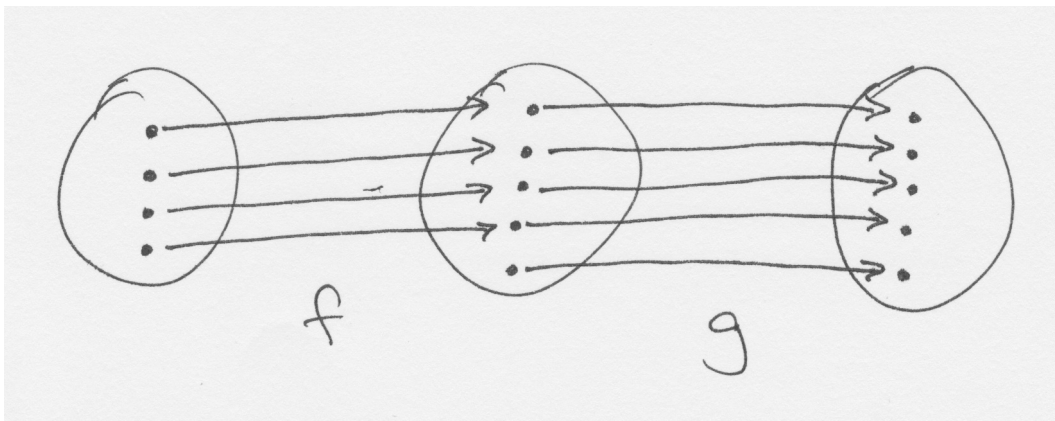
Figure 5.9: Diagram portraying why f must be surjective g of f to be surjective.

In addition to intuitive English language and mathematical symbolic

representations, the class also developed metaphorical language for some topics

of the course (Dawkins, 2009). During the first section, the class used three

distinct types of language to understand the definitions of one-to-one and onto.

The metaphor that the professor used to explain the two definitions portrayed the

function as arrows being shot by elements of the domain at elements of the

range or target. Thus the professor explained that a function on the natural

numbers that sends every odd number to 1 is not injective because "one guy gets hit by a bunch of arrows." According to the metaphor, onto means "everyone gets hit." All of this metaphorical language however was strictly verbal, while on the board the professor wrote:

f: A→B is one-to-one, or injective, if

$$f(x_1)=f(x_2)\Rightarrow x_1=x_2$$

… if distinct elements of A have different images in B.

That is: if $x_1 \neq x_2$ in A, then $f(x_1) \neq f(x_2)$

Equivalently: if $f(x_1)=f(x_2)$, then $x_1=x_2$. (Lect 8-25)

This excerpt from the board shows her juxtaposition of common English language in the third line with a symbolic formulation on the second, fourth, and fifth.

After the first test, I asked Tidus about his understanding of one-to-one and he replied:

> Injective, every input has a unique output. So if you have a set A and you put it through a function to set B, every input in the A set will have a unique output in the B set. (Ti 9-25)

Tidus initially explained his notion of one-to-one in the English formulation. Students often began with the common language form of definitions before translating into the symbolic form.

It was very common for students in the study to articulate one-to-one by saying "every input has a unique output" as Tidus did. However, because mathematics books often use this exact language for the definition of function

107

(this output is the unique output for this input among the set of possible outputs), the professor presented the third semester's class with an activity discussing these two meanings. Further questioning of Tidus and other students who used this language for one-to-one usually revealed that they meant the correct notion that the output is unique to that input. To understand what he meant, I questioned him further:

I: mmhmm. and so um, I think we had can you state that in another way? 'Cause I think we had other ways to formulate it.

Tidus: I can't remember. We haven't looked at it in a little while. That is how my brain interprets it. I can, if I had a definition I would be able to do it. Oh, like if f(x1)=f(x2)?

I: Yeah, yeah. So that was another way we talked about it. So can you finish that statement?

Tidus: If those two are equal, if $f(x_1)=f(x_2)$ and if it's 1:1, then $x_1=x_2$. (Ti 9-25)

Here, Tidus explicitly states that he thinks of the definition in terms of this intuitive English form ("That is how my brain interprets it."). Once he remembered the formal definition, he produced it correctly.

When I asked him about the two definitions, he explained his understanding of the relationship between the two saying:

[The first definition uses] more elementary words. Input has unique output; 1 has 1 output. Very simple to remember and I can see it that way. That more technical definition is something I will have to study, I won't be able to remember that right off of the bat, that is something I will have to think about. (Ti 9-25)

He also explained that even if we think in terms of the intuitive definition, we use the latter for proving. I then asked him about the notion of onto:

Tidus: Umm, everything in the target gets hit. [laughter] So let me remember. If I can remember how to say this: if there is, if you have a set A and a set B. Ok so if a is an element of a set A. So if I have this set A and this set B, and I guess g will be our function this time. Everything in here will have an output with this g. So everything in this set will have an output or have some kind of. So every element b will equal g of some input in A [writes g(a)=b]. I can't remember the exact definition, but

I: I see what you mean. Ok, very good. And so what was kind of the metaphor that she used to describe that?

Tidus: Everything gets hit. She had like a target, you know for arrows, and every arrow will hit it. Or every target will get hit. (Ti 9-25)

In the case of onto, Tidus along with several of his classmates initially articulated their understanding in terms of the arrow metaphor. He acknowledged the informal nature of the metaphor in his laughter before restating the definition in English and then writing a symbolic expression of the definition only leaving the quantifiers verbal. In this and similar cases, students employed all three classroom languages in tandem often transitioning from the less formal to the more formal.

The professor introduced the arrows metaphor into the class, but other elements of the classroom language originated with the students. During the second semester the class negotiated their understanding and definition of sequence convergence. The professor presented the class with an informal definition meant to represent common notions of convergence that said, "A sequence (a_n) converges to L if a_n gets closer and closer to L as n gets larger and larger" (Lect 9-29). However, they observed in the case of $(4 - 1/n)$ that 5 is a

109

possible limit of this sequence by this definition. As the class proposed different ways to amend the English definition to be more specific, Zell told the teacher that "if you want to find a party, see where everyone is at" (Lect 9-29). He argued that since no one was at 5, it could not be where the party was. This metaphor pleased the professor who seized upon the language.

She provided another example that was defined as $5 - \frac{1}{n}$ for the first million terms and every latter term was 4. She posed the question, "How many people have to be at the party," to which Locke responded infinitely many. The professor invited the students to talk for a while and share their ideas about how they should form their definition; she even stepped out of the room for a few minutes to let them discuss independently.

To bring the discussion together again, she wrote on the board a revised English definition that stated, "A sequence converges to the real number L if we can make the terms of the sequence stay as close to L as we wish by going far enough out in the sequence" (Lect 9-29). She translated this statement verbally into the metaphorical domain saying it is only a party if for any size party you pick, after some point everyone shows up at the party. In another formulation, she said "only finitely many guys can be outside the room for you to have a party." During later lectures, the professor began to refer to these terms outside the party as "stragglers."

The class went on to translate the English definition into symbolic form replacing "stay as close to L as we wish" with neighborhood representation and "going as far out as we wish" with index terminology. However, for the next several weeks of class meetings which covered the section on sequences, the professor continued to refer verbally to the party metaphor though everything written on the board was either graphical, common language, or symbolic.

In subsequent lectures and interviews, students persistently imitated Locke's language of convergence that "infinitely many terms" must be in a neighborhood of the limit rather than "all but finitely many terms" or "all of the terms after some point" as the professor phrased it (Lect 9-29). However, the professor continually pushed back on this language using an example such as (0,2,0,2,0,2…). When asked during interview whether having infinitely many terms in an epsilon neighborhood was sufficient for convergence, the students always pointed out such a counterexample and clarified that they meant all the terms after some point even though their infinitely many terms language persisted.

When asked three days after the introduction of the party metaphor what sequence convergence means, Tidus explained, "at some point all of the numbers will be in that neighborhood, so if it converges, some sequence converges to 4, eventually all of those numbers will be in 4's neighborhood" (Ti 10-2). Though he did not reference the party language, he said "at some point"

adopting the time-based metaphor for sequences rather than using index language and "numbers will be in 4's neighborhood" treating the neighborhood as a place rather than a set.

However, as did many of the students both in interviews and in class meetings, Tidus did not mention the arbitrariness of epsilon in his explanation of the definition, but rather treated it as a singular value or the neighborhood as a fixed object. Even when students began to pay more attention to epsilon, they sometimes tacked it onto the end of the definition ("and that's true for any epsilon") rather than stating up front that they had picked an epsilon. However, few students in interview could explain to me the professor's language of "arbitrary, but fixed" and some reported having been confused by the statement during class meetings.

When I asked Tidus later in that same interview to explain to me the role epsilon, K, and n played in the definition, he said, "K represents how far n has to go on the number line to get into the epsilon neighborhood. I don't know if that's true or not" (Ti 10-2). In this way, Tidus expressed a correct correspondence between the informal explanation he had previously provided of sequence convergence and its symbolic translation in terms of indices.

I then asked him to explain to me the previous class meeting's proof that limits of sequences are unique. He did not remember the logic of the argument, but did remember the image of two separate epsilon neighborhoods. He

112

appeared hindered by a desire to use the proof technique of using inequalities to establish equality—the technique that the class used to prove suprema are unique. In his attempt to reconstruct the argument about limit uniqueness, Tidus referred back to the party metaphor language. He said, "I remember when we had something, these are two epsilon neighborhoods and we had to show they were unique or she was saying something like they could be at both parties. Something like that" (Ti 10-2).

After the class took the test over sequence convergence, I asked Tidus to explain the definition of sequence convergence and he said, "you pick an epsilon. For any epsilon that you pick, an infinite amount of terms will be in that epsilon neighborhood and a finite amount of terms will be outside" (Ti 10-17). By this time, Tidus refined his understanding of the arbitrariness of epsilon and properly qualified his use of the infinitely many terms language. However, now when I asked him about the role of epsilon, K, and n, he explained in terms of their formal relationship in the definition saying, "If you let epsilon be greater than zero, there is a K in $\mathbb{N}$ such that for every n greater than K, all of the terms are going to be within the epsilon neighborhood. So x_n will be in, what is it, L minus epsilon, L plus epsilon" (Ti 10-17). I asked him whether he had memorized that expression or whether he was expressing it as he understood it, and he elaborated his understanding in terms of the party metaphor:

Tidus: So I think of K as that term where that's the last straggler and everything after this K are all in that neighborhood or in the party, as she puts it. Everyone before this K, they are out of the party.

I: Ok and then, so the n represents what?

Tidus: n represents the terms of the series. So when n gets greater than K, all of those terms from then on out are going to be in the epsilon neighborhood.

I: Ok and so what does epsilon represent in that case?

Tidus: Your, where you are going to make your, how big your party is going to be or how small your party is going to be. (Ti 10-17)

Thus by the time of the test, Tidus gave the symbolic definition a more prominent position in his concept image of sequence convergence, but still interpreted the relationships within the definition in terms of the class' party metaphor.

Cyan went beyond simply referring to the party metaphor and extended it. When he explained his understanding of sequence convergence during an interview after the test, he began with a graphical representation and explained himself in terms of the formal definition only using the metaphor to explain relationships within the formal definition. He said:

So I'll draw because I like to draw. Ok a sequence converges to L, and this being L… if I take a positive epsilon, or L plus epsilon and L minus epsilon… after some given point… after some given term in the sequence, so the sequence x sub n, if at some K less than or equal to n… after some point xk, everything after xk, xk plus 1, xk plus any number, any natural number, all of those x's are in this bound. So, like, we have this analogy in our class that we have been using that says that this is the party so everything inside of this, inside of this epsilon neighborhood is in the party, and this K, this guy, this term x sub k is the term, I like to call him the popular guy, so after this popular guy gets to the party, everyone else goes to the party. (Cy 10-14)

Figure 5.10 presents Cyan's accompanying drawing. He mistakenly wrote x_{-2K}

and x_{-K} instead of x_{K-2} and x_{K-1} to represent terms outside the epsilon interval.
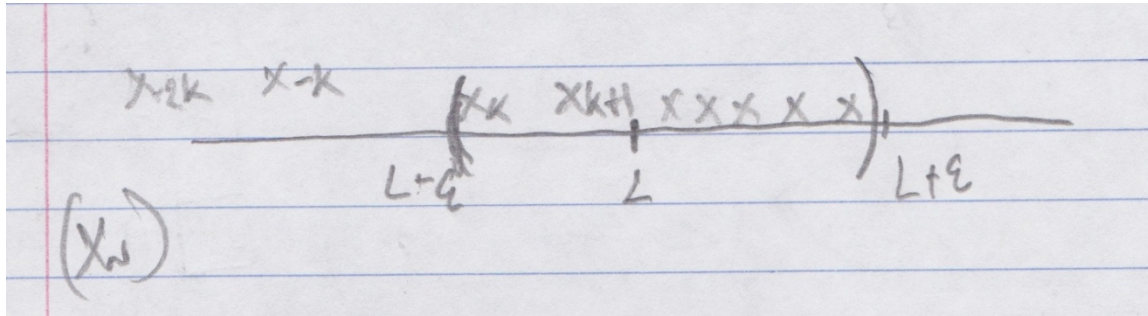
Figure 5.10: Cyan's depiction of the definition of sequence convergence (Cy 10-14).

Cyan not only used the metaphor to make sense of the formal definition, but he

extended the metaphor to represent the kth term of the sequence. This revealed

that he used the metaphor as a tool to make sense of the definition.

In contrast, Locke generally avoided referencing the metaphors unless

asked to, preferring more formal language. Another student in the study who

fared very poorly in the class wrote a definition on the exam in terms of the

metaphor revealing that she never moved beyond the metaphorical language.

5.2.3 Proposing, Arguing, and Reflecting

A third aspect of how mathematical communication developed as a norm

of classroom interaction was the presence of student proposal and

argumentation. The ways in which the professor invited explanations of student

reasoning have already been outlined, but several key instances provide insight

into how the class learned to formulate ideas or observations into hypotheses

and then critically evaluate each other's hypotheses by sharing counterexamples and arguments.

During the second semester introduction to limits of functions, the professor presented the students with the following prompt on the board:

What does this mean: $\lim_{x \to p} f(x) = L$

Could this have meaning in the following examples?

f:N->R $f(n) = \frac{1}{2} * n$ p=3 (Lect 10-27)

She drew a two dimensional graph of this discrete function and asked them to "make sense of this for me." Cyan and Locke quickly pointed out that $f(3) = \frac{3}{2}$ to which she agreed. Then Cyan began to explain that the image of f(x) approached 3/2 as x approached 3. She asked him what it meant for x to approach 3.

Using a table in the front of the classroom as a prop, she began to walk slowly towards the table maintaining at least one yard distance from it. She asked the class whether she could be said to be "approaching" the table, and they affirmed that she could. She then presented the class with a graph of a function which took on the value of x^2 on [-1,1], f(5)=0, and was undefined elsewhere. She told the class that by their logic x values approaching 1 from the left side also approached 5. So she asked them what the limit of the function is as x approaches 5. Cyan, who earlier used the language of "approaches," now suggested that they must get within an epsilon neighborhood of 5. Locke

116

suggests that they must approach from both sides and yet another student introduced the language of "arbitrarily close."

The professor agreed, combining this with Cyan's earlier statement to say they must find the "image" of the function arbitrarily close to 3. She asked the class what was the image of 2.75. When she pointed out it is not in the domain, Locke suggested that they pretend it is in the domain. The professor asked the class what more information the limit provided beyond f(3)=3/2. Cyan reiterated that the images are getting closer to 3/2, and the professor agreed that f(2) and f(4) were closer to 3/2 than were f(1) and f(5). Locke, however, observed that this would not be true of the sine function. The professor affirmed this idea by drawing the graph of a discrete function whose values at 1, 3, and 5 were relatively equal and the values at 2 and 4 were much different as in Figure 5.10.

Locke again revisited the idea that they assume the points in between be in the domain. Cyan affirmed this idea, and the professor, acting very surprised, asked them whether they really want to "pretend" as they said. She asked them whether they want to define analysis based on "pretending." She commented that she had never had a class go that far before, but that this was fun. She then described the process of picking an x, pretending to plug it in, and picking a value kind of close to what they wanted it to be using statements like, "so it's maybe kind of 5/4 or something." Cyan protested that just because a point was not in the domain doesn't mean that the function doesn't apply to it. However, other

students expressed their confusion because the point they were plugging in (5/2) was not in the domain. The professor wrote on the board "lim $_{x \to 3}$ f(x)=L is a 'pretend game'" (Lect 10-27).

Given $f: D \to \mathbb{R}$, $p \in \mathbb{R}$
What does this mean: $\boxed{\lim_{x \to p} f(x) = L}$
Could this have meaning in the following examples?
(1) $f: \mathbb{N} \to \mathbb{R}$ $f(n) = \frac{1}{2}n$ $p = 3$

$f(5) = 2$

$\lim_{x \to 5} f(x)$

$\lim_{x \to 3} f(x) = 3/2$
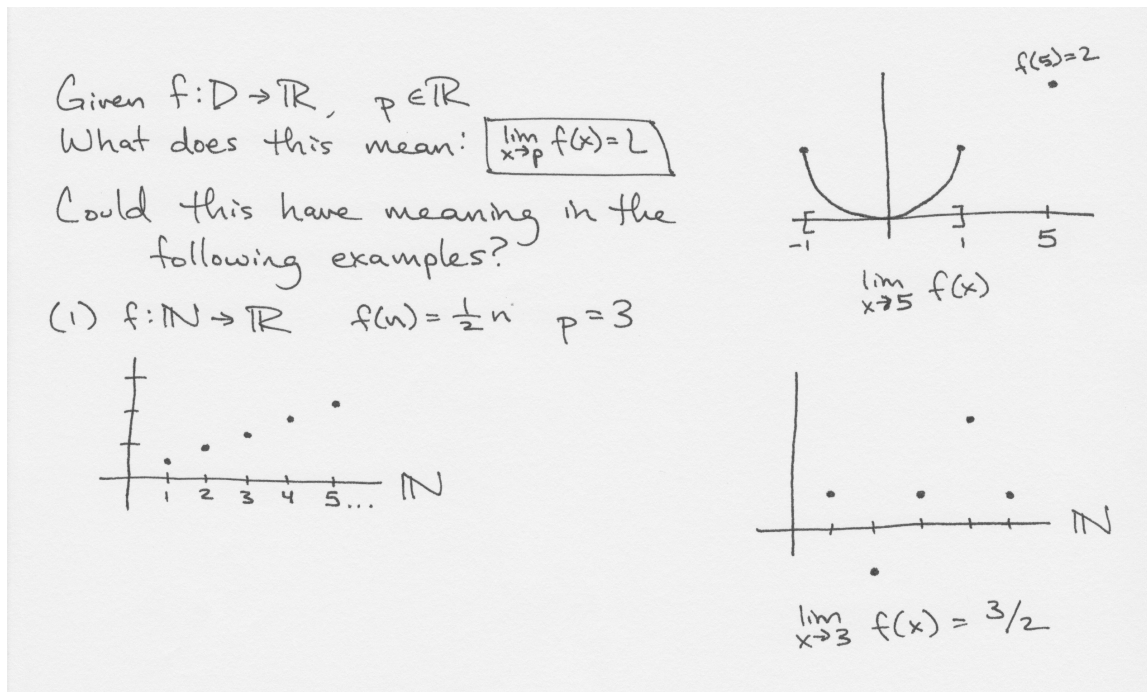
Figure 5.11: Examples that motivated the need for a cluster point condition (Lect 10-27).

At this point in the conversation, the professor introduced a new function which was equal to the line x+3 everywhere except for at x=3 where the value was zero. She asked the students anew what was the limit of the function at 3. They constructed tables of values approaching 3 from either side and the class affirmed that the value of the limit was 6.

She then asked them again how they decided that the limit of the previous function was 3/2. Cyan protested that they knew the behavior of the function.

Locke agreed that they were thinking of the function as if it were defined on all real numbers. The professor then drew a new graph which is equal to $^1/_2$ * n on the natural numbers, but she fills in the space between 2, 3, and 4 with asymptotic behavior such that lim $_{x \to 3}$ f(x)=∞. She asked the students then to pretend that the function is asymptotic about 3, but Cyan insists that they have more information than that. The other students at this point began to disagree with Cyan and his assertion. Locke points out that they are now forced to decide "what kind of pretend game are you playing?" The professor asked them, "Do you see how un-mathematical this is?" (Lect 10-27).

After further discussion of other issues related to function limits, the professor went to the sideboard and wrote the following summary of their previous discussion:

*In order to make sense of lim $_{x \to p}$ f(x), there must be points in the domain of f arbitrarily close to p.

Otherwise lim $_{x \to p}$ f(x) turns into a "pretend game."

*In order for the question of whether f has a limit at p or not to be relevant, p must be a cluster point of the domain of f. (Lect 10-27)

The class proceeded to define cluster point and observe how its integration into the definition of function limits solved the problem this discrete example caused.

I asked Locke about the "pretend game" conversation in a later interview and he said:

I liked it. I think it got in everybody's head… She never straight up said there is no limit for this function. She just kept saying it was pretend, so

that might have been a little shady for some people, but I liked it. It definitely, I don't know I would never make the mistake again of thinking that there was a limit for a function like that. Or where the domain was the natural numbers. (Lo 10-28)

Locke pointed out the teacher's refusal to give an authoritative answer regarding the definition of a limit in this situation. However, he indicated that he understood that making assumptions about the function where it was not defined was a mistake.

In the case of the pretend game discussion, the professor did not provide the students with an authoritative response to their false assumptions, nor did she ignore their suggestions and move on. She allowed the class to pursue the discussion until some level of consensus was reached, after which both Cyan and Locke articulated the errors of their arguments during interviews. This account highlights the manner in which the classroom conversation centered upon student thinking and proposals. The professor reflected the ideas the students presented back to them for the group to evaluate and make sense of. The next vignette shows an instance of how the mathematical practices of making hypotheses, testing hypotheses, and revising established themselves in the class' discourse.

During the third semester, the professor wanted the students to explore the algebraic theorem of sequence limits by applying the theorem (Shipman, in preparation). Throughout the whole subsequent discussion, she kept the algebraic theorem projected on the board for the class, which appeared:

Theorem: Suppose lim X = A, lim Y = B, and $c \in R$.
Then X + Y, X - Y, XY, and cX converge, and
lim X + Y= A + B ,   lim XY = AB
lim X – Y = A – B,   lim cX = cA,
and, if $y_n \neq 0$ for all n and if $B \neq 0$,
then X/Y converges and lim X/Y = A/B. (Lect 2-26)

She presented a set of true-false questions for the class to assess, discuss, and

prove. The questions were all typed on a sheet that she projected onto the board,

but she only revealed one question at a time. The first question she presented

said: "True or false? (a) If S+T and T converge, then S converges" (Lect 2-26).

The professor gave the class a few minutes to think about their response

and then took a vote that was about even between true and false responses. She

then encouraged the students to share their reasoning with those sitting around

them and the class discussed for a few minutes before a second vote. The

second vote revealed more on the side of true, so she asked someone to offer an

explanation.

Banon responded:

Okay, so if we take our theorem and suppose that these two converge to
some point B and A. [The professor asked him which two he meant, but
he did not respond.] Let's just look at the other part, we have lim X+Y =
A+B. The equality sign means that we can run the implication in both
directions. [The professor then restated his claim "so, this equals that and
that equals this" before he continued.] If that's the case, then that means
S+T converging means that S and T have to each individually converge.
And intuitively that makes sense. (Lect 2-26)

The professor thanked him for his explanation and wrote Banon's claim on the

board: "S+T converges $\Rightarrow$ S and T both converge." She pointed out that this was

121

in fact a different claim, and she revealed more of the sheet with the questions on it to show that this was her next true false statement.

She invited the class to offer another argument for or against the first statement that did not depend upon Banon's assertion. Banon now offered a different argument:

> For a sequence to diverge means it does not converge to a single value. So, if you are going to add a real number which is a limit to a value that is not a real number, that is it could be infinite or it could be multiple values, for example the series (sic) 1 and 0 [one common class example had been (0,1,0,1,0,1…)] goes to two different values. That's two separate epsilon neighborhoods; it doesn't make sense to add to a non-real number. (Lect 2-26)

The professor again rephrased his argument to say that adding something convergent with something divergent will yield something divergent. She asked the class how these assertions related to statement (a) and reminded the class of their algebraic theorem trying to encourage them to apply it. Vaan pointed out that Banon's statement would prove part (a) by contradiction. The professor affirmed this, but pointed out this is yet another unproven true-false statement, namely that S diverges and T converges implies that S+T diverges.

Tifa then questioned Banon's original argument pointing out that the algebraic theorem is only stated if-then, and so she expressed her difficulty in applying that theorem to the given situation when the conclusions was given as a hypothesis, namely that S+T converges. Vaan built upon Tifa's suggestion of applying the theorem by proposing they use the subtraction part of the theorem on (S+T) – T. Tifa, expressing some relief to her prior difficulty, affirms this idea

saying they could let S+T = X and T = Y and apply the theorem. The professor restated the full argument of applying the theorem to these two sequences.

Unsatisfied however that everyone understood, she restated the "problem" with applying the theorem, which is that the hypotheses of the theorem never mention any sums of sequences. She asked for another student to restate the argument to which someone responded, "Let's call S+T X." She accepted this restatement, but one student in the back spoke up expressing confusion. The professor restated the theorem to say if "some sequence" (X) converges to A, and "another sequence" (Y) converges to B, then the first sequence minus the second converges to A-B. The student who had expressed confusion assented to this explanation.

When the class turned their attention to statement (b), someone quickly provided the counterexample S = (n) and T = (-n). The class went on to use the algebraic theorem to prove "(c) If $-\frac{1}{2}$ S converges, then S converges" before considering "(d) If ST converges, then S and T converge." The professor asked for votes and many students vocalize "false" responses, but a few more timidly vote true. Banon proposed to let S = (n) and T = (0) such that ST = (0) and converges. Tifa then proposed that they alter the previous statement to say "If ST converges, then S or T converges." The professor affirms this suggestion changing the statement written on the board. She said, "This is good. This is how you make theorems. If that doesn't work, weaken it. Maybe that's true."

Responses are slower this time and Tifa tried to translate their proof from part (c) to the present situation. After a period of the class thinking, another student suggests letting S = (0,1,0,1…) and T = (1,0,1,0…) such that ST = (0). Barrett rather jokingly suggested that the statement be revised to, "If ST converges, then neither S nor T converge," which made the class laugh. The professor concluded the conversation suggesting that the students go home and see if they could make a true theorem from what they had been working on (Lect 2-26).

This classroom exchange presented examples of the students proposing arguments and hypotheses as well as

- restating (Vaan pointing out Banon's argument is proof by contradiction),
- extending (Vaan applying Tifa's suggestion to use the algebraic theorem), and
- testing (Tifa correcting Banon's if and only if assumption)

one another's propositions. The professor primarily took the role of reflecting student ideas back to the class for assessment. She restated what students expressed and organized their suggestions for class discussion.

5.3. Constructing Mathematics

The third socio-mathematical norm established in the classroom of study was constructing the mathematics. Throughout the course, the class considered the design and construction of all three major aspects of proof-based

mathematics: definitions, theorems, and proofs. The class constructed definitions from examples and their previous notions from calculus and considered the consequences of choices they made during this process. Theorems arose as the significant statements chosen out of sets of questions about relationships the class observed in examples. The class scaffolded proofs exploring both the argument itself and the means by which such an argument could be constructed. In the course of constructing the analysis material, the class considered aspects of how proof-based mathematics is generally structured yielding meta-mathematical lessons as well.

5.3.1. Constructing Definitions

As was previously mentioned, the professor began discussions of new definitions with an examination of examples and conversation about the intuitive notions that students had from calculus. The "pretend game" vignette displays these two tools the professor employed in that she chose specific examples which highlighted key ideas she wanted students to consider and she expressed intuitive formulations of the definition in the initial "English" definitions which they refined into rigorous symbolic forms. In the case of function limits, the class spent one whole meeting discussing issues related to the definition, only defining it during the subsequent class meeting.

The professor continually invited the class to consider whether they thought the definitions were correct or well-formulated emphasizing that past

mathematicians made these definitions and that does not mean that they are perfect or right. In the case of continuity at a point, she presented the class with four different definitions:

- the limit at the point and the function value at the point both exist and are equal,

- the epsilon delta definition,

- the calculus definition which requires that the limit exist on both sides and be equal to the function value at the point, and later

- the sequential definition.

The class used both of the first two definitions and discussed why the third definition lacked sufficient generality. She introduced language into the discussion that distinguished between the "idea" of a definition and the formal definition mathematicians used to capture that idea.

5.3.2. Students Constructing Definitions

During the first semester, a major part of the homework on continuity centered upon examining the function that is defined as 0 for every irrational input and is x for every rational input because it is only continuous at zero. The homework asked students in one section to prove that the function is not continuous at any non-zero point and in another section asked them to prove it is continuous at zero. A question on the exam over this section asked the students to provide an example of a function continuous at only one point or prove that no such example exists.

Edgar did not remember having worked on the homework problems either because he divided the homework with his classmates or he did not combine the two sections of proof to identify that the function from the homework is continuous at only one point. On the test, he convinced himself that no such example exists and gave an informal but thorough presentation of his reasoning. He argued that because continuity depends upon the limit at the point and because limits require the point in question to be a cluster point, then continuity at any point depended on the points around it and thus the point must have other points "to be continuous with." By this reasoning, no function could be continuous at only one point.

During a subsequent interview, he explained:

When you were saying, you take a point that's, if a is your supremum of the set, and you have a point in the set that's [less than the supremum], then there's another point in between meaning there is basically an infinite number of points in between… But the way we say it is there's one point in between. So that's sort of the way I was thinking of this continuous thing, you have to, I mean it would seem like you would have to have at least another point to be continuous with. It would seem like it. Which would mean it would have to be continuous over an infinite number in a very small interval. But that doesn't necessarily mean anything apparently, cause that's not true. (Ed 4-23)

Edgar referenced a previous proof regarding suprema to express the idea of cluster points, namely that there are infinitely many points in the domain in any epsilon neighborhood of a point of continuity. This notion of points very close to the point of continuity led him to think of continuity as a property that applied to the set of points locally. He knew however that he had been marked wrong and

127

had seen the counterexample, so he cited his mistake with some frustration. I

continued questioning him to learn the source of this frustration.

I: Yeah, I guess one of the questions is our definition of continuous. How do we define continuous?

Edgar: Well, the limit equaling the value.

I: And so is that a general property or a specific property of a point?

Edgar: Pretty specific, I guess. Seems like it cause you can evaluate it. And you can evaluate it. You can evaluate a point. So you would have two specific properties.

I: Yeah, so is our definition of continuous, is it on a line or is our definition of continuity at a point?

Edgar: Well it would be at a point. No, I get it. But I mean by the way we approach continuity we were talking about how, what should continuity mean. And it seemed to me continuity would have to be a relative thing. I mean continuous with what, you know? I mean continuous, am I continuous as a person? It's like, well, I don't know. What's it, you know continuous in what respect, what means? (Ed 4-23)

Edgar revealed that the source of his frustration was the definition

itself. The existence of a function that is continuous at only one point bothered

him because it did not match with his intuition of continuity. He cited the

professor's emphasis upon "what should continuity mean" and showed that

he had thought very seriously about this question. He went on to express his

discomfort with this counterintuitive example in even stronger terms:

The way I understood it was, ok, so it meets the definition. It still doesn't mean anything to me… You know, if you find this function where you have this point that's continuous at one place, that to me seems like a perversion of the definition. You know?... It's like a lawyer arguing a case where you just know that morally it's wrong, but by law it is ok. You see what I mean. It is the same kind of thing to me. (Ed 4-23)

He pointed out that he understood how the function was continuous at only one point according to the standard mathematical definition. However, he strongly felt like this example posed a problem with the definition, paralleling it to an ethical breach of law.

This strong issue Edgar took with the definition of continuity stood in stark contrast with his acceptance of the definition of cardinality in terms of bijections. Past research has shown that students strongly and persistently reject formal theories of transfinite numbers (McDonald & Brown, 2008). Some students in the present study similarly rejected the formal theories even when they understood them. However, Edgar showed far less resistance.

I: What have been some of the more interesting aspects of the course so far?

Edgar: Probably the several day spiel we went through on the quantity, I guess, of infinite numbers, and how when you have $2\mathbb{N}$ there's the same amount as $\mathbb{N}$. And uh, and it may be something you know, but it is still something you have to struggle with. I mean you can know it, but you know, there being a difference between knowing it and believing it. And it really is something to struggle with, you know cause you just automatically assume that if you take an infinite amount of numbers and you take a third of them away, you would have less, but you don't. I think that has been one of the most interesting because it has been more impacting as far as you know things that are really…

I: So tell me about that interface between knowing and believing.

Edgar: Well, for instance we, in [the intro to proofs course]… we were studying one to one and we used that arctan function and she said so, she demonstrated and proved that it is one to one. And then she kind of looked at us all and was like, do you realize what this means? This means that for every number, there's as many numbers on this whole number line as there are between negative pi halves and pi halves. And I was like, ok, it made sense, I could see it, and a couple of people had

129

problems with it, and then we went into it further. I guess it's just grasping it. (Ed 4-2)

Though he did mention that the theory of transfinite numbers is hard to comprehend or to "believe," Edgar described his response saying, "ok, it made sense, I could see it." He even proposed that students might not believe it because they are not yet "grasping it."

In these quotes Edgar displayed an ability to accept counterintuitive mathematical results when they made sense to him, but this contrasts all the more with his emotional dissatisfaction with the definition of continuity because of its inclusion of examples that do not match intuition. Ultimately, Edgar displayed the extent to which his acceptance of mathematical definitions depended more upon his own understanding and conception thereof than on external sources of authority. He understood that he had a role in the process of constructing, assessing, and accepting definitions. He even expressed hope that the professor had benefitted from the argument he made on his exam:

> She can also kind of see how we think by the tests. And so I think what I hope she got out of this, I hope I taught her something with this. We discussed what continuity meant and a lot of time the way she approaches things is "Well, as mathematicians, what do we think things should mean?" (Ed 4-23)

He went on to propose a different notion of what it means for a point to be "continuous with another point." Though he had not defined it, he felt that the mathematical definition was somehow insufficient and was searching for a different idea.

During the semester of study, Edgar and several classmates earned

honors college credit for this analysis course by completing extra research. In

their explorations, the students unsuccessfully sought an encyclopedia of

mathematics in which to find mathematical definitions. Reflecting on the fact that

the encyclopedia didn't exist, Edgar expressed a very sophisticated

understanding of the role of definitions in proof-based mathematics, saying:

> Apparently we don't have anything like [an encyclopedia for mathematics].
> We don't have definitions for a limit, definitions for continuity, definitions for
> this and that. There's things (sic) that people will accept, but it's a point of
> reference deal. If I am writing a book, and I am going to be using these
> things, I need to define them. And I guess where I'm at, if you are in a
> particular, I guess that's the best word, frame of reference, you know, you
> have to define what you are doing there and use a definition that other
> people can understand, even if they don't necessarily agree, that that
> definition is going to work. (Ed 3-26)

Edgar points out that mathematical definitions are not absolute, but rather exist

within a theory ("frame of reference") and are created to meet certain needs

("that definition is going to work"). He borrows the professor's language saying "if

I am writing a book," which is how she often framed her appeals to the students

to figure out what they think the definition should say, but Edgar displays here

that he is making sense of the mathematical community's practice of defining by

thinking from the standpoint of a creator of mathematics.

5.3.3. Students Questioning the Professor's Definition

During the first semester of study, of all the possible function limits the

professor only provided the class with the definition of a function limit at a point

and the definition of one-sided limits. After that, the students worked in their

groups during class to define when a function tends to infinity as x approaches

some point. The class presented their definitions and refined them, but the

professor left it to the class to write a final definition at home along with the rest

of the possible limit definitions.

During a later interview, I asked Ronso to define $\lim_{x \to p} f(x) = \infty$. He

worked to convince himself whether the notion of cluster point was truly

necessary for the definition of limits or whether the definition already excluded

problematic cases.

> I was thinking if it wasn't a cluster point, I am not sure there is any way you
> could possibly satisfy the definition, but I'm not sure. Cause I was thinking,
> let's say that it is not a cluster point and you have to find something in that
> little neighborhood around that little point that we are looking at. And it has
> to be, you always have to be able to make that greater than a certain
> number. So I guess if, let's see if I can draw an example where that
> wouldn't work. Let's say we have something with an asymptote over here,
> undefined over here. We are going to try and find the limit at p which is
> there, so we gotta find a delta neighborhood such that this thing is always,
> and I guess we always could here, you just make your delta neighborhood
> get closer and closer to, we call that c, you'd have to get it closer and
> closer to this distance, p plus c I guess since they're negative. So I guess
> p does have to be a cluster point. (Ro 4-8)

Figure 5.12 presents the example Ronso found to justify that the epsilon delta

definition of limits would have problems without the condition requiring the

point to be a cluster point. He recognized that nothing in the definition

required delta to be small, so the epsilon delta definition without a cluster

point condition would declare that $\lim_{x \to p} f(x) = \infty$ for this function.
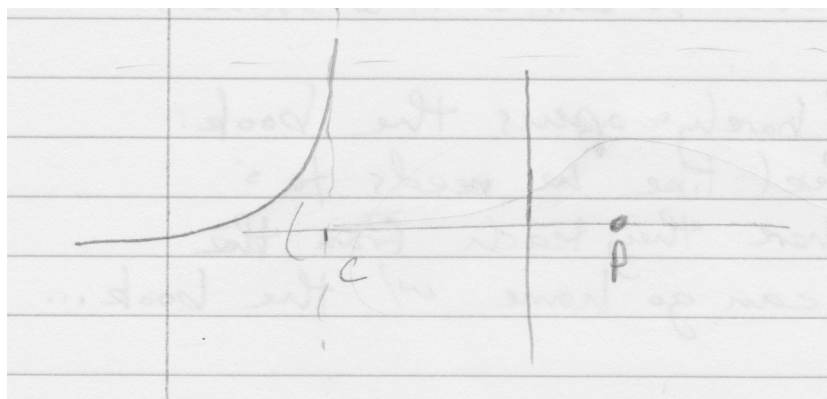
Figure 5.12: Ronso's example establishing the need for the cluster point condition (Ro 4-8).

Ronso went on clarifying his own understanding of the need for the cluster point condition in the definition.

I: So, do you then think that this fits the definition?

Ronso: Well if you ignore that p has to be a cluster point, I was trying to decide whether p actually had to be a cluster point or not and for a while I was thinking it didn't because I was thinking you weren't going to be able to satisfy this if p wasn't a cluster point, but you can actually satisfy it if p isn't a cluster point so that means you have to have one more condition that p is a cluster point…

I: Yeah, so why did we talk so much about cluster point here a few weeks ago?

Ronso: It helps clarify what we mean by approaching or a limit what it means to approach something. I mean, something has to be defined so we can find what it means to approach p, cause it wouldn't make sense to approach p if you weren't talking about a cluster point. So I guess the same would be true here. Yeah, that makes sense now. (Ro 4-8)

I then asked him to produce a definition for lim $_{x \to \infty}$ f(x) = L which had not been discussed in class.

x goes to infinity, so let's see, say, so we just state this stuff up front, [speaking and writing] from D to R, wouldn't be, don't need a cluster point anywhere, limit as x goes to infinity. We're gonna have to have it, and D is

133

also going to have to be defined past a certain point. We'll worry about that later. (Ro 4-8)

Initially, Ronso indicates that cluster points are not necessary in this situation, but he did observe the necessity for the domain to be unbounded above stating that "D is going to have to be defined past a certain point." He moved on and correctly produced the if-then portion of the proof. He then returned to how to properly require that the domain be appropriate for limits as x approaches infinity.

Ronso: I guess we would have to go back here and say that D is defined, or f is defined. Well, it wouldn't have to be for, I wouldn't know how to word it actually. So f has to be defined up to a certain point, and up to infinity, but it wouldn't have to be all the points. It could just be like the rationals or something. So densely defined, what would you say there? How would you word that?

I: I will let you think about that. It is a good question. That is good, so why do you think, or what made you think about it needs to be defined out there?

Ronso: Well if, actually, you couldn't, if it just stopped after two, you couldn't talk about what happens as you go to infinity because it is not really doing anything. But actually, even if it wasn't densely defined, even if it was defined for all the integers, seems like you could kind of talk about what it was doing as it tended to infinity like you can talk about a sequence. (Ro 4-8)

Ronso quickly observed both that the domain needed to be unbounded and that it does not need to be on the whole line after some point. However, he had trouble knowing how to express the condition properly and initially wanted to impose density upon the domain. He noticed, though, that "you could kind of talk about what it was doing as it tended to infinity" even if the set was not dense. He observed that a function defined only on the integers would be much like a sequence.

Ronso then tried to find a way to define a condition that was sufficiently general and that did not include bad examples. He began wanting to simply say that the domain contained infinitely many points beyond some given point, but found a counterexample for this quickly.

Ronso: So, but you want it to be defined on an infinite number of points past a certain point. So let's say your function was only defined past three, well that would be okay too, or past any finite number. That would be okay, as long as it has an infinite number of points past there. Well no, that wouldn't work because you could just make it tend towards; so I mean you could do something like this. [draws a graph with points approaching some finite supremum]… Then you couldn't talk about that one going to infinity either. So I guess if for every M you choose, there is always a number in the domain larger than it. Then you'd be ok I think. So if I can choose any number, real number, and I can find an element in the domain larger than that number, then I think you can talk about that.

I: Yeah, how did you think to do that?

Ronso: When you talk about like the bounded sets you get out of it being bounded by doing that. Yeah, I think that would work.

I: Are you trying to think of a counterexample?

Ronso: Yeah, I'm still thinking whether it matters actually whether it matters that it's not, that it can be defined with big gaps in it like that, but I think that's ok. I mean, just like sequences, because even though this is a function, I think that's still okay. I think that would be a reasonable definition. [writing] f is defined past every M, yeah, I am going to say that's reasonable. (Ro 4-8)

In searching for a way to describe the condition he wanted, Ronso was able to successfully translate a condition from their previous work with unbounded sets. It is unclear whether he reformulated the condition he wanted in terms of D being unbounded first, or whether he just recognized the need for the arbitrary point of reference, M. Even once he produced this statement, he continued to question

his definition to convince himself of its validity and sufficiency. Ultimately, he

revisited his earlier connection between functions and sequences to affirm his

claim. His final definition read: "f: D→R, f is defined past every M∈R.

lim $_{x→∞}$ f(x) = L if given a ε > 0, ∃ δ > 0 s.t. for all x ≥ δ, f(x) ∈ (L-ε, L+ε)" (Ro 4-8).

During the second semester, Locke was able to create a sequential

condition for this same issue of limits as x approaches infinity. I asked him to

define lim $_{x→∞}$ f(x) = -∞ which was a question on the test he had recently finished,

and while he produced his definition, he said:

> I guess the hardest part for me on this one, yeah I guess you need to say
> let f:D→R, but then you also have to say D needs to approach infinity.
> Where we normally say p is a cluster point, or something like that.
> Obviously we can't say infinity is a cluster point because it is not a cluster
> point because it's not a point. So I think I just said, I think I might have
> used something sequential to describe that. I think I said something like
> lim of x_n approaches infinity where, yeah x_n is an element of D for all n.
> And I think that, I hope that covers what I am trying to say. (Lo 11-26)

Unlike Ronso, Locke cited a direct parallel between the condition he wanted and

the notion of cluster point. He subsequently produced a correct definition of this

limit.

The class made very little reference to the textbook, but in deference to

the textbook the professor defined limits of functions as x goes to infinity with the

restriction that the domain of the function must contain some interval of the form

(a,∞). During both of the first two semesters of study, students in class and in

interviews showed dissatisfaction with the excessive restriction this placed upon

the definition observing that this excluded functions defined only on the set of

rational numbers. Cyan especially expressed his discomfort during interviews.

Cyan: So now you are just trying to get within a delta neighborhood of p. But if p, if our p value is actually negative infinity, then you want to constrain your function to be in some interval from some value to negative infinity. Now I know that that is not absolutely true, because when we do that we eliminate certain functions, like certain rational functions. If our function is just a function of just rational numbers, then I can never find an interval from p to infinity because. So I haven't actually resolved that yet. We were still discussing that the other day. And the same with inf. So if I have infinity, I want to put myself in some kind of neighborhood that takes me off to infinity, but I am not really certain how to define it without being exclusive of functions that aren't defined on all of R....

I: Did you just notice in class that the way she defined it in class is restrictive?...

Cyan: I think it was [another student] that asked her something about it. I was thinking it already… I went to her immediately after class with, "Well how do we address this issue with functions that aren't defined everywhere?" Like she never even mentioned rational functions...

I: So far what is the one we have used? Is it just has to be defined everywhere past a certain point?

Cyan: Yeah past a certain point, but what I want to say is… no matter how far out you go there is a point out there. That kind of follows I am not quite sure how to explain it. (Cy 11-18)

Though Cyan was unable to formulate a more general condition like Ronso and

Locke, the lack of generality bothered him enough to take his question to the

professor's office after class. He even cited discussions with his classmates

regarding the issue, but without any conclusion. Cyan successfully expressed his

idea in common language saying, "no matter how far you go there is a point out

there." However on the test that followed, he used the professor's condition

requiring the function to be defined on an interval of the form (a, ∞).

5.3.4. Student Participation in and Perception of Classroom Defining

Students reported that the class' process of constructing definitions benefitted both their recall of the definitions and their understanding of the intent of the definitions. In the course of describing their classroom practice of constructing definitions, several students also revealed strong understandings of the nature of mathematical definitions and the role and process of mathematical defining.

Celes described how the process of defining led students to think more actively about the definition or "struggle" and this promoted both their recall and understanding of the definitions. She said:

> I think because if you have to think about and try to figure out something for yourself, I think you remember it a whole lot more. I know you remember it a whole lot more than just telling you what it is. And I think that when she just gives you things to think about, I don't know, it helps me learn a whole lot more during her lectures than I have learned in other lectures cause most of the time I am learning, for me is when I sit down and have to struggle with it. And I think in her class she tries to make you struggle with it when she presents it, you know what I am saying?… I like when she asks true-false questions. I really like that, because I really feel like it is making me think, instead of just writing down everything that she says, and not really taking it all in, it makes me think about it, and so then I understand. Like, I would never remember that if you had just given me the definition, you know. Cause I don't even know what the definition is right now. But I know the concept behind it. But if she just said this is the definition, I would never remember that. (Ce 4-8)

Celes mirrored the language that the professor introduced of learning both the "concept behind it" as well as the "definition" meaning the formal expression thereof. In a later interview she reiterated the benefits of discovery over being told something outright:

138

I really like how she, how she really like, makes you think and makes you *try* to figure it out first. And I think for anybody, anybody learns better that way, instead of just being told something. But when you find it out, or when you look it up, or when you discover it, you know, it always sticks with you. And so I think her, she tried to do that, she tried to figure out, she tried to get it out of us I think instead of just saying this is how it is. And I like that. (Ce 5-7)

Celes' confession that she could not remember the definition offhand, but knew the idea behind it raises the question of how efficiently she could produce the formal definition from the concept. During a later interview, I asked her to write a function limit definition that she had previously produced on the test. She reported not having studied well enough for the test because she had been out of town. On the exam, she wrote a full definition, but showed hesitance fearing that she had omitted some portion thereof leaving a gap in the middle of the definition. When I invited her to look up the definition in her notes, she was surprised to find that she had been correct all along. She lamented the time she took during the test trying to figure out what was missing. A similar instance ensued when I asked her to produce a definition of uniform continuity. She wrote a full definition, but sensed that something was missing. Once she looked in her notes, she said:

Cyan: Oh, formal definition [sarcastically]. Look at that. I did memorize it. I am so proud of myself [still sarcastically].

I: Do you think you memorized it or do you think you figured it out from what you?

Cyan: Probably, well I remember that the distance between two things had to be close. And then I probably just figured the rest out thinking about it. (Ce 5-7)

In several instances, Celes was quite capable of producing the formal definition from her conceptual understanding, but she seemed to be assessing her definitions by some other criterion, which made her fear their inaccuracy. She stated in the latter instance that she felt like there needed to be an "H" in the definition, indicating attendance to surface details of the formal definition.

Celes reported that this course taught her about the variable or arbitrary nature of mathematical definitions, which she did not previously know.

> I thought a lot of things about math were really concrete, and I know, and I am not talking about the concepts cause I know the concepts can be abstract, but basic things about math I thought were very concrete. And then finding out that definitions are changed depending on the level of math you had, or things like that I think is what I was really shocked about… And coming up with definitions and the idea behind that, and all those kind of things were I think, was the ones that really changed about what I thought about math before. (Ce 5-7)

Her reference to definitions changing depending on the level of mathematics most likely refers to the professor's observation that the calculus definition of continuity required the function value equal both of the sided limit values, which was too restrictive for the purposes of real analysis. Celes changed her perspective about the process of mathematics from it being "concrete" or unchanging to it being context-dependent and created.

Ronso indicated that the process of constructing definitions gave him an understanding of why specific aspects of the definitions were necessary and what purpose they served.

> She wants the definition to, all the troubles and stuff that arises in trying to make a definition useful, she wants that to be apparent… I guess she just

wants it to be obvious why all the little parts from the definition come from. So instead of just saying, well, here's the definition, go use it. She says here's why we are, let's look at a problem and try to make a definition that's useful. Then after we see how mathematicians define a definition, it will be obvious why they define it this way: in order to satisfy or to solve particular problems. (Ro 3-28)

Ronso contrasted their classroom practice of constructing definitions with simply presenting definitions in final form much like Celes. He observed that this highlighted the fact that definitions fit within a body of theory and are formulated with a purpose in mind. He references "mathematicians" and "why they define it that way" indicating his understanding of defining as a human activity that he can comprehend in light of historical questions and intent. I invited him to speak more on that topic.

I: Why does she keep talking about "before we had definitions" and coming up with definitions?

Ronso: To me, it seems like she is trying to emphasize the point that these definitions can be arbitrary and they are created for a reason and if you just start making definitions, sure you are allowed to do that, but it is a matter of whether it is useful or not. I guess that by going through all the steps, we show why the definition is useful rather than just arbitrarily saying here is the definition, go use it, mathematicians created these definitions not just to have fun or whatever, but to make a useful thing for solving problems and stuff. (Ro 3-28)

In contrast with past research findings (Edwards & Ward; 2008), Ronso and others in this study expressed an understanding of mathematical definitions as stipulated rather than purely extracted calling them "arbitrary" and saying one can "just start making definitions, sure you can do that." However, he also acknowledged their purposeful nature in that definitions are measured according

to "whether it is useful or not." He reported that the classroom practice of defining revealed the utility of the definition.

Cyan also expressed an understanding of mathematical defining as a purposeful and human process in which mathematicians make choices that then affect the theory in which they work. Like Edgar, he identified himself as part of the community of defining saying:

Cyan: We can define it how we want to, but you know, by saying, but that is a lot of power to say that I am defining this how I want to. But if you do and you make an assumption, or you make a choice, you know it could have different results somewhere down the road... Maybe if we don't include it maybe we never come across any problems, but one day you could and you say that was the choice I made in my definition.

I: How do you think it is then that we choose to define things. What is the idea of how we choose to define things, or how we formulate definitions?

Cyan: Well we do our best to describe accurately what's, what the concept is, but I mean you have to talk about abstract things that you can't take for granted, well I guess we do take for granted certain things when we write definitions.

I: Like what, what do you mean?

Cyan: But that is the point, you are trying to minimize what people can and can't do with your definition. When you write a definition you want to make it where you can't just say well this because of the definition or this because of the definition. So you want to make your definition so that if you use it correctly, only certain things can happen. Like if I write my definition of limit a certain way, then if the limit really doesn't exist, when I use the definition it shows me that the limit doesn't exist. (Cy 11-11)

Cyan echoed Ronso's assessment that mathematical definitions are arbitrary insomuch as the definer has choices to make which influence the theory built upon that definition: "it could have different results somewhere down the road."

142

He added that mathematical definitions are also assessed according to the extent to which they capture the concept from which they are derived: "if the limit really doesn't exist, when I use the definition it shows me that the limit doesn't exist." In this way, Cyan argues for the quasi-extracted nature of mathematical definitions based upon the idea that definitions entail both a concept and the formal definition intended to capture that concept.

Edgar's strong understanding of mathematical defining and internal sense of authority in assessing definitions appeared earlier. He indicated that the process of defining in the classroom helped him remember it because he could reproduce the process to remember the definition rather than memorizing the formal definition:

> [We construct definitions in class] so we'll remember it more often. I don't know, I think it is pretty good. She even said, do you wish I would just stand up here and give you the definition? And I can see that. That makes, it is a lot more interesting to remember, not interesting to remember. But you are more able to remember it because, um, not just more interesting, but more constructed. You know what I mean? You can kind of replay it so to speak. I do that a lot in a lot of things. Not just this, I don't remember necessarily the end result of something, but I remember how to get there so that when I need it I can just recreate that and get that. And so that is good, that is kind of the way she built that, I think. (Ed 3-26)

Edgar introduced the language of construction to describe their classroom practice though the professor had not used that language. This displayed his sense that the class emulated the production of mathematics and that he could then emulate that process internally.

Vincent struggled more than many of his classmates in analysis and had

to retake the course after his participation in the study. He reported halfway

through the course that he generally refrained from participating in the classroom

construction of definitions. The following interchange occurred after the first

lecture on function limits, but before the professor provided the formal definition.

I: Usually we talk one class period about a subject, and then we define
          it the next class period… Why do you think she does that?

Vincent: Maybe cause she wants us to all try and develop our own definition
          of it, then once we come to the next class, she is going to show us
          the definition that is accepted by the people who argue over the
          definition. I don't know.

I: So do you feel like that has been a helpful process to you?

Vincent: Seriously, that process is I tend to define it, which I really haven't. I
          mainly just wait for her to define it for me.

I: Why is that?

Vincent: Because, I don't know. I guess, lack of motivation to do it on my own
          because I know she is going to do it. I don't know. I just don't think
          through it. I am just telling you how it is.

I: Yeah. Well do you feel like you understand, after she defines it, do
          you feel like you understand it differently than you would have, or?

Vincent: Well I mean I will sit there and think about all of these things she is
          bringing up and I will, like, you know, I don't know where all of this is
          going. I will just wait for next class. I don't know. That is how I have
          been doing it this semester. And when she finally gets us set, with a
          solid definition, sometimes I will look at the other stuff that she was
          talking about and try to relate 'em and everything. I don't go and find
          out on my own so much. (Vi 3-26)

Vincent cited a lack of motivation driving his classroom engagement. He

understood the classroom expectation of trying to produce his own definition that

he can then compare with the authoritative version produced by "the people who

144

argue over the definition." His reference to external authority indicated his lack of identification with that defining body.

However, just a few weeks later Vincent's attitude showed a strong change after participating in the classroom defining.

> She had us try to define in class, which I thought was pretty interesting. How, what was it, I think it was how a limit goes to infinity and try to write the definition for it as x approaches to a certain number. And that was kind of fun, I liked that cause I hadn't got, I usually try to look over the material before I came to class but I didn't get a chance to and I don't really sit there and try to define it because I would rather just look it up. And it was actually different trying to define it and getting pretty close to what was in the book. (Vi 4-8)

Vincent referenced his prior practice of avoiding participation saying, "I would rather just look it up," but expressed a very positive response to the activity of defining on his own, using language like "actually different", "interesting", and "kind of fun." His sense of success also seemed to influence his response to the activity.

5.3.5. Theorems Arising as Questions and Hypotheses

In addition to constructing definitions rather than presenting them, the class encountered theorems first as questions or hypotheses to be tested and proven true or false. The classroom vignette involving the algebraic theorem of sequences portrayed how the class discussed sets of true-false questions that the professor provided and the students participated by proposing arguments and alternate hypotheses. Most of the major theorems in the course were presented in such a fashion.

145

The aforementioned story from the second semester involving the Bolzano-Weierstrass theorem also displayed how the culture of inquiry in the classroom facilitated Locke proposing the Monotone Subsequence Theorem. The professor acknowledged his discovery by renaming the theorem after Locke for the rest of that semester.

In the same way that the professor invited students to use their calculus knowledge of definitions to help them construct formal definitions, each semester she invited the class to write out a statement of the Intermediate Value Theorem (IVT) based on their memory and intuition of what it says. The class worked in groups on the activity during the first semester and most of the class proposed that the IVT ensured that there exists a domain point in the interval of question whose function value is between the function values of the endpoints. Only one student proposed that the IVT ensures that for every point between the function values of the endpoint there exists a domain point that maps to it.

The professor ended the lecture without affirming either proposition, but rather instructing students to return to class with their own statement of the IVT, a statement of the theorem from a calculus book, and another from the analysis book. Before the following lecture, I asked Celes about which statement of the IVT she favored. She responded:

Celes: Well I think that it's, where most of the groups in the class were trying to take something in the domain and relate it to something in the range and I think you have to pick it in the range and then relate it to

the domain like that one guy was saying. So I really think that makes more sense to do it that way.

I: Why?

Celes: I think it's, I think it's more arbitrary in the sense. What am I trying to say? I don't know. It's like it's more stringent to say that. The requirement, I don't know. I can't get this out. It seems more inclusive, it seems like it puts more demands on the function if you are going to claim that the IVT holds, to choose that way. (Ce 4-25)

Celes conceptualized the two options enough to paraphrase them in a way that highlighted their differences: "take something in the domain and relate it to something in the range" versus "pick it in the range and then relate it to the domain." She also observed that the latter provided a stronger result and was thereby preferable. She went on to make a more global observation about the definitions and theorems of real analysis.

I: And so then you were favoring this because you said it is more stringent. So are both of these statements true do you think?

Celes: Well I think, I think they are both true, but I think this one. I don't know... And I think that follows more of what we have been doing in class anyway. I mean like when we are defining the limit, we never start with the delta interval, we always start with the epsilon, and then when we were defining continuity, it was the same thing, so I think it follows from that as well, but it would make sense that we would begin on the y axis. (Ce 4-25)

The activity of comparing possible theorem statements led Celes to observe patterns across analysis definitions and theorems that properly guided her assessment of those IVT statements.

5.3.6. Scaffolding Proof Construction

Several of the previously described instances from the class portray how the professor used pictures to scaffold proof construction. In the case of proving

147

g(x) $\geq$ f(x) for all x implies sup g $\geq$ sup f, the professor used a y-axis diagram to guide each of the major steps in the proof. The picture used to prove that the composition of continuous function is continuous helped students observe and recall the key idea of the proof which is setting the delta from the continuity of the latter function equal to the epsilon from the continuity of the former. In each of the first two semesters, the organizing image proved sufficient for some students in the class to propose this key idea. The persistent appearance of pictures on student exams shows that many students adopted this practice of proof construction via visual exploration.

Moreover, the professor scaffolded proving by eliciting student argumentation. For instance, the previously cited case of a student proving that f of g's surjectivity implies f's surjectivity highlighted how the professor guided the student to use both pictures and verbalization as tools to aid her in constructing a written argument. Once the student had articulated her reasoning verbally, she was able to translate this into a written argument.

Similarly, in the class' development of proof regarding the true-false questions regarding sums and products of sequences, the professor primarily played the role of eliciting student argument and ideas and then organizing them in such a way as to allow the class to either correct false assumptions or construct proofs. Banon presented an argument based on a false assumption that the algebraic theorem was an "if and only if" statement. He presented a

second argument that Vaan clarified as a proof by contradiction argument. Tifa

corrected Banon's initial false assumption, and then by articulating her struggle to

apply the algebraic theorem she seemed to trigger an idea for Vaan (subtract T

from S+T). Since Vaan's suggestion properly resolved the conflict Tifa

experienced with the proof, she immediately adopted the idea and made more

explicit this key step in the proof (let S+T = X and T = Y).

Edgar reported similar interchanges between his group members.

> [Celes had a] problem… and she had an idea of how to, of her problem.
> She had a really good idea, but she said, same problem I always have,
> how do you write this? And I knew how to write that one… but it was her
> idea, you know. I hadn't actually ever processed the problem, it was just
> the idea of how to write it and that is where the group work really helps,
> but I think she had a really good concept of what to do in the first place.
> (Ed 4-9)

Edgar, Celes, and Ronso comprised one homework group during the first

semster.

5.3.7. Ronso and Edgar's Proof Styles

Ronso and Edgar took the "Introduction to Proofs" course together before

participating in the study together in real analysis. They worked together some in

both classes and both commented on how their approaches to proof and their

insights were different but complementary. Edgar said,

> [We] would sit down right before class and we would share our problems,
> and it was like when he had a hang up, I could usually help him with it,
> when I had a hang up. And sometimes we would have a hang up together,
> but you know working on it, just kind of bouncing ideas around, it would
> work out really well. (Ed 4-9)

Ronso described, "it seems like he always has like a really neat, clean way of doing things, which is nice. Sometimes, a lot of times, mine are kind of roundabout" (Ro 4-3)

Their descriptions of one another's problem solving and performance during interviews identified Ronso as a verbal/algebraic thinking in the sense of Alcock and Simpson (2005a) and Edgar as a visualizer in the sense of Alcock and Simpson (2004). Both students earned nearly perfect grades in the proofs course, but Ronso tended to use what Alcock and Simpson (2005b) called a *syntactic* proof approach while Edgar used a *referential* approach. The proof approaches that appeared in the course of study, especially using visualization and informal argumentation as tools, generally fit within the referential style of proof construction.

Ronso perceived a shift in his proving style from the proofs class to analysis with pleasure. Early in the semester, he described his proof approach in terms consonant with a syntactic proof approach: "My problem is that I kind of fly by until I get somewhere. So it always takes me a while to [laughter]… yeah, just experiment around just a little bit" (Ro 2-22). Later, on the second exam of the semester, the students were asked to "Prove that if a sequence (x_n) converges to -1, then only finitely many terms of (x_n) are positive." Concerning this problem and the proof he produced, he said:

Ronso: This one was good, I feel like I, in our proofs class we always just did this step follows this step follows this step and eventually it is what we

wanted to show. But this type of proof is different, it seems like it is more the kind of argumentative proof kind of you show a couple of steps and then explain why that steps results in your conclusion. I felt like that was kind of, I'm kind of getting more to being able to do that kind of proof as opposed to step by step by step.

I: You are pretty comfortable with the step by step by step

Ronso: Yeah, I did really like that but I think that is also a good way to do it, maybe a little more flexible you know. (Ro 2-29)

Ronso contrasted the more "flexible" and "argumentative" proofs he found

himself producing in analysis with the more procedural or "step by step" proofs

he worked with in his introduction to proofs class. He described a shift from a

more syntactic proof approach to a more referential one. Figure 5.12 presents

Ronso's proof from the exam.

$\lim(x_n) = -1$, defined:

For any $\varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $\forall n \geq K$ where $n \in \mathbb{N}$, $x_n \in (-1-\varepsilon, -1+\varepsilon)$.

let $\varepsilon = 1$

according to the previous definition, $\exists K \in \mathbb{N}$ st $\forall n \geq K$ where $n \in \mathbb{N}$, $x_n \in (-1-1, -1+1)$

$\rightarrow x_n \in (-2, 0)$ where $n \geq K$

All terms previous to K may be positive or may be negative, but all terms after K are negative.

The set of possible positive terms of (x_n) must be a subset of $\{x_1, x_2, x_3, \cdots, x_K\}$ which itself is finite, therefore the set of positive terms is also finite.
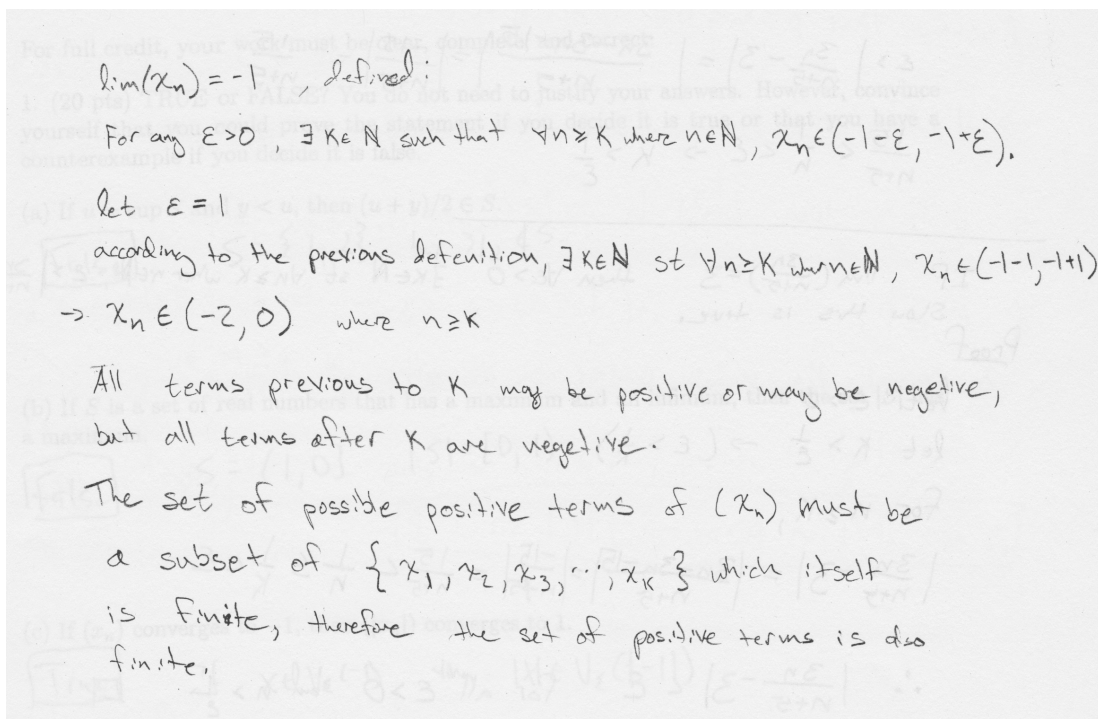
Figure 5.12: Ronso's proof that he described as more flexible.

151

CHAPTER 6

DISCUSSION AND CONCLUSIONS

The two central questions this study pursues involve the mathematical learning effects of this set of non-traditional socio-mathematical norms and how these norms contributed to students' transition to advanced mathematical thinking. To answer these questions, I first point out the parallels between the analysis classroom observed and the classroom in which Cobb et al. (1993) initially observed the emergence of socio-mathematical norms. Then, I address evidence of the establishment of each cluster of norms and the evidence therein of students' transition toward the five aspects of advanced mathematical thinking identified in the Theoretical Framework (Chapter 3). Third, I examine the modes of reasoning observed among the study participants and consider whether similar thinking might have been facilitated by traditional instruction, as defined by the previously cited literature. Next, I observe evidence of the emergence of students' mathematical autonomy promoted by these non-traditional norms. Then, I discuss contingencies in the instructional method and interrelationships between the clusters of norms. Finally, directions for further research follow some summative and concluding remarks.

6.1. Non-Traditional, Reform-Oriented Analysis Instruction

Balancing the influence of student thinking and mathematics community standards represents the unifying factor appearing throughout the curricular considerations presented in Ball & Bass's (2000) account of pedagogical content knowledge, Simon's (1995) Mathematics Teaching Cycle, Tall's (1991a) description of advanced mathematical thinking, and Cobb's (1989, 1991) synthesis of individual and communal views of constructivism. However, Cobb et al. (1993) presented three major aspects of instruction key to a classroom that balanced these influences and facilitated the communal establishment of socio-mathematical norms.

6.1.1. Rejecting the Assumption of Identical Mathematical Meanings

First, the teacher Cobb et al. (1993) observed rejected the assumption that every student will construct identical mathematical meanings (i.e. her meanings) and thus valued student input in order to address the varied meanings students constructed. The professor in the present study listened carefully to student input and incorporated it into class conversation in multiple ways:

- In the case of the "pretend game" conversation, the professor pursued the line of reasoning the students proposed until it became clear to the class that the reasoning was mathematically untenable. Locke commented that she never gave an authoritative response, but both Locke and Cyan, the primary proponents of "pretending," reported recognizing their error in light of the conversation.

153

- The professor gave students credit for their ideas as in the case when Locke proposed the Monotone Subsequence Theorem as a possible step toward proving the Bolzano-Weierstrass Theorem. The professor named it after him for the rest of the semester. She also allowed him to produce a diagram for the class to represent his thinking and she fleshed out his diagram to complete the proof.

- In the case of the party metaphor for sequence convergence, the professor extended Zell's suggested metaphor into a more complete tool for both explaining convergence and even constructing a proof that limits are unique. Cyan and Tidus both adopted this metaphor into their reasoning about sequences.

In this open culture of shared ideas and meanings, Cyan extended the party metaphor and Locke and Cyan together proposed a new theorem: that bounded divergent sequences contain subsequences that converge to two different limits.

The professors incorporation of student thinking into classroom activities while still guiding the class through many key elements of the historical development of real analysis shows that both sides of Simon's (1995) Mathematics Teaching Cycle were at work. The instructional activities that appeared in the classroom were chosen according to both the professor's knowledge of the mathematics and her hypotheses of student knowledge. She also redirected instruction according to her assessments of student knowledge

154

as in the "pretend" discussion, showing that the feedback aspect of the teaching cycle also affected her instructional design and trajectory.

This characteristic of the instruction observed also places it at the highest level of Cognitively Guided Instruction according to Fennema et al. (1996). The professor's instruction qualified for this classification because the professor made direct instructional decisions based upon her knowledge of student understanding. Even more, she made decisions about instruction based on the thinking specific students displayed during class and during other interactions.

6.1.2. *Promoting Students to the Role of Validators*

Second, Cobb et al. (1993) observed that the teacher did not stand as the class' sole validator of mathematical knowledge. The professor in the present study invited student argument and even championed false arguments to promote class discussion and elicit proper counter-arguments from students. For instance, during the discussion of the algebraic theorem of sequence limits (from the *Proposing, Arguing, and Reflecting* section) the professor argued that the hypothesis of the algebraic theorem never mentioned a sum of sequences. That same episode displayed how the professor reflected student ideas back to the class in an elicit-respond-elaborate interaction pattern rather than an initiate-respond-evaluate pattern (Nickerson & Bowers, 2008), the former pattern affirming the student input while the latter establishes the professor as the one

who dictates what is correct. The teacher thereby shared the intellectual authority in the classroom.

The professor also invited students to question definitions by considering "what it should mean." She referred to the people who created the mathematics pointing out that they were not perfect and inviting students to create better definitions when they go write their own mathematics books. Several students exhibited high-level mathematical reasoning in the course of taking her up on this invitation:

- Edgar questioned the continuity definition because of its inclusion of a counter-intuitive example and even proposed a possible alternate notion for continuity established on a set of points rather than individual points.

- Ronso went through several iterations of proposing and testing regarding the proper domain conditions for function limits at a point and as x approaches infinity.

- Locke produced a valid alternate domain condition for limits as x approaches infinity.

It should also be noted that the professor equipped students to properly evaluate various proposals by presenting large sets of examples and asking true-false questions that helped students develop and hone their concept images. Selden and Selden (2008) pointed out that the success of this type of open instruction in a rigorous definition-based course depended on students being able to develop robust concept images.

6.1.3. Mediating Enculturation and Student Meaning in Representations

The third aspect of instruction Cobb et al. (1993) observed involved how the teacher translated the representations and expressions students presented into forms that were both recognizable to the students who proposed them, but also were compatible with the shared meanings and conventions of the mathematical community at large. The professor observed in this investigation similarly translated student propositions in the case of Zell's party metaphor and during the algebraic theorem discussion. In the latter case, the teacher wrote student propositions on the board as well-formed mathematical statements for the class to assess. She restated and clarified Vaan's assertion that Banon's argument was proof by contradiction, and she attended to Tifa's suggestion of a weaker form of the final statement, bringing it to the attention of the entire class.

The teacher also provided mediation in the other direction, taking problematic conventions from the mathematical community's shared representations and discussing them directly with a class. She criticized the problems presented by the notation for sequences tending to infinity, but also noted the parallels that motivated the use of such notation and used standard notation in deference to the community at large.

6.1.4. Flexibility

Both Ball & Bass (2000) and Cobb et al. (1993) considered the flexibility teachers must exhibit to identify student conceptions as they are expressed and

to incorporate student input into their instruction. The professor in this study displayed great flexibility in several instances. The "pretend" discussion could not have been planned or planned for by the professor, but she devoted at least 20 minutes of class time to addressing the students' reasoning and the fallacy therein. She similarly adopted the party metaphor spontaneously as it arose from student suggestions. These instances indicate that the professor possessed a deep and functioning pedagogical content knowledge of real analysis. The extent to which her classroom structure provided her with access to student thinking and reasoning almost certainly contributed much to this end.

6.1.5. Talking About Talking About Mathematics

Cobb et al. (1993) observed the teacher in their study alternating between talking about mathematics with the students and talking about talking about mathematics. In the former case she was very open, centering the conversation upon student thinking and feedback, while in the latter she became more directive, telling the students explicitly what she expected of the classroom culture. The professor in the present study similarly alternated between mathematical conversation and meta-mathematical conversations, promoting discussion in the former case and often expounding in the latter.

She invited students to consider directly the thoughts and purposes of the mathematicians who originally produced the mathematics they studied, which shifted the conversation from the content to the human operations underlying the

content. Regarding proposing theorems, the teacher provided direct commentary, "This is good. This is how you make theorems. If that doesn't work, weaken it. Maybe that's true." She asked students to consider why she sometimes spent one class period talking about an idea without defining it instead of presenting the definition to start with, and students gave substantial answers when I repeated this question during interviews.

In the words of Cobb et al. (1993), my observations in the classroom support the claim that "explaining, justifying, and collaborating had become objects of reflection in a consensual domain." Thus, the classroom of study exhibited each major aspect of Cobb et al.'s description of a student-centered or reform-oriented classroom. The particular vignettes chosen from the class of course represent patterns that repeated themselves throughout instruction. These specific accounts were chosen for their clarity and rich content. Next, I outline the establishment of each cluster of norms individually and examine them in light of the two primary research questions.

6.2. Visualization

Though use of visualization does not represent a "non-traditional" norm of mathematics in general, it does not appear to be inherent to all Definition-Theorem-Proof teaching (Weber, 2004). The role visualization played in the classroom of study especially appears "non-traditional" against the backdrop of prominent mathematical views of proving in which it holds a secondary rank to

analytical methods (Eisenberg, 1991; Eisenberg & Dreyfus, 1991). Just as the professor continually called upon visualization as a tool for communication, explanation, and problem solving, students integrated her pictorial frames into their problem solving on tests and during interviews. They developed visual representations to communicate with one another. Some of the meta-mathematical discussions in the class even focused on strengths and drawbacks of visual images, satisfying the research literature's recommendation that students be trained directly in using visual schemas (Aspinwall et al., 1997; Eisenberg & Dreyfus, 1991, 1994).

Cyan reported a shift in his understanding of graphical representations precipitated by considering a novel function defined differently on rational and irrational numbers. He realized that the function had to be drawn using two lines and that the lines did not represent every point in the domain, but were "dotted." Upon this observation, he stopped identifying the output value on the line itself, which he considered could be misleading, and began to think of the output values along the y-axis. His shift corresponds to transitioning from an action view of graphs in which the physically drawn line represents the function to a process view in which the graph represents a correspondence between the input values and function values (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). This example represents the powerful ways in which students reflected upon visual representations and used them as tools to construct mathematical meaning, in accordance with Aspinwall et al.'s call for instruction in which visualization is a

tool for reasoning and mathematical construction rather than an isolated curricular goal.

The visual representations the class used such as the diagram for the proof that the composition of continuous functions is continuous facilitated recall of those proofs as well as providing global understanding of the proofs. During the first two semesters, the diagram led a student rather than the professor to propose the key idea of the proof and during two interviews, the diagram reminded students of that key idea. This instance also highlighted the contingent aspects of visual reasoning in that as student comprehension and then recall of the diagram and the associated argument interacted with the student's concept definition of continuity. Those students who had evoked the limit definition of continuity did not understand the argument nor fully remember the diagram.

In the case of uniform continuity, the professor's visual explanations regarding delta's varying with the steepness of the graph and uniform continuity corresponding to a graph containing a steepest point both led to misconceptions. The professor attempted to provide students with an intuitive tool by which they could identify which functions are and are not uniformly continuous as well as a way to understand why the square root and natural log functions are different according to this property. However, since her explanation led students to reason about tangential slopes and to focus on the steepest point, students misapplied

161

the arguments. Thus these arguments may not have been cognitively appropriate or may require further direct training in their application.

One student, though, was able to overcome this intuitive misconception when she participated in the class practice of formulating definitions in common English. Once Aerith articulated uniform continuity saying, "this thing basically is saying when x_1 and x_2 getting closer, the image of, I mean the value of these two points will getting close, too," she reevaluated her previous false conclusion. This action emulated the mathematical practice of evaluating visual insights and intuitions according to analytical justifications. As Alcock & Simpson (2002) say regarding the parallel intuitive approach of prototypes, "[mathematicians] are aware that, in order to ensure universal validity for their arguments, they must eventually formulate these in terms of appropriate definitions" (p. 32).

One socio-mathematical norm that the class established was "Visualization is an acceptable and helpful tool for sense-making, defining, and proof production." Visualization became an object of class discussion and instruction providing students with training in visual schemas as the research literature recommends. Though the professor did encourage students to draw pictures, students were never assigned to draw pictures or assessed based upon them. Thus the students observed in the data displayed participation in the second aspect of my definition of mathematical activity, which is to Use Visualization for Sense-Making and Problem Solving.

Visual representations in the classroom of study also contributed to the next set of socio-mathematical norms about communication. In addition to students using pictures as tools for reasoning, the adoption of common visual frames, such as arrow diagrams for functions or the three number line diagram for the composition of functions, contributed to establishing a shared language in the classroom. This shared language did not establish identical meanings in the classroom, but most likely did help students construct meanings with sufficient compatibility for effective communication and normative understandings.

6.3. Meaningful Mathematical Communication

The structure of the class facilitated the sharing of ideas inside and outside of class meetings both between the teacher and students and among students. Students shared ideas and arguments, built upon one another's ideas, and assessed what one another shared. The professor often took the roles of clarifying student ideas, reflecting them to the class, or arguing a student's position even if it was mathematically incorrect in addition to or instead of standard lecturing.

6.3.1. Negotiating Meaning Through Sharing and Assessing

During the algebraic theorem discussion, the teacher and the class produced three separate statements of the idea behind the proof that T converges and S+T converges implies S converges. Vaan first said that they should use the subtraction part of the algebraic theorem. Tifa made this more

163

specific saying let S+T=X and let T=Y. Another student later offered, "Let's call

S+T X." Vaan articulated the strategy. Tifa explained how to apply that strategy to

the given situation. The third student's comment explained the acceptability of

this strategy indicating it represented a matter of naming: "call S+T" by a different

name "X." The professor's final explanation built upon this same idea from the

third student by restating the algebraic theorem replacing the name "X" with

"some sequence" emphasizing the arbitrary nature of the label.

This repetition and reformulation of the same idea by different members of

the class served two purposes. First, the students spoke in order to teach or

convince one another. They communicated their ideas for the purpose of sharing.

Second, they reformulated the proposition to negotiate the communal

understanding. As students continued to express their understandings, the class

verified the level of compatibility among their individual meanings. Even after the

third expression, one student verbalized confusion because he could tell that his

understanding did not match what his classmates were describing. The professor

stepped in at that point to further clarify the meaning that she observed emerging

in the class.

6.3.2. Classroom Languages

The class also developed three distinct mathematical languages that were

used in tandem. Cyan extending the party metaphor represents one example of

how students used the metaphorical language to construct mathematical

164

meaning. Tidus' move from the formal language to the metaphorical to express intuitive understanding showed the different purposes the languages possessed for the students.

Repeated questioning about definitions revealed shifts in students' primary choice of language. Tidus initially spoke in metaphorical terms about sequence convergence, and then was successful in translating those terms into the class' intuitive or symbolic language. After the test when he had more time to develop understanding of the formal definition, he began instead with the English definition and then the symbolic form.

The three languages exist on a continuum of formality with metaphorical language being the most informal or figurative, English or intuitive language standing in the middle, and symbolic formulations representing the most formal language. Students shifted over the course of instruction toward the more formal end of the spectrum. However, when Tidus could not recall a proof or when I asked him to explain his understanding of a definition, he hearkened back to metaphorical terms. This shows the permanence and coordination of each language in his thinking, because he did not abandon the metaphor altogether.

Tidus' successful shift from metaphorical to symbolic establishes the value of the metaphorical and English languages as scaffolds for his construction of understanding of the formal, symbolic definition. Tidus also displayed structural understanding of the coordination of the languages when he explained regarding

the arrows metaphor that the informal language helped him reason while the formal definition should be used in formal proving. On the other hand, Locke seemed to avoid using these tools preferring more formal language, while an unsuccessful student was unable to construct understanding beyond the metaphor. Though this study does not compare directly student understanding under instruction with and without these three languages, Cyan commented that simply reading the book's definitions (representing symbolic language) proved very difficult for him when compared to the development across the languages in the classroom discussion.

6.3.3. Direct Instruction and Contingency

In addition to developing the three languages, the professor provided direct instruction in comprehending formal language itself and translating ideas into the formal language. During the study session, she specifically guided a student to organize her ideas in a diagram and then translate her findings into common language. Then she invited the student to articulate her ideas in English before translating them into symbolic form.

Vincent's experiences qualify the communicative aspect of the class, however. The relative ease with which his partners understood the mathematics and solved problems frustrated Vincent and disenfranchised him from the group. He expressed a sense of futility and lack of motivation as a result. Despite this drawback, Vincent was one of the only students to cite collaboration as one of his

favorite aspects of the class. He had a very positive sense of working together and communicating about mathematics because he had never experienced it in other classes, despite the fact that he maintained his critical stance toward his particular homework group. On the other hand Ronso, Celes, and Edgar, who were in a homework group together, reported a very mutually beneficial intellectual relationship.

One social norm about communication observed during the study was, "Each student's input is valued and they should ask questions when they do not understand what is being shared." Some examples of socio-mathematical norms identified in the communication cluster include: "Mathematics is a communal and collaborative activity" and "Mathematicians develop and express their ideas informally before making them more rigorous."

6.4. Constructing Mathematics

Through reconstructing the (possible) thought processes behind the original construction of elements of real analysis, students gained insight into the content itself and displayed evidence of advanced mathematical thinking as will be explained below. In addition, lessons about the structure of proof-based mathematics came implicit in this (re)construction process. We examine the reconstruction process in each of the three major aspects of proof-based mathematics: definitions, theorems, and proofs.

6.4.1. Constructing Definitions

Students articulated the understanding that formal definitions represented one half of a dual construct consisting of the idea or the concept and the definition used to capture that idea. In other courses, idea or concept might be replaced with set of mathematical objects, but almost all real analysis definitions represent properties of sets, sequences, or functions rather than mathematical objects themselves. Cyan stated in this regard, "we do our best to describe accurately what's, what the concept is." Thus he articulated an understanding that when in formation, mathematicians extract definitions from the set of objects they wish to define. He associated the choices made in this describing process with the "power" the mathematician has in defining something. However, he went on to reveal his understanding that choices made in defining are binding in a sense because later theory flows from the definition chosen. In this sense, mathematical definitions, once they are formulated, have a stipulated nature.

In considering and coordinating multiple definitions of continuity (limit, epsilon-delta, and sequential), the class engaged in the process of choosing defining ideas described by Alcock & Simpson (2002). Celes showed surprise at the idea that mathematical definitions vary with context, such as the difference between the calculus definition of continuity, which is based on the equality of the two-sided limits and the function value, versus the analysis definition, which is more general. The case of proving the composition of continuous functions is continuous introduced one contingency in such instruction. When one definition

became more prominent in their minds (possibly because it was used in the proof prior), it inhibited some students' ability to understand and remember a proof using an alternate, equivalent definition.

Many of the students expressed the understanding that definitions ultimately fit into a body of theory and thus do not stand isolated, but are constructed for a specific purpose. Edgar used the term "constructed" to describe the way the class treated definitions. Ronso referenced the theoretical context of definitions when he spoke of definitions being useful to solve problems. He acknowledged that mathematical definitions are ultimately arbitrary (stipulated), but argued that the practical reality is that they are adopted for their utility. Edgar similarly said that mathematicians accept definitions if they are "going to work." This notion of definitions' relationship to theory mirrors the assertions made by Mariotti (2006) and Boero et al. (1996) regarding proofs being inseparable from the theory in which they are couched.

Thus in several such ways, students expressed very sophisticated understandings of the nature of mathematical definitions and the process of mathematical defining. These constitute evidence of the students' transition toward advanced mathematical behavior according to the third aspect of my definition: Create Definitions Within a Body of Theory. Some students adopted this role to the extent that they questioned the defining choices the professor made regarding function limits as x approaches infinity.

Along the same lines, Edgar vigorously questioned the definition of

continuity, comparing its acceptance of a counter-intuitive example to a crooked

lawyer's perversion of the law. Several other students protested the domain

restriction in a limit definition that they assessed as insufficiently general for the

context of real analysis. As Cyan said, "you want to make your definition so that if

you use it correctly, only certain things can happen. Like if I write my definition of

limit a certain way, then if the limit really doesn't exist, when I use the definition it

shows me that the limit doesn't exist." Edgar considered the continuity definition

to fail this criterion and appeared to be trying to formulate an alternate notion of a

point being "continuous with other points." Cyan's description of building a

definition to properly capture their notion of limit reveals the compatibility

between this class' quasi-empirical approach to definitions (modeling their ideas

in a rigorous definition) and Boero et al.'s (1995, 1996) and Mariotti's (2006)

modeling approach to proof instruction in the context of shadows.

In contrast to these students who adopted very seriously the role of

defining, Vincent initially reported that he often chose not to participate in the

classroom activities of proposing definitions. Instead, he would simply wait for the

professor to present the definition and then go back and look over the previous

conversation in light of the definition. This disengagement most likely contributed

to the fact that he is one of the only interview participants who did not pass the

course on his first try. However, after some time he reported trying to formulate

one of the definitions that the teacher posed to the class and expressed enthusiastic pleasure in the activity and his relative success therein.

In addition to the pleasure Vincent expressed, others reported how this construction yielded benefits for definition understanding and recall. Ronso pointed out that constructing the definition made the necessity of each piece clear, which appeared as a strong value he held during his extensive exploration of the notion of cluster point. Celes said struggle and discovery helped her remember the definitions much like Edgar indicated that he could remember definitions better because he could recreate the thought process by which the class originally created the definition.

The process of defining appeared powerful both for students' development of understanding of the definitions and their understanding of mathematical definitions, in accordance with the advice of the cited research literature (Branford, 1908; De Villiers, 1998; Dreyfus, 1991). Students became aware that mathematical definitions are constructed and contextually dependent by taking part in their construction. They also understood the interplay between the extracted and stipulated aspects of definitions by transitioning from the process of describing ideas in definitions and then proving based upon those definitions. Some students also expressed a set of values for mathematical definitions very compatible with those more widely held in the mathematical community.

One socio-mathematical norm about constructing mathematics that developed is "We, as mathematicians, construct mathematical definitions, often to capture a property or behavior in a rigorous way that is useful for proof and problem solving." Another socio-mathematical norm in that same cluster is "Mathematical definitions are constructed for a purpose and thus are not absolute, but should be questioned either to be refined or more fully understood."

6.4.2. Constructing Theorems

Theorems appeared in the class most often as one among a set of true-false questions the class either proved or disproved. This fostered a "culture of why questions" as described by Jahnke (2005). Students adopted this practice proposing possible theorems themselves such as Locke's Lemma, any bounded and divergent sequence contains subsequences that converge to different limits, and Tifa's revision of the algebraic theorem statements. The class then tested these statements through a scientific debate format as described by Alibert & Thomas (1991) or Cobb et al. (1993). Each of these processes emulated the activity of the mathematical community at large.

One of the more significant conversations that arose involving the proposing of theorems came from the activity in which the class produced a statement of the Intermediate Value Theorem from their memory of calculus. Two major options arose, one a weaker case of the true IVT. Celes made a powerful, global observation about analysis definitions and theorems during her attempts to

identify the correct IVT. In order to identify the theorem, she examined the structure of other statements and noted that most of them choose some object or quantity related to the range and then found some object or quantity in the domain that corresponded thereto. She was thereby successful in identifying that the IVT ensured for any chosen element between the function values of the endpoints of the domain interval, there exists a point in the domain interval that maps to that element.

Also, Alibert and Thomas (1991) and Mason and Watson (2008) indicated that the process of formulating and proposing theorems can prove beneficial for students' development of proof because students are able to retrace the conceptual pathway of discovery. Mariotti (2006) espoused group work as a powerful context for engaging students in the creative process of proposing and proving. Both of these recommendations were likely influential in the classroom of study because it integrated the process of proposing in a group-oriented atmosphere. In this way, students showed participation in the fourth aspect of advanced mathematical behavior, which is to Propose, Test, and Validate Statements. One socio-mathematical norm about constructing mathematics that arose was "We, as mathematicians, ask questions and observe patterns in the pursuit of general theorems."

6.4.3. Constructing Proofs

Rather than presenting proofs line-by-line, the professor often scaffolded proofs using diagrams or structural presentations. The conversation about the Bolzano-Weierstrass Theorem allowed Locke to propose the Monotone Subsequence Theorem and to share with the class a generic picture of a sequence with finitely many peaks. The professor's diagram for the proof that the composition of continuous functions is continuous led students during the first two semesters to propose the key idea to the proof without her suggesting it. These processes of scaffolding also provided students with more global understanding of the proofs and insight into the reasoning by which the proof could be produced in accordance with Raman (2003).

Also, students shared ideas in ways that proved formative for proof construction. Edgar reported an instance of Celes providing an idea that he then translated into a proof. Vaan and Tifa sparked one another's insights into the class' proof during the algebraic theorem discussion. Through sharing ideas, Ronso and Edgar appreciated their complementary proving abilities. However, through the class' influence Ronso described a shift in his proving practice from more syntactic and procedural proof styles to include more semantic and conceptual proof styles. As the literature review indicated, adopting a referential or semantic proof approach hinges upon students developing more robust and refined concept images, which the class of study accomplished by looking at sets of varied examples and true-false questioning.

The class of study also attended to both the convincing and validating parts of proof, both the epistemological and the didactic purposes. As in the case of proving that sup f ≤ sup g whenever $f(x) \leq g(x)$ for all x in the domain, the class tried to find a counterexample before attempting a proof. That particular activity did not move the students toward producing a proof, but the students began to suspect that the statement was true when they could not find any such counterexample. The professor allowed the students to consider counterexamples for the purpose of letting them convince themselves that the statement should be true.

This evidence also displays the students' participation in the fourth aspect of advanced mathematical behavior: Propose, Test, and Validate Statements. One socio-mathematical norm about constructing mathematics that arose in the classroom of study was "The process of proving begins with exploring the situation described by a mathematical statement and convincing one's self that it is true or false." Students were able at times to connect the exploration of examples and mathematical relationships to the production of proof's key ideas.

6.5. Present Findings Compared with Traditional Instruction

Several of the previously described aspects of the instruction observed in the classroom of study qualify it as "non-traditional." Weber (2004) described traditional instruction according to four main characteristics:

1) lecture dominates class meetings,

2) students are predominately silent,

3) the curriculum is structured according to the logical flow of the course

     material, and

4) students learning how to prove is the primary goal of the course.

The classroom of study would be better described by the following

characteristics:

1) class meetings are comprised largely of discussion and dialogue,

2) students provide large amounts of input and respond to one another,

3) the curriculum is structured according to the logical construction of the course

     concepts and the way in which students come to experience and understand

     those concepts, and

4) proficiency in proof and conceptual understanding were dual goals pursued in

     the classroom.

    In addition, several of the results from this study contrast with previous

descriptions of traditional teaching from the literature or their particular research

findings. Students in this study expressed proper understanding of the stipulated

nature of definitions contrary to the findings reported by Alcock and Simpson

(2002) or Edwards and Ward (2008). Many students in this study exhibited more

global understanding of proofs pointing to key ideas rather than line-by-line

explanations as previous researchers have found (Selden & Selden, 2003).

Cyan directly compared the accessibility of the classroom treatment of definitions and the book's presentation thereof, which mirrors traditional, definition-theorem-proof teaching's style of presenting definitions in symbolic and complete form. He reported that the class' treatment was much easier, but the book's was useful once he had been introduced to the idea in class. The study participants spoke of the benefits yielded by the defining process that would seemingly be absent in a course where definitions were initially presented complete in symbolic form.

The culture of collaboration also appeared to benefit students. Vincent spoke positively about his experiences doing mathematics communally despite his poor group dynamic. Ronso and Edgar described the benefits their interactions yielded due to their differing approaches and insights. Cyan spoke very positively of Locke's insights as well, thankful for his input in the class. A traditional lecture context may have inhibited all of these synergistic interactions between peers in the class.

Several instances of high-level reasoning appeared as a direct result of the defining process the class adopted. Celes made a global observation about the structure of analysis definitions and theorems in the course of reasoning about forming statements rather than proving or disproving them. Ronso went through several iterations of proposing and testing with himself in the course of constructing definitions. This line of reasoning also engendered his observance

of the parallel between function limits as x approaches infinity and sequence limits. Locke might never have created a sequential condition for cluster points had he not engaged in the process of forming his own definitions. I hold that these examples of reasoning are epistemologically valuable for these students, are instances of advanced mathematical thinking, and would not be facilitated by traditional modes of analysis instruction.

6.6.   Students as Validators and Mathematical Autonomy

The emergence of mathematical autonomy represents another significant result of the non-standard set of socio-mathematical norms identified in the classroom of study. As was previously stated, the professor avoided setting herself up as the sole validator of mathematical meaning and helped the students adopt that role themselves. Their command of that role appeared in several instances as well as evidence of their sense of autonomy.

The professor avoided authority through use of the ERE interaction pattern as in the algebraic theorem discussion. She could have corrected Banon's false assumption about the algebraic theorem, but she waited for the rest of the class to provide the questioning. Locke noted after the pretend discussion a similar effect that the professor did not give an authoritative answer but left it up to the students to find clarity from their discussion. The professor also equipped students to become validators by training them in mathematical language and visualization and by modeling the process of translating their

178

intuitive ideas into symbolic form. She also presented them with examples so that they had rich concept images against which to assess the validity of mathematical statements and arguments. The professor also gave students credit for their ideas as in the case of Locke's Lemma or Zell's party metaphor.

Most significantly, Edgar and Cyan both used personal language when talking about the process of defining. Edgar used the language, "If I am writing a book, and I am going to be using these things, I need to define them." Cyan said, "that is a lot of power to say that I am defining this how I want to." Both of these students also expressed emotion when they disagreed in some way with choices the teacher made: Edgar regarding continuity and Cyan regarding her less general domain condition on limits. They spoke referring to themselves as the definers and showed a very personal investment in the process of defining. Edgar so believed his argument that, even though he knew that logically he missed a question on the test based on the standard definition, he hoped his conceptual reasoning had taught the professor something about the problems with the standard definition itself. Instances like these reveal the powerful sense of autonomy and ownership that these students developed through their mathematical experiences in real analysis. This evidence reveals the students' participation in the first aspect of advanced mathematical behavior, which is to Develop a Sense of Mathematical Autonomy.

6.7. Contingencies and Interactions

I identified visualization, mathematical communication, and developing mathematics as three clusters of non-traditional socio-mathematical norms established in the classroom of study. Students exhibited participation in each cluster of norms as well as significantly using each cluster to construct mathematical understanding and meaning. Each cluster also carried contingencies for which the class had to account. In many ways, these contingencies stemmed from the interactions between the norms themselves.

Student use of visualization sometimes depended upon students' concept definitions. Since the class treated definitions as mathematical objects to be created and honed, students had to coordinate various concepts associated with possible definitions. In the case of continuity, some students simply recalled the wrong form of the definition and it inhibited their understanding of a proof. In the case of uniform continuity, students reasoned in terms of informal definitions that yielded misconceptions. However, the classroom practice of translating definitions helped one student circumvent her initial misapplication of the informal notion.

Student use of visualization, their ability to test mathematical hypotheses and arguments, and their ability to construct rigorous definitions all depended upon their development of robust concept images. The class began discussion of each section of instruction with an examination of examples and honed their

understanding through true-false questioning. The professor thus guided students to consider salient issues involved with abstraction in real analysis facilitating students' construction of strong concept images and key counter-examples. For instance, the class discussed at length sequences which tend to infinity but are not monotonic and functions that are continuous at only one point.

Students used visualization and the multiple classroom languages as tools to construct definitions and proofs. The classroom languages appeared to correspond to different levels of formality in student thinking that students continually coordinated. The informal, English language embodied the idea aspect of a definition while the formal, symbolic language embodied the formal-logical aspect of a definition. The informal language often corresponded to mathematical argument while the formal language expressed stages of mathematical proof. In both cases, the two languages interacted and I observed direct transitions between the two. In this way, the first two clusters of socio-mathematical norms facilitated the third.

6.8. Summary and Conclusions

The establishment of these three clusters of non-traditional socio-mathematical norms transformed the students' classroom experience from one of acquisition of externally imposed mathematical abstractions into one of construction, participation, and advanced meaning making. The students learned about and participated in many cognitive and social activities that characterize

181

the greater mathematical community. In the ways previously described, the class emulated the construction of each major aspect of proof-based mathematics. Through emulating the thought processes of mathematicians, they appeared to move toward more advanced modes of thought. This term "emulation" should be qualified, though, by the observation that any discoveries the students made through class discussion and scaffolded exploration appeared truly novel psychologically. In such cases, they were not emulating discovery but truly discovering. I thereby argue that the students in this study gained access to the Platonic Experience of Discovery, which was the fifth aspect of advanced mathematical thinking outlined in my theoretical framework. Though no students directly espoused a Platonic view of mathematics, Edgar's comparison of a counterintuitive entailment of a definition to a moral breach did display a strong, personal sense of the reality of real analysis. Students also communicated a sense of pleasure and beauty about their mathematical explorations that I would argue is integral to the mathematician's experience.

Cobb (1989) asserted the existence of parallel mechanisms for the institutionalization of mathematical meaning in the research community at large and the institutionalization in the individual mathematics classroom. The discovery of mathematical meaning for the class or for the individual student mirrors historical discovery. The class developed their own language and engaged in many of the key activities of the mathematical community thereby becoming a microcosm of that community. I assert thus that the classroom of

study successfully facilitated students' movement toward advanced mathematical thinking by operating as a relatively self-contained mathematical community. The students adopted modes of reasoning mathematicians exhibit because they participated in the activities mathematicians do.

In the course of so doing, students were able to construct significant understanding of both advanced mathematical thought and advanced mathematical thinking. Students reported shifts in their proof approaches as well as benefits for recall and depth of insight. The reasoning students displayed during interviews often mirrored their reported perception; some students surpassed their own expectations in producing definition.

Several significant instances of student reasoning and learning arose in ways that call into question whether traditional instruction could have facilitated similar constructions. Whereas many lessons about the nature of mathematics are implicit in traditional instruction, issues regarding the nature and process of formal mathematics became objects of discussion and instruction in the classroom of study. Thus the students simultaneously gained access to the thought processes behind the construction of the mathematical content and the structural parameters that guide that construction. These findings indicate that even though traditional instruction pays more direct attention to institutionalized meanings or enculturation than to student thinking, this classroom that balanced

the two influences provided students greater access to and insight into the meanings and practice of the mathematics institution.

Guiding students to participate in a mathematical community also promoted their sense of mathematical autonomy. Helping them develop richer concept images facilitated their ability to make and assess hypotheses and arguments, allowing them to act as a community of mathematical validators. If advanced mathematics instruction intends to help students transition into the mathematical community itself, this would appear a necessary step in that direction. I observed significant instances of mathematical reasoning involving multiple iterations of proposing, interpreting, and assessing, both between students and a student processing alone.

Finally, the multiple interactions between the norms and contingencies that I identified reveal the unity this classroom exhibited. In other words, these socio-mathematical norms as I observed and described them do not represent modular units, but aspects of a greater whole.

6.9.   Directions for Future Study

Many questions still exist regarding the emergence, nature, and effects of the socio-mathematical norms here described.

1. The professor leveraged the students' prior exposure to class topics from calculus to have them reason about the concepts behind the definitions. This parameter of the present instruction calls into question whether and how such

norms could arise in an abstract algebra course, where students have little or no prior intuition about course topics.

2. These norms appeared reliably over the three semesters of study, but there has been little follow-up investigation regarding how the effects of this instruction are sustained into future courses.

3. How do students who have acclimated to this non-traditional environment respond to later traditional instruction?

4. The professor observed displayed a strong pedagogical content knowledge and flexibility that facilitated the emergence of the socio-mathematical norms. Is this flexibility a necessary condition for facilitating these socio-mathematical norms? How can research inform the appropriate development of pedagogical content knowledge needed for facilitating emergence of these socio-mathematical norms?

APPENDIX A

EXAMPLE PROFESSOR INTERVIEW QUESTIONS

1. Tell me about the students' performance on the test.

2. What strengths and weaknesses do you see in the students?

3. What do you want the students to think of when they think of "one-to-one?"

4. When you encourage the students to speak "in English," what do you want the students to get out of that practice?

5. What misconceptions are you trying to test on this exam?

6. What do you expect them to have trouble with?

7. What was your purpose in assigning [this homework question]?

8. What big ideas do you want the class to be coming away with from this section of the course?

9. Why do you emphasize the changeable nature of mathematical definitions?

10. Tell me about any instances where you saw a student have an epiphany lately.

11. Why do you think [this student] made [this comment] during class?

12. Why do you think the students missed [this test question]?

APPENDIX B

EXAMPLE STUDENT INTERVIEW QUESTIONS

1. What is the definition of "injective" or "one-to-one?"

2. (If they acknowledge both the intuitive notion and the formal notion of injective) Why do we define it as we do instead of the intuitive way?

3. What does "surjective" mean?

4. How do we show a function is surjective?

5. (If they did not refer to it voluntarily) What metaphor did we use for surjective? What was the point of this metaphor?

6. Prove that if g and f are bijections, then g composed with f is a bijection.

7. Explain to me the proof that the set of real numbers is not countable.

8. What does it mean for a sequence to converge to a point?

9. In the definition of convergence of a sequence, what are the roles of ε, K, and n?

10. Prove that $(^{n+1}/_n) \to 1$. (not proven in class)

11. Prove that the limit of a convergent sequence is unique. (proven in class)

12. What are the different ways a sequence can diverge?

13. How can we prove that a sequence diverges?

14. Explain how we proved that every sequence has a monotone subsequence.

15. Why did the professor complain about the notation $\lim(x_n)=\infty$?

16. Explain the relationship between diverging to infinity and being monotone.

17. Explain to me the definition of $\lim_{x \to p} f(x)=L$.

18. Draw me a picture of what that limit definition means.

19.	What is the difference between saying "the limit approaches L" and "the limit is L" as she talked about in class? Which one is correct and why?

20.	Is there a difference between the statements: "f(x) gets larger than M on some interval for any M that you pick" and "for any M that you pick, f(x) gets larger than that on some interval?" (statements taken from lecture)

21.	Explain to me the proof of the statement "If f is continuous at p and g is continuous at f(p), then g composed with f is continuous at p."

22.	What does the Intermediate Value Theorem say?

23.	What is uniform continuity? What does it look like for a function to be uniformly continuous?

24.	Give me examples of functions that are (not) uniformly continuous?

25.	How did you study for the test?

26.	What was your reasoning on [this question] on the test?

27.	What parts of the material did you still not feel like you understood when you took the test and why?

28.	What aspects of the professor's teaching have been most helpful to you?

29.	Have your beliefs about mathematics changed in any way this semester?

APPENDIX C

STUDY PARTICIPATION PROTOCOL

1. Oral Study Participation Invitation

[To be read by the researcher:] This semester, with the cooperation of Dr. Shipman, I Paul Dawkins will be conducting a mathematics education study regarding certain aspects of classroom communication and student study habits in Mathematical Analysis. Each of you is invited to volunteer to participate. Participation includes up to nine interviews with me, spaced roughly once per week, each for about 30-45 minutes. These interviews will cover material from the course and in this way will constitute a guided study time, so it is my expectation that participating in the study may benefit your course performance. If you study consistently with one or more of your classmates, you may opt to be interviewed jointly. All interviews will be audio recorded for later transcription to preserve the integrity of your statements. All data gathered will be secured under lock and key to preserve anonymity. No presentations of the research findings will identify you as a participant. Please indicate on the written participation form, if and with whom you would like to be interviewed. In addition, if you participate I will gather copies of your written class notes, written homework assignments, and class exams, to better understand your understanding and growth over the semester.

If you agree to participate, please read and sign the informed consent form you have been given and sign the volunteer roster. Please provide your contact information by which I can most easily make appointments with you for

interviews and indicate whether you would like to be interviewed together with your study partners.

2.   Informed Consent Form Contents

This Informed Consent will explain about being a research subject in an experiment. It is important that you read this material carefully and then decide if you wish to be a volunteer.

Purpose of the Study:

The purpose of this study is to investigate certian aspects of classroom communication and student learning as it occurs during class meetings, study sessions, personal study time, homework completion, and class exams in first-semester undergraduate analysis.

Duration:

Participation in this study will consist of your normal class meetings and study sessions with the addition of a series of 30-45 minute interviews outside of class totaling up to 6-7 hours of time. These interviews will consist of discussing course material and course work and are thus designed to replicate your personal study time.

Procedures:

The procedures, which will involve you as a research subject, consist of you explaining your understanding of selected portions of your notes, the class

presentation, the textbook, and the home work problems. We will also gather

copies of your written notes, copies of your written class work, and copies of your

exams.

Possible Risks/Discomforts and Possible Benefits:

There are believed to be no substantial potential risks from participation in

this study. There are believed to be no substantial benefits from participation in this

study, except possibly gaining a fuller understanding of the course material and

benefits to your course performance due to guided study sessions with the

researcher.

Alternative Procedures/Treatments

Participation in this study does not constitute an experimental treatment.

You are free not to participate in the study, which will yield no consequences to

your course assessment or performance.

Confidentiality:

Every attempt will be made to see that your study results are kept

confidential. A copy of the records from this study will be stored in [room number]

under lock and key for at least three (3) years after the end of this research. The

results of this study may be published and/or presented at meetings without

naming you as the subject. Although your rights and privacy will be maintained, the

Secretary of the Department of Health and Human Services, the UTA IRB, and

personnel particular to this research (Mr. Paul Dawkins, Dr. James Epperson) have access to the study records. Your interview manuscripts and written work and records will be kept completely confidential according to current legal requirements. They will not be revealed unless required by law, or as noted above.

Voluntary Participation

Participation in this research experiment is voluntary. You may refuse to participate or quit at any time. If you quit or refuse to participate, the benefits to which you are otherwise entitled will not be affected. You may quit by calling Paul Dawkins, whose phone number is [phone number]. You will be told immediately if any of the results of the study should reasonably be expected to make you change your mind about staying in the study.

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BIOGRAPHICAL INFORMATION

Paul Christian Dawkins was born on July 31, 1983 in San Angelo, TX to Ross Clinton Dawkins, Jr. and Keri Louise Whittle Dawkins. Paul graduated from San Angelo Central High School in May of 2001. He was the distinguished student of the college of sciences at Angelo State University when he earned a Bachelors of Science degree in mathematics in August of 2005. He then earned his Ph.D. in mathematics at the University of Texas at Arlington in December of 2009. He received the Distinguished Graduate Student Award from the UT Arlington Mathematics department in 2007.

Paul's research in undergraduate mathematics education focuses on problem solving and the transition to advanced mathematical thinking. As a graduate student, he worked on a grant for the Texas Math Initiative reviewing statewide professional development materials for mathematics teachers and was a consultant for the development of precalculus curriculum at Arizona State University. Paul also was a fellow of the Graduate Assistantships in Areas of National Need (GAANN) fellowship from the US Department of Education. He will pursue a university position in August of 2010 where he will continue research on undergraduate mathematics education. Paul also pursues the teaching and learning of religion and theology.