THE UNIQUENESS OF MINIMAL ACYCLIC COMPLEXES

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2009

ACKNOWLEDGEMENTS

This thesis is dedicated to the memory of the most beautiful mind I have ever known. DMH, Jr.

I am forever grateful to Dr. David Jorgensen. Not only is he the most patient advisor on Earth, he's also a really fun guy. Thank you for sharing your brilliance. My thankfulness also goes to Dr. Jianping Zhu and the entire UTA math faculty. Their dedication to the profession and accessibility to the students have set the highest standard for me to follow. In addition, I'd like to acknowledge my fellow graduate students, who contributed to making my stint at UTA such an incredible experience. I'm so glad we had this time together.

Marshall, my all-star, and Ranger, my joy, thank you for making this a wonderful life. Because of you, I am the luckiest mom in the world. It's true. I am also fortunate to be born into a lineage of strong women, the strongest of all being my mama, Trema Hughes. Thank you for being my rock. I'm grateful for my family of friendship. Lynette, Dudley, David, Craig, and Erin, thank you for loving and encouraging me. I am so very blessed.

The steadfast love of the Lord never ceases, his mercies never come to an end; they are new every morning; great is thy faithfulness. Lamentations 3:22-23

July 17, 2009

ABSTRACT

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The University of Texas at Arlington, 2009

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In this paper, we discuss conditions for uniqueness among minimal acyclic complexes of finitely generated free modules over a commutative local ring which share a common syzygy module. Although such uniqueness exists over Gorenstein rings, the question has been asked whether two minimal acyclic complexes in general can be isomorphic to the left and non-isomorphic to the right. We answer the question in the negative for certain cases, including periodic complexes, sesqui-acyclic complexes, and certain rings with radical cube zero. In particular, we investigate the question for graded algebras with Hilbert series $H_R(t) = 1 + et + (e-1)t^2$, and such monomial algebras possessing a special generator.

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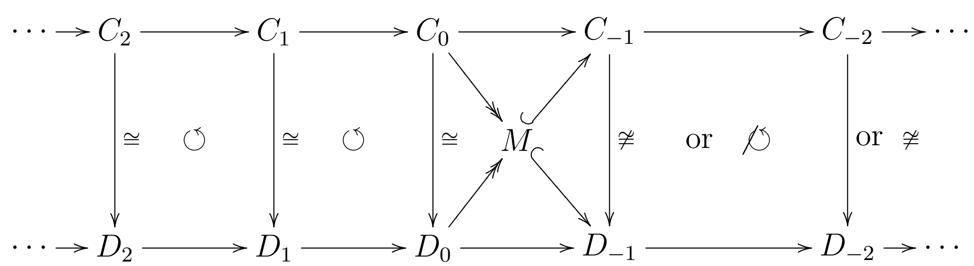
CHAPTER 1

INTRODUCTION

With R a local ring and M a finitely generated R-module, it is well known that minimal free resolutions of M are unique up to isomorphism. If we consider M as a syzygy module in a minimal acyclic complex, the left-hand side of the complex is precisely its unique free resolution. As a result, minimal acyclic complexes having a syzygy module M in common are "unique to the left" of that module. However, whether M can exist as a syzygy module in two non-isomorphic complexes has yet to be concluded. In other words, is it possible for the following to exist, where $\cong$ refers to the fact that modules are isomorphic and $\circlearrowleft$ refers to the fact that the squares commute:



Thus, one major focus of this paper is to determine the conditions under which a minimal acyclic complex is "unique to the right", or "has no branching".

Definition 1.1.1. A nonzero minimal acyclic complex of free modules

$$C: \quad \cdots \to C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{d_0^C} C_{-1} \to \cdots ,$$

branches if there exists a minimal acyclic complex

$$D: \quad \cdots \to D_1 \xrightarrow{d_1^D} D_0 \xrightarrow{d_0^D} D_{-1} \to \cdots$$

such that $C_{\geq s} \cong D_{\geq s}$ some $s \in \mathbb{Z}$, but $C \not\cong D$. If this is the case, we say that C branches at r if r is the minimal integer such that $C_{\geq r} \cong D_{\geq r}$.

We attempt to explore this concept for specific types of rings and complexes. After defining the question of branching and considering the necessary properties for the ring in chapter three, we examine the traits of a periodic complex in chapter four. Chapter five provides branching results that relate to periodicity. We define the concept of pushing a matrix in a minimal acyclic complex forward in chapter six, while chapters seven, eight, and nine explore the branching results of monomial algebras with Hilbert series $H_R(t) = 1 + et + (e-1)t^2$. We conclude the paper with a discussion of sesqui-acyclic complexes.

CHAPTER 2

PRELIMINARIES

Throughout this paper, assume R is a commutative ring with unity. This chapter contains preliminary concepts which will be used in the later chapters. Most of these definitions can be found in [3], [5], [6], and [8].

2.1 Properties and Classes of Rings

The types of rings we study in this paper are noetherian local rings that are also graded. To define this idea, we start by defining certain useful properties of rings, and use these definitions to expand our classification of rings.

2.1.1 Properties of General Commutative Rings

- A ring is *noetherian* if every ideal is finitely generated.
- The *Krull dimension* of a ring R, $\dim(R) = d$, also called simply the *dimension*, is the length of the longest chain of prime ideals in R,

$$p_0 \subset p_1 \subset \cdots \subset p_d.$$

  We always assume that $\dim R$ is finite.

2.1.2 Local Rings

- A *maximal ideal* of a ring is a proper ideal that is not contained in any other proper ideal.
- A Noetherian ring that has a unique maximal ideal $\mathfrak{m}$ is called a *local ring.* In this case, we let k denote the residue field $R/\mathfrak{m}$.

- The *embedding dimension* of a local ring R is the finite number $\operatorname{edim}(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. In other words, the minimal number of generators of the maximal ideal of R.
- The *codimension* of a local ring R is the number $\operatorname{edim}(R) - \dim(R)$.
- If M is a finitely generated R-module, a *regular sequence* on M is a sequence of elements $x_1, \ldots, x_n$ in R such that x_1 is a nonzerodivisor on M and each x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$, where $(x_1, \ldots, x_n)M \neq M$.
- Let M be a module over a local ring $(R, \mathfrak{m}, k)$. Then $\operatorname{Soc} M = (0 : \mathfrak{m})_M \cong \operatorname{Hom}_R(k, M)$ is called the *socle* of M. In other words, it is the annihilator in M of the maximal ideal.
- The *depth* of a local ring R is the length of the longest regular sequence on R contained in $\mathfrak{m}$.

2.1.3 Classes of Local Rings

- For a local ring $(R, \mathfrak{m})$ of dimension d, R is a *regular ring* if $\mathfrak{m}$ can be generated by exactly d elements. When $\dim_k(R) = \operatorname{edim}(R)$, R is regular.
- A local ring is a *complete intersection* if it is the quotient of a regular local ring by an ideal generated by a regular sequence.
- A local ring R is called a *Gorenstein ring* if it has finite injective dimension as a module over itself.
- When $\operatorname{depth}(R) = \dim(R)$, the ring R is a *Cohen-Macaulay ring,* which is the class of rings studied in this paper.

2.1.4 Graded Rings

In addition, we focus on noetherian local rings that are also graded. We only consider non-negatively graded rings.

- A *graded ring* is a ring R together with a direct sum decomposition $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, where R_i is called the ith homogeneous component of R, such that $R_i R_j \subset R_{i+j}$, for $i, j \geq 0$. The ring of polynomials $R = R_0[x_1, \cdots, x_e]$ is a graded ring where R_i consists of homogeneous polynomials of degree i and R_0 is a ring. The examples used in this paper are primarily quotients of polynomial rings.
- If R is a graded ring, then a *graded module* over R is an R-module M that can be decomposed as $M = \oplus_{i=0}^{\infty} M_i$, where M_i is called the ith homogeneous component of M, such that $R_i M_j \subset M_{i+j}$ for all i, j.
- Such a module is *free* if it has a linearly independent generating set over the graded ring consisting of homogeneous elements.
- If R is graded, $R_0 = k$ is a field, and M is a graded R-module, then the formal power series $H_M(t) = \Sigma_{i=0}^{\infty} \dim_k(M_i) t^i$ is called the *Hilbert Series of M.*
- The module $R^n := \oplus^n R$ is a graded free R-module with *standard basis* consisting of $e_i, 1 \leq i \leq n$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ has 1 in the ith component and zero elsewhere.
- A homomorphism $\phi : M \to N$ of graded R-modules is *homogeneous of degree d* if $\phi(M_i) \subseteq N_{i+d}$ for every i. An R-module homomorphism $\phi : R^n \to R^m$ of free R-modules is represented with respect to the standard bases of R^n and R^m by a matrix $[f_1 \cdots f_n]$ where the $f_i \in R^m$ are columns. If R is graded, then ϕ being homogeneous implies that the entries of $[f_1 \cdots f_n]$ are homogeneous elements of R.
- Moreover, if $R_0 = k$, then $\phi|_{R_i^n} : R_i^n \to R_{i+d}^m$ is also a linear transformation of vector spaces, and thus fixing vector space bases of R_i^n and R_{i+d}^m one may represent the linear transformation $\phi|_{R_i^n}$ by a matrix T, the associated linear transformation of ϕ of degree i.

2.2 Homological Algebra

We now give some general definitions from homological algebra, which is the study of chain complexes of algebraic structures, in our case, R-modules.

2.2.1 Complexes

- A *complex* is a sequence of R-modules and R-linear maps

$$C: \quad \cdots \to C_{i+1} \xrightarrow{d_{i+1}^C} C_i \xrightarrow{d_i^C} C_{i-1} \to \cdots$$

  with $d_i^C \circ d_{i+1}^C = 0$ for all i, alternately, the image of d_{i+1}^C is contained in the kernel of d_i^C. The maps d_i^C are called the *differentials*.

- Let C and D be complexes. A *homomorphism of complexes* $f : C \to D$ is a set of homomorphisms $f_n : C_n \to D_n$ such that for every n the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n & \xrightarrow{d_n^C} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & D_n & \xrightarrow{d_n^D} & D_{n-1} & \longrightarrow & \cdots , \end{array}$$

  more formally, $f_{n-1}d_n^C = d_n^D f_n$.

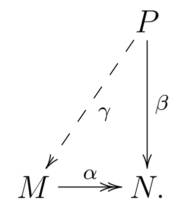
- A complex is *exact at* C_i if image $d_{i+1}^C = \ker d_i^C$. If C_i is exact for each i, then the complex is said to be *exact*.

- The *homology* $H_i(C)$ at C_i is the module $\ker d_i^C / \operatorname{image} d_{i+1}^C$.

- The homology of the complex is given by $H(C) = \oplus H_i(C)$.

2.2.2 Projective and Free Resolutions

- An R-module P is *projective* if for every epimorphism of R-modules $\alpha : M \to N$ and every map $\beta : P \to N$, there exists a map $\gamma : P \to M$ such that $\beta = \alpha\gamma$, as in the following figure:



- Free modules are projective. To see this, if P is free on a set of generators p_i, then choose elements q_i of M that map to $\beta(p_i) \in N$, and let γ send p_i to q_i.
- A *projective resolution* of an R-module M is a complex

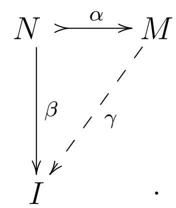
$$F : \cdots \to F_n \xrightarrow{d_n^F} \cdots \to F_1 \xrightarrow{d_1^F} F_0$$

of projective R-modules such that $\operatorname{Coker} d_1^F \cong M$ and F is an exact complex. If, in addition, each F_i is free, F is called a *free resolution of* M. If M is finitely generated and R is noetherian, then each F_i can be chosen to be finitely generated. Free resolutions serve to compare projective modules with free modules.
- If for some $n < \infty$, we have $F_{n+1} = 0$, but $F_i \neq 0$ for $0 \leq i \leq n$, then F is a *finite resolution of length n.*
- Assume that R is either local or graded with (homogeneous) maximal ideal $\mathfrak{m}$. Then F, as above, is *minimal* if entries of each matrix representing the d_i^F are in $\mathfrak{m}$.
- Assuming that $\mathfrak{m}$ is finitely generated and R is neotherian, the $\operatorname{rank} F_i = b_i$ are called the *Betti numbers* of M.
- The *Poincare series* of the $R-$module M is the power series in t,

$$P_M^R(t) = \Sigma_{i \geq 0} b_i t^i = b_0 + b_1 t + b_2 t^2 + \cdots ,$$

where the b_i are the Betti numbers.

- An R-module I is *injective* if for every monomorphism of R-modules $\alpha : N \to M$ and every homomorphism of R-modules $\beta : N \to I$, there exists a homomorphism of R-modules $\gamma : M \to I$ such that $\beta = \gamma\alpha$, as in the following figure:



- If M is an R-module, we may embed M in an injective module I_0. We may then embed the cokernel, I_0/M, in an injective module I_1. Continuing in this way, we get an *injective resolution*

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$

of M; that is, an exact sequence of the given form in which all the I_i are injectives.

2.2.3 Minimal Acyclic Complexes

- An *acyclic complex* of free R-modules is a complex

$$C \qquad \cdots \to C_2 \xrightarrow{d_2^C} C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{d_0^C} C_{-1} \xrightarrow{d_{-1}^C} C_{-2} \to \cdots$$

with C_i finitely generated and free for each i and $\mathrm{H}(C) = 0$.

- If M and N are R-modules, then $\mathrm{Hom}_R(M, N)$ is the abelian group of all homomorphisms from M to N. Since R is commutative, it is itself an R-module by the property $(rf)(m) = rf(m) = f(rm)$ for $r \in R$ and $f \in \mathrm{Hom}_R(M, N)$.

- If

$$f : M \to N$$

is a homomorphism of R-modules, then we have the mapping

$$f^* : \mathrm{Hom}(N, R) \to \mathrm{Hom}(M, R),$$

where

$$f^*(\theta) = \theta \circ f.$$

We set $M^* = \mathrm{Hom}(M, R)$ and call M^* the *dual* of M and f^* the *dual* of f. We have the dual $C^* = \mathrm{Hom}_R(C, R)$:

$$C^* = \cdots C^*_{n-1} \xrightarrow{d^*_n} C^*_n \xrightarrow{d^*_{n+1}} C^*_{n+1} \to \cdots .$$

- If $R^n \xrightarrow{f} R^m$ is represented by A with respect to the dual bases of R^n and R^m, then $\mathrm{Hom}(R^m, R) \xrightarrow{f^*} \mathrm{Hom}(R^n, R)$ is represented with respect to the standard bases of $\mathrm{Hom}(R^m, R)$ and $\mathrm{Hom}(R^n, R)$ by A^T.
- An acyclic complex of free R-modules C satisfying $H(C^*) = 0$, where $C^* = \mathrm{Hom}_R(C, R)$ is called *totally acyclic*, or a *complete resolution.*
- For an acyclic complex C, if we have $H_i(C^*) = 0$ for $i \gg 0$ then C is called a *sesqui-acyclic* complex.
- Assume $(R, \mathfrak{m})$ is local with maximal ideal $\mathfrak{m}$, or assume $(R, \mathfrak{m})$ is graded with homogeneous maximal ideal $\mathfrak{m}$. An acyclic complex is *minimal* if image $d_i \subseteq \mathfrak{m}R^{d_{i-1}}$ for all i.
- The *shift functor*, notated Σ^r, takes complexes over R to complexes over R, acting on both the modules and the morphisms, i.e. $C \mapsto \Sigma^r C$, and $f : C \to D \mapsto \{\Sigma^r f : \Sigma^r C \to \Sigma^r D\}$, where $(\Sigma^r C)_i = C_{i-r}$, and $d_i^{\Sigma^r C} = (-1)^r d^C_{i-r}$.
- For a complex C we define the *truncated complex* $C_{\geq r}$ to be the complex with $(C_{\geq r})_i = C_i$ if $i \geq r$, and 0 if $i < r$.

2.2.4 Syzygies

- The element $z \in M$ is called a *syzygy* of an R-module homomorphism $\phi : M \to N$ if $z \in \mathrm{Ker}(\phi)$. If $\phi : R^n \to R^m$ is represented by $[s_1 \cdots s_n]$ then a syzygy takes the form $(c_1, \cdots, c_n) \in R^n$ such that $c_1 f_1 + \cdots + c_n f_n = 0$ in R^m.
- Let M and N be finitely generated R-modules. Then M is called an *nth syzygy module* (of N) if there is an exact sequence

$$\cdots \to C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1^C} C_0 \to N \to 0$$

  with the C_i finitely generated and free, and $M \cong \mathrm{image}\, d_n^C$.
- We say that M is an *infinite syzygy module* if there exists a minimal acyclic complex of projective R-modules

$$\cdots \to C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{d_0^C} C_{-1} \to \cdots$$

  such that $M \cong \mathrm{image}\, d_i^C$ for some $i \in \mathbb{Z}$.

2.2.5 Koszul Homology

- The *tensor product of two complexes*

$$C : \cdots \to C_i \xrightarrow{\alpha_i} C_{i+1} \to \cdots$$

  and

$$D : \cdots \to D_i \xrightarrow{\beta_i} D_{i+1} \to \cdots$$

  is defined to be the complex

$$C \otimes D : \cdots \to \bigoplus_{i+j=k} C_i \otimes D_j \xrightarrow{d_k} \bigoplus_{i+j=k-1} C_i \otimes D_j \to \cdots$$

  where the map d_k on $C_i \otimes D_j$ (with $i + j = k$) is given by

$$d_k : \bigoplus_{i+j=k} F_i \otimes D_j \to \bigoplus_{i+j=k-1} C_i \otimes D_j,$$

where for $a \otimes b \in C_i \otimes D_j$, the differentials are given by

$$d_k(a \otimes b) = d_i^C(a) \otimes b + (-1)^i a \otimes d_j^D(b).$$

- Let x be an element in R. The *Koszul complex* $K(x; R)$ on x is the complex

$$K(x) : 0 \to R \xrightarrow{x} R \to 0,$$

  with R situated in homological degrees 0 and 1.

- Suppose we are given a sequence $\mathbf{x} = x_1, \dots, x_n$ of elements in R. The *Koszul complex on* $\mathbf{x}$ is the complex

$$K(\mathbf{x}; R) = K(x_1; R) \otimes_R \cdots \otimes_R K(x_n; R).$$

  The nonzero modules in this complex are situated in degrees 0 to n.

- The *Koszul homology* of a local ring R, is the homology of the Koszul complex on a minimal set of generators $\mathbf{x} = x_1, \dots, x_n$ of the maximal ideal:

$$H(\mathbf{x}; R) = H(K(\mathbf{x}; R)).$$

2.2.6 Linkage

- Let I and J be ideals in a ring R. Then I and J are said to be *linked,* written $I \sim J$, if there exists a regular sequence $g_1, \dots, g_d$ in $I \cap J$ such that $(g_1, \dots, g_d) : I = J$ and $(g_1, \dots, g_d) : J = I$.
- We also say I is one link from a complete intersection if $I \sim J$ and J is generated by a regular sequence.

CHAPTER 3

DEFINING THE QUESTION

We now pose the issue of branching in the form of a question, and attempt to answer it for specific types of rings and complexes throughout this paper.

3.1 The Question

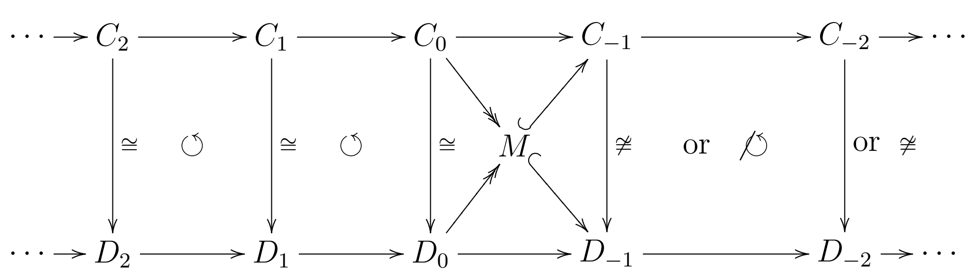
Question 3.1.1. Given a nonzero minimal acyclic complex of free modules

$$A: \quad \cdots \to A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \to \cdots ,$$

does there exist a minimal acyclic complex of free modules

$$B: \quad \cdots \to B_1 \xrightarrow{f_1} B_0 \xrightarrow{f_0} B_{-1} \to \cdots$$

such that $A_{\geq s} \cong B_{\geq s}$ some $s \in \mathbb{Z}$, but $A \not\cong B$?



Before exploring this question, we give conditions for R for the remainder of this paper.

3.2 Cohen-Macaulay Rings

We have the classical chain of inclusion for classes of local rings:

$$regular \subset complete\,intersections \subset Gorenstein \subset CohenMacaulay$$

In the case of a Gorenstein ring, any acyclic complex of free modules is totally acyclic. We point out in the final chapter that totally acyclic complexes do not have branching. Thus Question 3.1.1 is answered for Gorenstein rings. Gorenstein rings are part of the larger class of Cohen-Macaulay rings. Therefore, the next logical case to consider is that of Cohen-Macaulay rings.

3.3 Dimension Zero

A natural starting point is to investigate rings of dimension zero. One justification for this is the following fact. Consider a minimal acyclic complex C over a Cohen-Macaulay ring R. If $\mathbf{x} = x_1, \cdots, x_d$ is a maximal R-sequence, then $R/(\mathbf{x})$ is a ring of dimension zero. If C is a minimal acyclic complex over R, then $C \otimes R/(\mathbf{x})$ is a minimal acyclic complex over $R/(\mathbf{x})$.

We note, however, that it is possible for two non-isomorphic acyclic complexes over R to become isomorphic when modding out by an R-sequence. As an example, let $R = k[[x, y]]/(x^2 - y^2)$. Consider the minimal acyclic complexes over R :

$$C \qquad \cdots \to C_2 \xrightarrow{x-y} C_1 \xrightarrow{x+y} C_0 \xrightarrow{x-y} C_{-1} \to \cdots,$$

and

$$D \qquad \cdots \to D_2 \xrightarrow{x+y} D_1 \xrightarrow{x-y} D_0 \xrightarrow{x+y} D_{-1} \to \cdots,$$

where $C_i \cong R$ and $D_i \cong R$, all $i \in \mathbb{Z}$.

To see that C and D are not isomorphic, assume the following square commutes:

$$\begin{array}{ccc} C_1 & \xrightarrow{x+y} & C_0 \\ \downarrow v & & \downarrow u \\ D_1 & \xrightarrow{x-y} & D_0, \end{array}$$

where u and v are units in R. On the one hand,

$$1 \mapsto x + y \mapsto u(x + y),$$

on the other hand

$$1 \mapsto v \mapsto v(x-y).$$

However, for no units u and v do we have

$$u(x+y) = v(x-y),$$

which contradicts commutativity. Thus C and D are not isomorphic complexes.

Note that y is a non-zero divisor on R. Quotienting by (y),

$$\begin{array}{lccccccccc}
C/(y): & \cdots \longrightarrow & C_2/(y) & \xrightarrow{\quad x \quad} & C_1/(y) & \xrightarrow{\quad x \quad} & C_0/(y) & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 1} & \circlearrowleft & & \\
D/(y): & \cdots \longrightarrow & D_2/(y) & \xrightarrow{\quad x \quad} & D_1/(y) & \xrightarrow{\quad x \quad} & D_0/(y) & \longrightarrow \cdots,
\end{array}$$

we obtain

$$C/(y) \cong D/(y),$$

giving the minimal acyclic complex

$$\cdots \longrightarrow E_2 \xrightarrow{x} E_1 \xrightarrow{x} E_0 \xrightarrow{x} E_{-1} \rightarrow \cdots,$$

over $R/(y) \cong k[x]/(x^2)$.

Consider a minimal acyclic complex C over a Cohen-Macaulay ring R. If $\mathbf{x} = x_1, \cdots, x_d$ is a maximal R-sequence, then $R/(\mathbf{x})$ is a ring of dimension zero. If C is a minimal acyclic complex over R, then $C \otimes R/(\mathbf{x})$ is a minimal acyclic complex over $R/(\mathbf{x})$. Many properties of C over R are transferred to those of $C \otimes R/(\mathbf{x})$ over $R/(\mathbf{x})$. In particular, non-uniqueness for minimal acyclic complexes over $R/(x)$, implies the same for R. If we can answer question 3.1.1 positively for $R/(x), x$ a non-zero divisor, then we can also answer it positively for R.

3.4 $\mathfrak{m}^3 = 0$

Assuming R has dimension zero, then $\mathfrak{m}^n = 0$ for some n. It turns out, the first interesting case to investigate the uniqueness of minimal acyclic complexes is $\mathfrak{m}^3 = 0$. For iff $n = 1$, then R is a field, and the only minimal acyclic complexes is the zero complex. For the $n = 2$ case, nonzero minimal acyclic complexes also do not exist. To see this, suppose $(R, \mathfrak{m})$ has $\mathfrak{m}^2 = 0$. Let Ω be a finitely generated R-module with $\mathfrak{m}\Omega = 0$, that is, Ω is a finite dimensional vector space over $k = R/\mathfrak{m}$, and

$$0 \to \Omega' \to F \to \Omega \to 0,$$

an exact sequence where F is free, $\mu(F) = \mu(\Omega)$, where $\mu(X)$ denotes the minimal number of generators of the module X. By minimality, $\Omega' \subseteq \mathfrak{m}F$. So $\mathfrak{m}\Omega' \subseteq \mathfrak{m}^2 F = 0$, which gives $\mathfrak{m}\Omega' = 0$.

We know $\dim_k \Omega' = \operatorname{length} F - \dim_k \Omega$, and $\operatorname{length} F = \operatorname{rank} F(\operatorname{length} R)$, giving

$$\dim_k \Omega' = \operatorname{rank} F(\operatorname{length} R) - \dim_k \Omega.$$

By exactness, $\operatorname{rank} F = \dim_k \Omega$, so

$$\dim_k \Omega' = (\dim_k \Omega)(\operatorname{length} R) - \dim_k \Omega$$

$$= \dim_k \Omega(\operatorname{length} R - 1).$$

Now take a minimal acyclic complex

$$C \qquad \cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots ,$$

and apply inductively to C, we have the short exact sequence

$$0 \to \Omega_{i+1} \to C_i \to \Omega_i \to 0,$$

$$\operatorname{rank} C_i = \dim \Omega_i = \dim \Omega_{i-j}(\operatorname{length} R - 1)^j,$$

for all $j \leq 0$. This is absurd unless length $R \leq 2$. If the length of R is 1, then R is a field. If the length of R is 2, then R is isomorphic to the ring $k[x]/(x^2)$, which is Gorenstein. Any acyclic complex over a Gorenstein ring is totally acyclic, and therefore, unique.

For the $\mathfrak{m}^3 = 0$ case, it is possible to construct minimal acyclic complexes over non-Gorenstein rings, therefore this is the first case where 3.1.1 is open. For example, over the ring $R = k[x, y, z]/(x^2, y^2, z^2, xy + yz)$, we have minimal acyclic complexes

$$\cdots \to R \xrightarrow{z} R \xrightarrow{z} R \to \cdots,$$

and

$$\cdots \to R^2 \xrightarrow{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}} R^2 \to \cdots.$$

3.5 Required Properties

The following theorem from [4] maintains that these complexes can exist only if R has the following properties:

Theorem 3.5.1. *Let $(R, \mathfrak{m}, k)$ be a local ring that is not Gorenstein and has $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If there exists a non-zero minimal acyclic complex A of finitely generated free R-modules, then the ring has the following properties:*

(a) $(0 : \mathfrak{m}) = \mathfrak{m}^2$.

(b) $e = r + 1$ *with* length $R = 2e$.

(c) Poincare series $P_k^R(t) = 1/(1-t)(1-rt)$.

It is assumed that the rings we study in the following sections will be $\mathfrak{m}^3 = 0$ and possess these properties.

CHAPTER 4

PERIODICITY OF MINIMAL ACYCLIC COMPLEXES

In the next chapter, we will prove, among other results, that periodic complexes have no branching. Once periodicity is established, it of course remains periodic to the left. It is not known, however, that a minimal complex with periodicity to the left must be periodic everywhere.

4.1 Family of Complexes

This first lemma builds a new complex from a family of isomorphic complexes. We will use the representative complex to establish isomorphisms among all complexes containing the given period.

Lemma 4.1.1. *Let* $\{A^i\}_{i\in\mathbb{Z}}$ *be a family of complexes, and* $\{f^i : A^i \to A^{i+1}\}_{i\in\mathbb{Z}}$ *a family of chain isomorphisms. For any sequence* $\{n_j\}_{j\in\mathbb{Z}}$*, define a new complex*

$$A\{n_j\}: \qquad \cdots \to A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} \cdots$$

where $A\{n_j\}_i = A_i^{n_i}$ *and*

$$d_i = \begin{cases} (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}d_i^{n_i} & \text{for } n_{i-1} < n_i \\ f_{i-1}^{n_{i-1}-1}f_{i-1}^{n_{i-1}-2}\cdots f_{i-1}^{n_i}d_i^{n_i} & \text{for } n_{i-1} \geq n_i \end{cases} \tag{4.1.1.1}$$

for $i \in \mathbb{Z}$*. Then* $A\{n_j\} \cong A^i$ *for all* $i \in \mathbb{Z}$*.*

Proof. Consider the diagram:

$$
\begin{array}{ccccccccccc}
\vdots & & & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
A^{-1}: & & \cdots & \longrightarrow & A_1^{-1} & \xrightarrow{d_1^{-1}} & A_0^{-1} & \xrightarrow{d_0^{-1}} & A_{-1}^{-1} & \xrightarrow{d_{-1}^{-1}} & \cdots \\
\downarrow{\scriptstyle f^{-1}} & & & & \downarrow{\scriptstyle f_1^{-1}} & & \downarrow{\scriptstyle f_0^{-1}} & & \downarrow{\scriptstyle f_{-1}^{-1}} & & \\
A^{0}: & & \cdots & \longrightarrow & A_1^{0} & \xrightarrow{d_1^{0}} & A_0^{0} & \xrightarrow{d_0^{0}} & A_{-1}^{0} & \xrightarrow{d_{-1}^{0}} & \cdots \\
\downarrow{\scriptstyle f^{0}} & & & & \downarrow{\scriptstyle f_1^{0}} & & \downarrow{\scriptstyle f_0^{0}} & & \downarrow{\scriptstyle f_{-1}^{0}} & & \\
A^{1}: & & \cdots & \longrightarrow & A_1^{1} & \xrightarrow{d_1^{1}} & A_0^{1} & \xrightarrow{d_0^{1}} & A_{-1}^{1} & \xrightarrow{d_{-1}^{1}} & \cdots \\
\downarrow{\scriptstyle f^{1}} & & & & \downarrow{\scriptstyle f_1^{1}} & & \downarrow{\scriptstyle f_0^{1}} & & \downarrow{\scriptstyle f_{-1}^{1}} & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & .
\end{array}
$$

Since each of the A^i are isomorphic to each other, it suffices to show the new complex A is isomorphic to the complex A^0. To simplify notation, let $A_i = (A\{n_j\})_i = A_i^{n_i}$. We need to define maps $f_i : A_i^0 \to A_i$ and show that the squares in the following diagram commute:

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & A_i^0 & \xrightarrow{d_i^0} & A_{i-1}^0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \longrightarrow & A_i & \xrightarrow{d_i} & A_{i-1} & \longrightarrow & \cdots .
\end{array}
$$

In other words, show $f_{i-1}d_i^0 = d_i^{n_i} f_i$. For fixed i we have six cases to consider:

(1) $n_i \geq n_{i-1} \geq 0$

(2) $n_i \geq 0 \geq n_{i-1}$

(3) $0 \geq n_i \geq n_{i-1}$

(4) $n_{i-1} \geq n_i \geq 0$

(5) $n_{i-1} \geq 0 \geq n_i$

(6) $0 \geq n_{i-1} \geq n_i$.

We examine the first three cases, and recognize the remaining three are symmetrically similar.

Case 1: $n_i \geq n_{i-1} \geq 0.$

From commutativity of

$$\begin{array}{ccccccc}
\cdots \longrightarrow & A_i^{n_{i-1}} & \xrightarrow{d_i^{n_i-1}} & A_{i-1}^{n_{i-1}} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
& \vdots & & \vdots & \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & A_i^{n_i} & \xrightarrow{d_i^{n_i}} & A_{i-1}^{n_i} & \longrightarrow \cdots .
\end{array}$$

we rewrite $d_i : A_i^{n_i} \to A_{i-1}^{n_{i-1}}$ as

$$d_i = (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1} \cdots (f_{i-1}^{n_i-1})^{-1} d_i^{n_i} = d_i^{n_{i-1}}(f_i^{n_{i-1}})^{-1}(f_i^{n_{i-1}+1})^{-1} \cdots (f_i^{n_i-1})^{-1}.$$

The chain maps $f_i : A_i^0 \to A_i^{n_i}$ are given by

$$f_i = f_i^{n_i-1} f_i^{n_i-2} \cdots f_i^0.$$

We need to show that $f_{i-1}d_i^0 = d_i f_i$:

$$\begin{aligned}
f_{i-1}d_i^0 &= f_{i-1}^{n_{i-1}-1} f_{i-1}^{n_{i-1}-2} \cdots f_{i-1}^0 d_i^0 \\
&= f_{i-1}^{n_{i-1}-1} f_{i-1}^{n_{i-1}-2} \cdots f_{i-1}^1 d_i^1 f_i^0 \\
&= f_{i-1}^{n_{i-1}-1} f_{i-1}^{n_{i-1}-2} \cdots d_i^2 f_i^1 f_i^0 \\
&\ \vdots \\
&= f_{i-1}^{n_{i-1}-1} d_i^{n_{i-1}-1} f_i^{n_{i-1}-2} \cdots f_i^1 f_i^0 \\
&= d_i^{n_{i-1}} f_i^{n_{i-1}-1} f_i^{n_{i-1}-2} \cdots f_i^1 f_i^0 .
\end{aligned}$$

Since $d_i^{n_{i-1}} = d_i^{n_i} f_i^{n_i-1} \cdots f_i^{n_{i-1}+1} f_i^{n_{i-1}}$, we then have

$$f_{i-1}d_i^0 = d_i^{n_i} f_i^{n_i-1} \cdots f_i^{n_{i-1}+1} f_i^{n_{i-1}} f_i^{n_{i-1}-1} f_i^{n_{i-1}-2} \cdots f_i^1 f_i^0 = d_i f_i.$$

Case 2: $n_i \geq 0 \geq n_{i-1}$.

Define $f_i : A_i^0 \to A_i^{n_i}$ as $f_i = f_i^{n_i-1} f_i^{n_i-2} \cdots f_i^0$. Since $n_{i-1} \leq 0$, define

$$f_{i-1} = (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1} \cdots (f_{i-1}^{-1})^{-1}.$$

We again need $f_{i-1}d_i^0 = d_i f_i$:

$$d_i f_i = (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{-1})^{-1}(f_{i-1}^{0})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}d_i^{n_i}f_i^{n_i-1}f_i^{n_i-2}\cdots f_i^0$$

$$= (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{-1})^{-1}(f_{i-1}^{0})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}f_{i-1}^{n_i-1}d_i^{n_i-1}f_i^{n_i-2}\cdots f_i^0$$

$$= (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{-1})^{-1}(f_{i-1}^{0})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}f_{i-1}^{n_i-1}f_{i-1}^{n_i-2}d_i^{n_i-2}\cdots f_i^0$$

$$\vdots$$

$$= (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{-1})^{-1}(f_{i-1}^{0})^{-1}\cdots(f_{i-1}^{0})d_i^0$$

$$= (f_{i-1}^{n_{i-1}}\cdots(f_{i-1}^{-1})^{-1}d_i^0 = f_{i-1}d_i^0.$$

Case 3: $0 \geq n_i \geq n_{i-1}$.

Define $f_i = (f_i^{n_i})^{-1}\cdots(f_i^{-2})^{-1}(f_i^{-1})^{-1}$ and

$$f_{i-1} = (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{n_i})^{-1}(f_{i-1}^{n_i+1})^{-1}\cdots(f_{i-1}^{-2})^{-1}(f_{i-1}^{-1})^{-1}.$$

So we have,

$$d_i f_i = (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}d_i^{n_i}(f_i^{n_i})^{-1}\cdots(f_i^{-2})^{-1}(f_i^{-1})^{-1}$$

$$= (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}(f_{i-1}^{n_i})^{-1}d_i^{n_i+1}\cdots(f_i^{-2})^{-1}(f_i^{-1})^{-1}$$

$$\vdots$$

$$= (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}(f_{i-1}^{n_i})^{-1}(f_{i-1}^{n_i+1})^{-1}\cdots d_i^{-1}(f_i^{-1})^{-1}$$

$$= (f_{i-1}^{n_{i-1}})^{-1}(f_{i-1}^{n_{i-1}+1})^{-1}\cdots(f_{i-1}^{n_i-1})^{-1}(f_{i-1}^{n_i})^{-1}(f_{i-1}^{n_i+1})^{-1}\cdots(f_{i-1}^{-1})^{-1}d_i^0 = f_{i-1}d_i^0.$$

□

4.2 Periodicity

Definition 4.2.1. $(\Sigma^p C)_n = C_{n-p}$. A complex C is *periodic* of period p if there exists an isomorphism $f : C \to \Sigma^p C$ and $C \ncong \Sigma^s C$ for $0 < s < p$.

We now establish that a periodic complex is isomorphic to a complex PA which is periodic in a stronger sense.

Lemma 4.2.2. *If a complex C is periodic of period p, then it is isomorphic to the complex PC defined by $(PC)_i = C_j$, $d_i^{PC} = d_j^C$, provided $i \equiv j \mod p$ with $i \leq j \leq p-1$, and $d_i^{PC} = d_0 f_p^{-1}$ for $i \equiv 0 \mod p$. In other words,*

$$
\begin{array}{ccccccccccccc}
\cdots \to & (PC)_p & \xrightarrow{d_p^{PC}} & (PC)_{p-1} & \xrightarrow{d_{p-1}^{PC}} & (PC)_{p-2} & \to \cdots \to & (PC)_1 & \xrightarrow{d_1^{PC}} & (PC)_0 & \xrightarrow{d_0^{PC}} & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & C_0 & \xrightarrow{d_0^C f^{-1}} & C_{p-1} & \xrightarrow{d_{p-1}^C} & C_{p-2} & \longrightarrow \cdots \longrightarrow & C_1 & \xrightarrow{d_1^C} & C_0 & \xrightarrow{d_0^C f_p^{-1}} & \cdots .
\end{array}
$$

Proof. By definition, $C \xrightarrow{f} \Sigma^p C$ is an isomorphism, and $(\Sigma^p C)_n = C_{n-p}$. Consider the diagram:

$$
\begin{array}{lccccccc}
\vdots & & & & & & & \vdots \\
\big\downarrow & & & & & & & \big\downarrow{\scriptstyle (\Sigma^{-3p}f)_{-2p}} \\
\Sigma^{-2p}C: & & & & & & \cdots \to C_p \to \cdots \to & C_0 \to \cdots \\
\big\downarrow{\scriptstyle \Sigma^{-2p}f} & & & & & & \big\downarrow{\scriptstyle (\Sigma^{-2p}f)_{-p}} & \\
\Sigma^{-p}C: & & & & & \cdots \to C_p \to \cdots \to & C_0 \to \cdots & \\
\big\downarrow{\scriptstyle \Sigma^{-p}f} & & & & & \big\downarrow{\scriptstyle (\Sigma^{-p}f)_{0}} & & \\
C: & & & & \cdots \to C_p \to \cdots \to & C_0 \to \cdots & & \\
\big\downarrow{\scriptstyle f} & & & & \big\downarrow{\scriptstyle f_p} & & & \\
\Sigma^{p}C: & & & \cdots \to C_p \to \cdots \to & C_0 \to \cdots & & & \\
\big\downarrow{\scriptstyle \Sigma^{p}f} & & & \big\downarrow{\scriptstyle (\Sigma^{p}f)_{2p}} & & & & \\
\Sigma^{2p}C: & C_p f \to \cdots \to & & C_0 \to \cdots & & & & \\
\big\downarrow{\scriptstyle \Sigma^{2p}f} & \big\downarrow{\scriptstyle (\Sigma^{2p}f)_{3p}} & & & & & & \\
\vdots & \vdots & & & & & & .
\end{array}
$$

By Lemma 4.1.1, C is isomorphic to the complex

$$
\cdots \to C_0 \xrightarrow{d_0^C f_p^{-1}} C_{p-1} \xrightarrow{d_{p-1}^C} C_{p-2} \xrightarrow{d_{p-2}^C} \cdots \to C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{d_0^C f_p^{-1}} C_{p-1} \xrightarrow{d_{p-1}^C} C_{p-2} \xrightarrow{d_{p-2}^C} \cdots
$$

which is what we wanted to show. □

Finally, we show that once periodicity is established, the length of the period does not change.

Lemma 4.2.3. *If D is a periodic complex of period p, then for all $r \in \mathbb{Z}$, there is an isomorphism $D_{\geq r} \to (\Sigma^p D)_{\geq r}$ and $D_{\geq r} \ncong (\Sigma^q D)_{\geq r}$ for $q < p$.*

Proof. Assume r is the smallest integer such that there exists $D_{\geq r} \cong (\Sigma^q D)_{\geq r}$ for $q < p$. Since D is periodic of period p, there exists a chain isomorphism $f : D \to \Sigma^p D$ and $D \ncong \Sigma^s D$ for any $0 < s < p$. From f, we see that $D_{\geq r} \cong (\Sigma^p D)_{\geq r}$:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & D_{r+i} & \longrightarrow \cdots \longrightarrow & D_{r+1} & \longrightarrow & D_r & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f_{r+i}} & & \Big\downarrow{\scriptstyle f_{r+1}} & & \Big\downarrow{\scriptstyle f_r} & & & & \\
\cdots & \longrightarrow & D_{r-p+i} & \longrightarrow \cdots \longrightarrow & D_{r-p+1} & \longrightarrow & D_{r-p} & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

There exists a chain isomorphism $g : D_{\geq r} \to (\Sigma^q D)_{\geq r}$, and we have the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & D_{r-p+1} & \longrightarrow & D_{r-p} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f_{r+1}^{-1}} & & \Big\downarrow{\scriptstyle f_r^{-1}} & & & & \\
\cdots & \longrightarrow & D_{r+1} & \longrightarrow & D_r & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle g_{r+1}} & & \Big\downarrow{\scriptstyle g_r} & & & & \\
\cdots & \longrightarrow & D_{r-q+1} & \longrightarrow & D_{r-q} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f_{r-q+1}} & & \Big\downarrow{\scriptstyle f_{r-q}} & & & & \\
\cdots & \longrightarrow & D_{r-q-p+1} & \longrightarrow & D_{r-q-p} & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

Since each of the squares commute, $D_{\geq r-p} \cong (\Sigma^q D)_{r-p}$, which contradicts the choice of r. $\square$

CHAPTER 5

UNIQUENESS RESULTS THAT FOLLOW FROM PERIODICITY

In this chapter, we first answer 3.1.1 in the negative for periodic complexes. From this result, we additionally prove that complexes over k-algebras with $\mathfrak{m}^3 = 0$ and Hilbert series $H_R(t) = 1 + et + (e-1)t^2$ such that k is a finite field, and complexes over rings of codimension ≤ 3 are also unique to the right.

5.1 Periodic Complexes

Theorem 5.1.1. *Let C be a periodic complex of periodicity p and D be a periodic complex of periodicity q such that $C_{\geq r} \cong D_{\geq r}$. Then $C \cong D$.*

Proof. Since C is periodic, $C \cong \Sigma^p C$. Likewise, $D \cong \Sigma^q D$. We have

$$D_{\geq r+p} \cong C_{\geq r+p} \cong (\Sigma^p C)_{\geq r+p} \cong (\Sigma^p D)_{\geq r+p}.$$

By 4.2.3, this implies $q \leq p$ since q is the smallest q such that $D_{\geq r+p} \cong \Sigma^q D_{\geq r+p}$ for all r. Symmetrically,

$$C_{r+p} \cong D_{r+p} \cong (\Sigma^q D)_{\geq r+p} \cong (\Sigma^q C)_{\geq r+p}.$$

Again, by 4.2.3, $p \leq q$. Thus, $p = q$ and the complexes have the same period. This gives by Lemma 4.0.3 C isomorphic to the complex

$$\cdots \to C_r \to C_{r+p} \xrightarrow{d^C_{r+2}} C_{r+1} \xrightarrow{d^C_{r+1}} C_r \xrightarrow{d_r} C_{r+p} \to \cdots,$$

with D isomorphic to

$$\cdots \to D_r \to D_{r+p} \to \cdots \xrightarrow{d^D_{r+2}} D_{r+1} \xrightarrow{d^D_{r+1}} D_r \to D_{r+p} \to \cdots.$$

Since $D_r \cong C_r$, and the complexes have the maps and modules repeated:

$$\begin{array}{ccccccccccccc} \cdots \to & C_r & \to & C_{r+1} & \to \cdots \to & C_{r+1} & \xrightarrow{d^C_{r+1}} & C_r & \to & C_{r+p} & \to \cdots \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\ \cdots \to & D_r & \to & D_{r+1} & \to \cdots \to & D_{r+1} & \xrightarrow{d^D_{r+1}} & D_r & \to & D_{r+p} & \to \cdots , \end{array}$$

they are isomorphic everywhere. □

5.2 k-algebras with $\mathfrak{m}^3 = 0$ and k finite

Lemma 5.2.1. *If $\mathfrak{m}^3 = 0$ and $|k| < \infty$, then any minimal acyclic complex C is periodic.*

Proof. For a minimal acyclic complex with $\mathfrak{m}^3 = 0$, the negative Betti numbers are constant by [11], say equal to n. The "negative differentials" are thus represented by $n \times n$ matrices with linear entries. For the codimension of R being e, there are $|k|^e$ possible linear forms, allowing $(k^e)^{n^2}$ possible matrices representing the d^i. Since there are infinitely many d_i for $i \ll 0$,we must have $d_i^C = d_j^C$ for some $i \neq j$. Assume $i < j$. Then $f : C_{\geq i} \to \Sigma^{j-i} C_{\geq i}$ is a chain isomorphism. Thus periodicity is established. □

Since periodicity is established for minimal acyclic complexes over local rings with $\mathfrak{m}^3 = 0$ over a finite field, we have no branching.

Theorem 5.2.2. *For minimal acyclic complexes C and D, let $(R, \mathfrak{m}, k)$ be a local ring such that $\mathfrak{m}^3 = 0$. If $|k| < \infty$, then $C_{\geq r} \cong D_{\geq r}$ implies $C \cong D$.*

Proof. From Theorem 5.2.1, a minimal acyclic complex over R is periodic, and by Theorem 5.1.1, $C \cong D$. □

5.3 Codimension 3

This next result involving minimal acyclic complexes with bounded Betti numbers follows easily from a result by Avramov, [2]..

Theorem 5.3.1. *Suppose R is a local ring of codimension ≤ 3, or, suppose R is 1 link from a complete intersection. Let C and D be minimal acyclic complexes of finitely generated free R-modules such that $C_{\geq r} \cong D_{\geq r}$ and the Betti numbers are bounded for each complex. Then $C \cong D$.*

Proof. From Avramov [2], these complexes are periodic of period 2. Since they are periodic, we know $C \cong D$. □

CHAPTER 6

THE PUSH FORWARD METHOD

We now examine the branching of a complex from a *push forward* perspective, i.e., given a d_i as part of a complex

$$\cdots \to R^n_{i+1} \xrightarrow{d_{i+1}} R^p_i \xrightarrow{d_i} R^q_{i-1} \xrightarrow{d_{i-1}} \cdots ,$$

determine all possibilities (up to isomorphism) for d_{i-1}.

6.1 Push Forward

Definition 6.1.1. Given an R-linear map between free modules $R^p \xrightarrow{d} R^p$, to *push d forward* is to find another map $R^n \xrightarrow{d'} R^q$ such that $R^n \xrightarrow{d} R^p \xrightarrow{d'} R^q$ is exact, i.e., $\ker d' = \operatorname{image} d$.

The motivation behind this process is that if a minimal acyclic complex has a syzygy module that can be pushed forward to two non-isomorphic modules, then branching of the complex is a possibility. On the other hand, the major theorem in the next chapter maintains that branching is impossible for a particular class of rings. This result is based on the conclusion that a given syzygy module can be pushed forward to only one module up to isomorphism. We discuss this method for finding the next module to the right in a complex.

Assume that the local ring (R, n) is also a graded k-algebra with Hilbert series $H_R(t) = 1 + et + ft^2$. Suppose we have a R-linear map between free modules,

$$R^n \xrightarrow{d} R^p,$$

such that $\operatorname{image} d \subseteq \mathfrak{m}R^p$.

Let A be the $p \times n$ matrix representing d as an R-linear map.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix},$$

with respect to the standard bases of R^n and R^p. We want to determine B such that B is the $q \times p$ matrix representing d' :

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qp} \end{pmatrix},$$

with respect to the standard bases of R^p and R^q. We know the composition is a complex when $BA = 0$, $A^T B^T = 0$. Thus the columns of B^T are syzygies of A^T. To push A forward, first determine the syzygies of $A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{pmatrix}$ by computing $\ker A^T$. We build B^T from the syzygies of A^T. Let $z_i = \begin{pmatrix} z_{i1} \\ z_{i2} \\ \vdots \\ z_{ip} \end{pmatrix}$, $1 \leq i \leq m$, be a minimal generating set for the syzygies of A^T. The columns of B^T can be written as linear transformations of the z_i, i.e.:

$$\begin{pmatrix} b_{j1} \\ b_{j2} \\ \vdots \\ b_{jp} \end{pmatrix} = r_{j1}z_1 + r_{j2}z_2 + \cdots r_{jm}z_m, 1 \leq j \leq q.$$

Now we want to consider

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qp} \end{pmatrix},$$

and regard it as a k-linear map $R^p \to R^q$.

Fix a k-vector space basis $v_1, \cdots, v_e$ of R_1, and for all i, j. Let $T_{b_{ij}}$ be the $f \times e$ block matrix with entries in k representing the k-linear map $R_1 \xrightarrow{T_{b_{ij}}} R_2$, multiplication by b_{ij}. Form the matrix of linear transformations of B, T_B, with these submatrices $T_{b_{ij}}$ as the blocks of the $qf \times pe$ matrix T_B.

So as a k-linear map, d' is represented by

$$T_B = \begin{pmatrix} T_{b_{11}} & T_{b_{12}} & \cdots & T_{b_{1p}} \\ T_{b_{21}} & T_{b_{22}} & \cdots & T_{b_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{b_{q1}} & T_{b_{q2}} & \cdots & T_{b_{qp}} \end{pmatrix}.$$

If $R^n \xrightarrow{A} R^p \xrightarrow{B} R^q$ is exact, the matrix T_B needs to have rank $pe - n$. This is assuming T_A is surjective. Then, in turn, B can only be pushed forward if T_B is surjective.

The following diagram illustrates the direct sum decomposition of a graded complex and the roles of the matrix of linear transformations T_A and T_B within the complex.

$$
\begin{array}{ccccc}
R_2^n & \longrightarrow & 0 & & \\
\oplus & & \oplus & & \\
R_1^n & \xrightarrow{T_A} & R_2^p & \longrightarrow & 0 \\
\oplus & & \oplus & & \oplus \\
R_0^n & \longrightarrow & R_1^p & \xrightarrow{T_B} & R_2^q \\
 & & \oplus & & \oplus \\
 & & R_0^p & \longrightarrow & R_1^q \\
 & & & & \oplus \\
 & & & & R_0^q \\
R^n & \xrightarrow{A} & R^p & \xrightarrow{B} & R^q.
\end{array}
$$

6.2 Examples

The task is to find equations involving the r_{jk} such that at least one $qf \times qf$ minor is nonzero. From these, we will determine the conditions needed for B to exist. We start by taking a matrix A and attempting to push it forward. In this first example, it turns out A is not part of an infinite acyclic complex.

Example 6.2.1. *Given the ring $R = k[x,y]/(x^2, xy, y^2)$ and the map $R^1 \xrightarrow{(x)} R^1$, find the matrix B such that the previous conditions are met.*

Since $A = (x)$ is a 1×1 matrix, let B be a $q \times 1$ matrix. By computing $\ker A^T$, we get a minimal generating set for the syzygies: $z_1 = x, z_2 = y$. The q columns of B^T can then be written $b_j = r_{j1}x + r_{j2}y$, which forms

$$B = \begin{pmatrix} r_{11}x + r_{12}y \\ \vdots \\ r_{q1}x + r_{q2}y \end{pmatrix}.$$

We need to fix a basis for R as a vector space over k and form the matrix of linear transformations of B.

The b_{ij} are 3×3 matrices representing the linear transformations $R_1 \xrightarrow{b_{ij}} R_2$, $\dim_k R = 3$, $x : R_1 \to R_2$ is represented by $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $y : R_1 \to R_2$ is represented by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. So we get

$$T_B = \begin{pmatrix} 0 & 0 & 0 \\ r_{11} & 0 & 0 \\ r_{12} & 0 & 0 \\ \cdots & \cdots & \cdots \\ & \vdots & \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ r_{q1} & 0 & 0 \\ r_{q2} & 0 & 0 \end{pmatrix},$$

which has rank 1.

But exactness can only be achieved when $\operatorname{rank} r = q \dim r - \operatorname{rank} A = 1 \times 3 - 1 = 2$. Thus, this map does not have a matrix B satisfying these conditions.

The next example is a scenario where a matrix is pushed forward one step. The attempt to push the new matrix forward, however, fails.

Example 6.2.2. *Again,consider the ring $R = k[x,y]/(x^2, xy, y^2)$ with map*

$$d : R^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^2,$$

find another map $d' : R^2 \to R^q$ such that $R^4 \xrightarrow{d} R^2 \xrightarrow{d'} R^q$ is exact.

Given $A = \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}$, find the $q \times 2$ matrix B that makes $BA = 0$. From $\ker A^t$, we have the syzygies $\begin{pmatrix} x \\ 0 \end{pmatrix}$, $\begin{pmatrix} y \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x \end{pmatrix}$, and $\begin{pmatrix} 0 \\ y \end{pmatrix}$. Thus the q columns of B^t can be written as

$$\begin{pmatrix} b_{j1} \\ b_{j2} \end{pmatrix} = r_{j1} \begin{pmatrix} x \\ 0 \end{pmatrix} + r_{j2} \begin{pmatrix} y \\ 0 \end{pmatrix} + r_{j3} \begin{pmatrix} 0 \\ x \end{pmatrix} + r_{j4} \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Since $\dim(\operatorname{image} d') + \dim(\ker d') = \dim(R^2) = 4$, by the rank-nullity theorem, $q < 4$.

Let's consider the possibilities for $q = 1 : B^T$ is the 2×1 matrix $\begin{pmatrix} r_1 x + r_2 y \\ r_3 x + r_4 y \end{pmatrix}$.The matrix of linear transformations of B has 2 submatrices having size $\dim_k R \times \dim_k R = 3 \times 3$.

$$T_B = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & r_3 & 0 & 0 \\ r_2 & 0 & 0 & r_4 & 0 & 0 \end{array}\right),$$

which needs to have $\operatorname{rank} r = p \dim R - \operatorname{rank} A = 2 \times 3 - 4 = 2$ to be exact. This possibility occurs when at least one 2×2 minor is nonzero and all 3×3 minors are 0.

As an example, exactness occurs when $r_1 = r_4 = 1, r_2 = r_3 = 0$, giving B full rank with $B = \begin{pmatrix} x & y \end{pmatrix}$.

This gives us a step further than the previous example, but is this B part of an infinite complex? In other words, for $R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R^1$, can we push forward again from here?

Find the map $R^1 \xrightarrow{d'} R^{q'}$ making

$$R^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R^1 \xrightarrow{d'} R^{q'}$$

exact.

$B = \begin{pmatrix} x & y \end{pmatrix}$, and C is the $q' \times 1$ matrix such that $CB = 0$. The kernel of B^t gives syzygies $\begin{pmatrix} x \end{pmatrix}, \begin{pmatrix} y \end{pmatrix}$. So $c_j = r_{j1}(x) + r_{j2}(y), 1 \leq j' \leq q'$. Since $q' < 2$, we know $q' = 1$, and $c_1 = r_1 x + r_2 y$. The matrix of linear transformations is

$$T_C = \begin{pmatrix} 0 & 0 & 0 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix},$$

needing rank $1 \times 3 - 2 = 1$, or $r_1 = 1$.

So we know $C = \begin{pmatrix} x \end{pmatrix}, \begin{pmatrix} y \end{pmatrix}$, or $\begin{pmatrix} x + y \end{pmatrix}$ works. But we have already determined from the previous example that this cannot be pushed forward. Thus, the complex terminates when $q = 1$.

6.3 Possibility of Branching

Throughout this section, we consider rings with Hilbert series $H_R(t) = 1 + et + (e-1)t^2$. In this case, according to Theorem B of [4], the ranks of the free modules in a minimal acyclic complex become constant to the right. We investigate pushing forward in this context. Specifically, we consider a piece of a minimal acyclic complex of this form:

$$R^n \xrightarrow{A} R^p \xrightarrow{B} R^p.$$

As the following theorem shows, the occurrence of A pushing forward to two non-isomorphic choices for B is possible only when T_A has full rank and T_{A^t} has at least $p+1$ linear syzygies. Assuming that T_A has full rank, T_{A^t} can have at most $pe-1$ linear syzygies. We will see in the next chapter that sometimes T_{A^t} only has p linear syzygies, and as already mentioned, this makes branching impossible.

As an example, A may be a 2×3 matrix with T_A having full rank. Then T_{A^t} could have 3 linear syzygies, making it possible to construct two non-isomorphic B's. On the other hand, T_{A^t} could have only 2 linear syzygies, allowing only one B up to isomorphism.

Theorem 6.3.1. *For pieces of minimal acyclic complexes*

$$R^n \xrightarrow{A} R^p \xrightarrow{B} R^p.$$

and

$$R^n \xrightarrow{A'} R^p \xrightarrow{B'} R^p,$$

assume both A and A^t have p linear syzygies. If $\operatorname{Coker} A \cong \operatorname{Coker} A'$, *then* $\operatorname{Coker} B \cong \operatorname{Coker} B'$.

Proof. Let

$$s_1 = \begin{pmatrix} s_{11} \\ \vdots \\ s_{p1} \end{pmatrix}, \cdots, s_p = \begin{pmatrix} s_{1p} \\ \vdots \\ s_{pp} \end{pmatrix},$$

be the p linear syzygies of A^t. Since $\operatorname{Coker} A \cong \operatorname{Coker} A'$, we have p linear syzygies of ${A'}^t$,

$$s'_1 = \begin{pmatrix} s'_{11} \\ \vdots \\ s'_{p1} \end{pmatrix}, \cdots, s'_p = \begin{pmatrix} s'_{1p} \\ \vdots \\ s'_{pp} \end{pmatrix}.$$

Form B^t from the first set of syzygies:

$$B^t = \begin{pmatrix} r_{11}s_1 + \cdots + r_{1p}s_q & \cdots & r_{p1}s_1 + \cdots + r_{pp}s_p \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}s_{11} + \cdots + r_{1p}s_{1p} & \cdots & r_{p1}s_{11} + \cdots + r_{pp}s_{1p} \\ \vdots & \ddots & \vdots \\ r_{11}s_{p1} + \cdots + r_{1p}s_{pp} & \cdots & r_{p1}s_{p1} + \cdots + r_{pp}s_{pp} \end{pmatrix},$$

which is row and column operations away from $\begin{pmatrix} s_1 & \cdots & s_q \end{pmatrix}$, giving

$$\operatorname{Coker} B \cong \operatorname{Coker} \begin{pmatrix} s_1^t \\ \vdots \\ s_p^t \end{pmatrix}.$$

To show that $\operatorname{Coker} B \cong \operatorname{Coker} B'$, we need to find the isomorphism f that makes the following diagram commute:

$$\begin{array}{ccccccccc} R^n & \xrightarrow{A} & R^p & \xrightarrow{B} & R^p & \longrightarrow & \operatorname{Coker} B & \longrightarrow & 0\,. \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{f} & & & & \\ R^n & \xrightarrow{A'} & R^p & \xrightarrow{B'} & R^p & \longrightarrow & \operatorname{Coker} B' & \longrightarrow & 0 \end{array}$$

First, take surjective mappings

$$\epsilon_0 : R^p \to s_1R + \cdots + s_pR$$

and

$$\epsilon_1 : R^p \to (r_{11}s_1 + \cdots + r_{1p}s_p)R + \cdots + (r_{p1}s_1 + \cdots + r_{pp}s_p)R,$$

where $\{s_1, \cdots, s_p\}$, and $\{r_{11}s_1 + \cdots + r_{1p}s_p, \cdots, r_{p1}s_1 + \cdots + r_{pp}s_p\}$ both generate $\ker A^t$, together with the inclusion maps

$$\alpha_0 : s_1R + \cdots + s_pR \hookrightarrow R^p$$

and

$$\alpha_1 : (r_{11}s_1 + \cdots + r_{1p}s_q)R + \cdots + (r_{p1}s_1 + \cdots + r_{pp}s_p)R \hookrightarrow R^p.$$

Since there exists an induced isomorphism between the kernels, compose the $\alpha_i(\epsilon_i)$ to get

$$\begin{array}{ccccccc} R^p & \xrightarrow{(s_1 \cdots s_p)} & R^p & \xrightarrow{A^t} & R^n & \longrightarrow & \cdots, \\ \uparrow{\scriptstyle g*} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} & & \\ R^p & \xrightarrow{B^t} & R^p & \xrightarrow{A^t} & R^n & \longrightarrow & \cdots \end{array}$$

where $B^t = \begin{pmatrix} r_{11}s_1 + \cdots + r_{1p}s_p & \cdots & r_{p1}s_1 + \cdots + r_{pp}s_p \end{pmatrix}$, and $g* : R^p \to R^p$ is the isomorphism that makes the diagram commute.

With respect to the standard basis, $g*$ is represented by the matrix $\begin{pmatrix} r_{11} & \cdots & r_{p1} \\ \vdots & \ddots & \vdots \\ r_{1p} & \cdots & r_{pp} \end{pmatrix}$.

Dualizing, we get the commutative diagram:

$$\begin{array}{ccccc} R^n & \xrightarrow{A} & R^p & \xrightarrow{\begin{pmatrix} s_1^t \\ \vdots \\ s_p^t \end{pmatrix}} & R^p \;. \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle g} \\ R^n & \xrightarrow{A} & R^p & \xrightarrow{B} & R^p \end{array}$$

Similarly we have

$$\begin{array}{ccccc} R^n & \xrightarrow{A'} & R^p & \xrightarrow{\begin{pmatrix} (s_1')^t \\ \vdots \\ (s_p')^t \end{pmatrix}} & R^p \;. \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle g'} \\ R^n & \xrightarrow{A'} & R^p & \xrightarrow{B'} & R^p \end{array}$$

Basically, given a set of generators of the kernel, $s_1, \cdots, s_p$, there exists an isomorphism between any other set of generators, $r_{11}s_1 + \cdots + r_{1p}s_p, \cdots, r_{p1}s_1 + \cdots + r_{pp}s_p$ provided $\begin{vmatrix} r_{11} & \cdots & r_{p1} \\ \vdots & \ddots & \vdots \\ r_{1p} & \cdots & r_{pp} \end{vmatrix} \neq 0$, and in this case g and g' are isomorphisms.

Now, given the surjections

$$R^p \to s_1R + \cdots + s_pR$$

$$R^p \to s'_1R + \cdots + s'_pR,$$

since Coker $A \cong$ Coker A', we get an induced commutative diagram

$$\begin{array}{ccccc} R^p & \xrightarrow{(s_1 \ \cdots \ s_p)} & R^p & \xrightarrow{A^t} & R^n \\ \uparrow{\scriptstyle h*} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\ R^p & \xrightarrow{(s'_1 \ \cdots \ s'_p)} & R^p & \xrightarrow{(A')^t} & R^n \end{array} .$$

Dualize back to get

$$\begin{array}{ccccc} R^n & \xrightarrow{A} & R^p & \xrightarrow{\begin{pmatrix} s_1^t \\ \vdots \\ s_p^t \end{pmatrix}} & R^p \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle h} \\ R^n & \xrightarrow{A'} & R^p & \xrightarrow{\begin{pmatrix} (s'_1)^t \\ \vdots \\ (s'_p)^t \end{pmatrix}} & R^p \end{array} ,$$

where $f = g'hg^{-1}$. □

Finally, we provide an example illustrating the impetus of the paper. Take a matrix A and push it forward to two non-isomorphic choices for B, making non-uniqueness of minimal acyclic complexes a possibility.

Example 6.3.2. *Let* $R = k[x,y,z,w]/(x^2, y^2, z^2, w^2, xy+xz, zw+xw, zy)$*. Then* R *has Hilbert series* $H_R(t) = 1 + 4t + 3t^2$ *with* R_2 *having* k*-basis* $\{xy, zw, yw\}$*. Given the map*

$$d : R^5 \xrightarrow{\begin{pmatrix} 0 & 0 & y+z & zw & xz \\ w & x+z & 0 & 0 & 0 \end{pmatrix}} R^2,$$

find another map $d' : R^2 \xrightarrow{d'} R^2$ *such that* $R^5 \xrightarrow{d} R^2 \xrightarrow{d'} R^2$ *is exact.*

Given the matrix $A = \begin{pmatrix} 0 & 0 & y+z & zw & xz \\ w & x+z & 0 & 0 & 0 \end{pmatrix}$, determine a matrix B for which A is the syzygy matrix. The syzygies of A^T are given by

$$\operatorname{syz}(A^T) = \begin{pmatrix} 0 \\ w \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}, \text{and} \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

Form the matrix B^T from linear combinations of the syzygies of A^T.

$$B^T = \left(r_{1,1} \begin{pmatrix} 0 \\ w \end{pmatrix} + \cdots + r_{1,12} \begin{pmatrix} z \\ 0 \end{pmatrix} \quad r_{2,1} \begin{pmatrix} 0 \\ w \end{pmatrix} + \cdots + r_{2,12} \begin{pmatrix} z \\ 0 \end{pmatrix} \right).$$

One possibility is $B^T = \begin{pmatrix} w & 0 \\ z & x \end{pmatrix}$. Another gives $B^T = \begin{pmatrix} w & 0 \\ y & x \end{pmatrix}$. So we have

$$\cdots R^{16} \xrightarrow{\begin{pmatrix} w & x+z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & y+z & 0 & x-z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & y & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z & y & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z & y & x \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} 0 & 0 & y+z & zw & xz \\ w & x+z & 0 & 0 & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} w & z \\ 0 & x \end{pmatrix}} R^2$$

and

$$\cdots R^{16} \xrightarrow{\begin{pmatrix} w & x+z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & y+z & 0 & x-z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & y & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z & y & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z & y & x \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} 0 & 0 & y+z & zw & xz \\ w & x+z & 0 & 0 & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} w & y \\ 0 & x \end{pmatrix}} R^2,$$

and conclude that A can be pushed forward to two non-isomorphic choices for B.

CHAPTER 7

CLASSIFICATION OF MONOMIAL ALGEBRAS

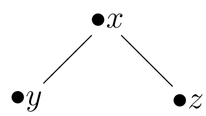
A monomial algebra is defined as a polynomial ring modulo an ideal generated by monomials. In the next chapter, we explore the possibilities for branching of minimal acyclic complexes over this type of ring. First, we will classify monomial algebras with Hilbert series $H_R(t) = 1 + et + (e-1)t^2$, where $e \geq 1$. We limit the exploration to those algebras whose defining ideals are generated by monomials of degree 2, and whose socle consists only of elements of degree 2.

We begin by noting that the ideal of definition must include the squares of the variables. Otherwise, the socle will contain a cubic element. In addition, no variable is included in the ideal of definition $e-1$ times, for example, $x_1^2, x_1x_2, \cdots, x_1x_e$. This scenario would force x_1 to be a socle element, again in violation of the socle requirement.

First, look at $e = 2$, so that $H_R(t) = 1 + 2t + t^2$. Then there is up to algebra isomorphism one choice only for the algebra: $R = k[x, y]/(x^2, y^2)$, with socle $\{xy\}$. Since the socle is 1-dimensional, this is a Gorenstein ring, and so there is nothing to do.

For $e = 3, H_R(t) = 1 + 3t + 2t^2$, there is (up to algebra isomorphism) still only one possibility, $R = k[x, y, z]/(x^2, y^2, z^2, yz)$.

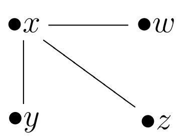
At this point, we consider the monomial algebras from a combinatorial perspective. Each figure has e vertices representing the R_1 basis with the $(e-1)$ connected edges representing the R_2 basis elements. The "missing" edges represent the square-free elements in the ideal of definition. For example, the $e = 3$ case is represented by

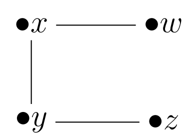


$$R = k[x,y,z]/(x^2,y^2,z^2,yz).$$

It is also useful to consider each algebra as an e-tuple given in terms of the number of edges extending from each of the e vertices. The $e=3$ case is represented by the 3-tuple $(2,1,1)$, with the 2 representing the x_1 vertex, and then continuing in a counter-clockwise direction.

For $e = 4$ we have $H_R(t) = 1 + 4t + 3t^2$, and we show that there are two non-isomorphic monomial algebras:

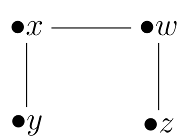




$R^{(1)} = k[x,y,z,w], (x^2,y^2,z^2,w^2,yz,yw,zw)$ $R^{(2)} = k[x,y,z,w]/(x^2,y^2,z^2,w^2,xz,yw,zw)$.

$(3,1,1,1)$ $(2,2,1,1)$

Notice also that



$(2,1,1,2)$,

or any 4-tuple with two 2's and two 1's would represent a ring isomorphic to $R^{(2)}$.

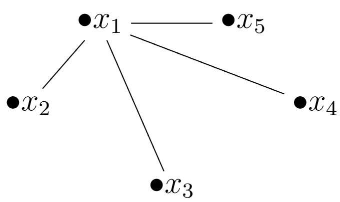
By examining the Koszul Homology, $H(R) \cong \operatorname{Tor}_i^Q(R,k)$, where Q is the polynomial ring $Q = k[x,y,z,w]$, we find that the Betti numbers, $\dim_k \operatorname{Tor}_i^Q(R,k)$, are different for each of the two algebras, giving two non-isomorphic structures. Using Macaulay 2, resolve the maximal ideal of each ring over Q to obtain:

$$R^{(1)}: \qquad 0 \to Q^3 \to Q^{11} \to Q^{14} \to Q^7 \to Q^1$$

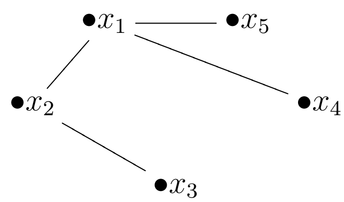
and

$$R^{(2)}: \qquad 0 \to Q^3 \to Q^{10} \to Q^{13} \to Q^7 \to Q^1.$$

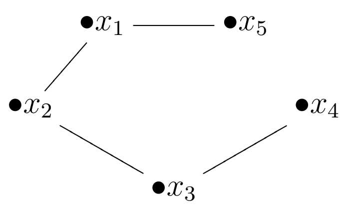
Continuing with $e = 5$, we get 3 non-isomorphic structures.



$R^{(1)} = (4, 1, 1, 1, 1)$



$R^{(2)} = (3, 2, 1, 1, 1)$



$R^{(3)} = (2, 2, 2, 1, 1)$

with resolutions over $Q = k[x_1, \cdots, x_5]$:

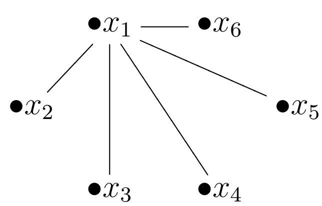
$$R^{(1)}: \qquad 0 \to Q^4 \to Q^{19} \to Q^{35} \to Q^{30} \to Q^{11} \to Q^1,$$

$$R^{(2)}: \qquad 0 \to Q^4 \to Q^{18} \to Q^{32} \to Q^{28} \to Q^{11} \to Q^1,$$

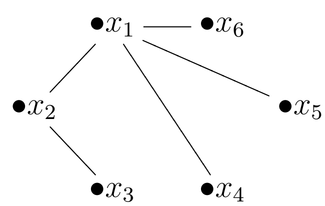
and

$$R^{(3)}: \qquad 0 \to Q^4 \to Q^{17} \to Q^{30} \to Q^{27} \to Q^{11} \to Q^1.$$

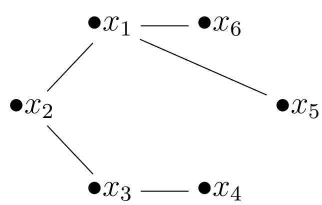
For $e = 6$ we might predict 4 non-isomorphic structures. However, $e = 6$ is the first case where the question arises as to whether we distinguish among two algebras with the same vertex edges, but in a different order.



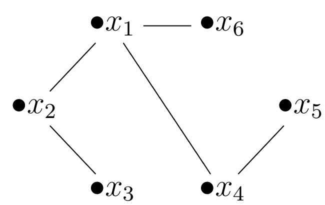
$(5, 1, 1, 1, 1, 1)$



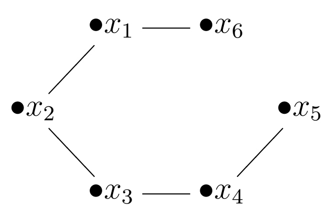
$(4, 2, 1, 1, 1, 1)$



$(3, 2, 2, 1, 1, 1)$



$(3, 2, 1, 2, 1, 1)$



$(2, 2, 2, 2, 1, 1)$

For example, are the two rings in the second row isomorphic? By examining their Koszul homology as represented by their resolutions over Q :

$$0 \to Q^5 \to Q^{28} \to Q^{64} \to Q^{77} \to Q^{51} \to Q^{16} \to Q^1$$

for the first algebra, and

$$0 \to Q^5 \to Q^{27} \to Q^{61} \to Q^{74} \to Q^{50} \to Q^{16} \to Q^1$$

for the second, we see they are not isomorphic.

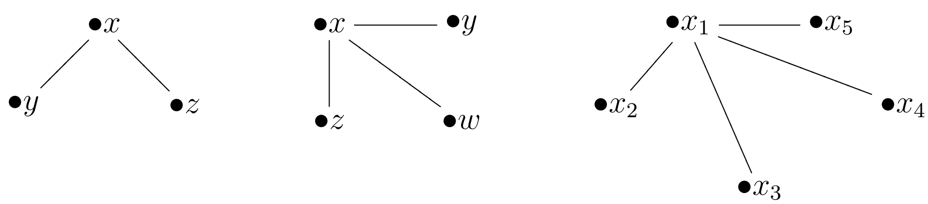
We thus classify monomial algebras of codimension e into $e-2$ distinct rings for $3 \leq e \leq 5$, and we attempt to answer the question of branching by working through this classification.

CHAPTER 8

CONCA RINGS

8.1 Definition

Notice that for each $e \geq 3$, there exists one ring that has the property (from a visual perspective) that one vertex is connected to each of the other vertices, and these vertices are only connected to the one vertex, as seen here:



For each e, these cases represent the Conca generator case. A Conca generator is an element that generates the square of the maximal ideal.

Definition 8.1.1. The maximal ideal $\mathfrak{m}$ has a *Conca generator* l when $l^2 = 0$ and $l\mathfrak{m} = \mathfrak{m}^2$.

Notice the Conca generator appears in the R_2 basis $e - 1$ times. For example, the $e = 4$ case $R = k[x, y, z, w]/(x^2, y^2, z^2, w^2, yz, yw, zw)$, with R_2 basis $\{xy, xz, xw\}$ has Conca generator x.

Definition 8.1.2. Let R be a monomial, quadratic algebra with an indeterminate x such that x is a Conca generator. Then R is called a *Conca algebra*.

8.2 Uniqueness of Minimal Acyclic Complexes over Conca Algebras

We show that for Conca algebras, the answer to Question 3.1.1 is negative. This is based on the stronger result, provided by the following lemma, that given an infinite syzygy module in a minimal acyclic complex over this type of ring, the next syzygy module is unique up to isomorphism. In other words, since a syzygy module cannot even be pushed forward one step to two non-isomorphic modules, branching is impossible.

Theorem 8.2.1. *Let R be a Conca algebra with $H_R(t) = 1+et+(e-1)t^2$. Given a piece of a minimal acyclic complex*

$$R^n \xrightarrow{A} R^p \xrightarrow{B} R^p$$

and

$$R^n \xrightarrow{A'} R^p \xrightarrow{B'} R^p$$

if $\operatorname{Coker} A \cong \operatorname{Coker} A'$, *then* $\operatorname{Coker} B \cong \operatorname{Coker} B'$.

Proof. Let A be $p \times n$ and B be $p \times p$ matrices of the minimal acyclic complex

$$\cdots \to R^n \xrightarrow{A} R^p \xrightarrow{B} R^p \to \cdots ,$$

representing maps with respect to the standard basis.

By assumption, $R = k[x_1, ..., x_e]/I$ where I is generated by quadratic monomials such that $H_R(t) = 1+et+(e-1)t^2$, R has a Conca generator, and socle $R \subseteq \mathfrak{m}^2$. Let l be a Conca generator for R. Without loss of generality, we can assume $l = x_1$. If we let $I = (x_1^2, \cdots, x_e^2; x_ix_j | 2 \leq i < j \leq e)$, with $\mathfrak{m} = (x_1, \cdots, x_e)$, then $\mathfrak{m}^2 = (x_1x_2, \cdots, x_1x_e)$, $\mathfrak{m}^3 = 0$, and the Conca condition is satisfied. We show that $\operatorname{Coker} B \cong \operatorname{Coker} B'$.

Since

$$\cdots \to R^n \xrightarrow{A} R^p \xrightarrow{B} R^p \to \cdots$$

is a minimal acyclic complex, the matrix of linear transformations for A, T_A, must have full rank. If T_A did not have full rank, A would not represent a surjective mapping.

For this ring, the linear transformations $R_1 \xrightarrow{x_i} R_2$, where R_1 is with respect to the basis $\{x_1, \cdots, x_e\}$ and R_2 is with respect to the basis $\{x_1x_2, \cdots, x_1x_e\}$, have the form:

$$T_{x_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, T_{x_i} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

which are $(e-1) \times e$ matrices, and for $T_{x_i}, 2 \le i \le e$, the 1 in the first column occurs in the $(i-1)^{st}$ row. Notice 1 appears as an entry $e-1$ times for T_{x_1} and one time for $T_{x_i}, 2 \le i \le e$.

T_{A_ij} is an $(e-1) \times e$ block associated to a linear form l_{ij}, the ij^{th} entry of A, and $T_{A_{ij}}$ has full rank if and only if the coefficient of x_1 in l_{ij} is nonzero. Examine each of the np blocks $T_{A_{ij}}, 1 \le i \le p, 1 \le j \le n$. If T_{x_1} is not a component, $(T_A)_{ij}$ will have rank $1 \neq (e-1)$. Thus for $(T_A)_{ij}$ to have full rank $(e-1)$, T_{x_1} must be included, giving $(T_A)_{ij}$ the form

$$(*) \qquad \begin{pmatrix} b_1 & a & 0 & 0 & \cdots & 0 \\ b_2 & 0 & a & 0 & \cdots & 0 \\ b_3 & 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_e & 0 & 0 & 0 & \cdots & a \end{pmatrix}, a \neq 0,$$

where a is the coefficient of the x_1-term in l_{ij}.

For T_A to have full rank $p(e-1)$, a $T_{A_{ij}}$ block of the form $(*)$ must appear for each i and for each j. One such possibility for all $1 \le i \le p$ to have this form is $T_{A_{ii}}$, where all linear entries on the diagonal of A have a nonzero x_1 term.

We have already determined that T_A must have full rank. When this is the case, we now show that T_{A^t} must also have full rank. For each $(T_A)_{ij}$ of the form $(*)$, $(T_{A^t})_{ji}$ is of the form $(*), 1 \leq i \leq p, 1 \leq j \leq n$. So for T_{A^t}, for each of the p rows of block matrices, at least one of the n $(e-1) \times e$ matrices has full rank. Also, for each of the n columns of blocks, at least one of the p blocks has full rank. We can use row and column operations to rearrange T_{A^t} to a diagonal block matrix that has full rank, giving p linear syzygies. By Theorem 6.3.1, we have $\operatorname{Coker} B \cong \operatorname{Coker} B'$. □

So we can conclude that if R is a Conca algebra, then every push forward is unique. By induction on this theorem, we have the following corollary:

Corollary 8.2.2. *Let R be a Conca algebra with $H_R(t) = 1 + et + (e-1)t^2$. There is no branching.*

Proof. Assume $A_{\geq r} \cong B_{\geq r}$. Then $\operatorname{Coker} A_{r-1} \cong \operatorname{Coker} B_{r-1}$ by 8.2.1. Given $A_{r-n} \cong B_{r-n}$, we have by 8.2.1 that $\operatorname{Coker} A_{r-(n-1)} \cong \operatorname{Coker} B_{r-(n-1)}$. Using induction, since true for all n, $A \cong B$. □

8.3 Necessary Conditions for Uniqueness

It is natural to ask whether the conditions Conca generator and monomial algebra are both necessary for the given result. Although there is no evidence of the existence of two non-isomorphic minimal acyclic complexes A and B such that $A_{\geq r} \cong B_{\geq r}$, as the following example illustrates, removing the monomial condition allows us to push forward in two different ways.

Example 8.3.1. *Let $R = k[x, y, z]/(x^2, y^2, z^2, xy + xz)$. Then R has Hilbert series $H(t) = 1 + 3t + 2t^2$ and a k-basis of R_2 is given by $\{xy, yz\}$. Let $A = \begin{pmatrix} y & x \\ z & 0 \end{pmatrix}$. We determine those matrices B for which A is the syzygy matrix.*

Notice y is the Conca generator, and R is not a monomial algebra. As the "left-hand side" of the minimal acyclic complex containing A, the unique free resolution of A begins as

$$\cdots \to R^{10} \xrightarrow{\begin{pmatrix} y-z & x & 0 & 0&0&0&0&0&0&0 \\ 0 & 0 & y+z & x&0&0&0&0&0&0 \\ 0&0&0&0&z&0&y&x&0&0 \\ 0&0&0&0&0&z&0&0&y&x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} 0 & 0 & yz & xz \\ y+z & x & 0 & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \\ z & 0 \end{pmatrix}} R^2 \to R.$$

The conditions on the linear syzygies for non-uniqueness of the "right-hand side" to occur are met: A has two linear syzygies and A^T has at least three. In this case, the syzygy generators of A^T are $\begin{pmatrix} 0 \\ z \end{pmatrix}, \begin{pmatrix} y+z \\ -y \end{pmatrix}, \begin{pmatrix} x \\ x \end{pmatrix}$. From these, we find

$$B = \begin{pmatrix} by+bz+cx & az-by+cx \\ ey+ez+fx & dz-ey+fx \end{pmatrix}.$$

As linear transformations,

$$x = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix of linear transformations for B, T_B, determines possibilities for B that have full rank. Two of the non-isomorphic choices that give the next step in the complex are

$$\cdots \to R^{10} \xrightarrow{\begin{pmatrix} y-z & x & 0 & 0&0&0&0&0&0&0 \\ 0 & 0 & y+z & x&0&0&0&0&0&0 \\ 0&0&0&0&z&0&y&x&0&0 \\ 0&0&0&0&0&z&0&0&y&x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} 0 & 0 & yz & xz \\ y+z & x & 0 & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \\ z & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y+z+x & -y+x \\ 0 & z \end{pmatrix}} R^2$$

and

$$\cdots \to R^{10} \xrightarrow{\begin{pmatrix} y-z & x & 0 & 0&0&0&0&0&0&0 \\ 0 & 0 & y+z & x&0&0&0&0&0&0 \\ 0&0&0&0&z&0&y&x&0&0 \\ 0&0&0&0&0&z&0&0&y&x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} 0 & 0 & yz & xz \\ y+z & x & 0 & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \\ z & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y+z & -y \\ x & z+x \end{pmatrix}} R^2.$$

To see that an isomorphism between the two matrices does not exist, try to find non-singular matrices so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y+z & -y \\ x & z+x \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} y+z+x & -y+x \\ 0 & z \end{pmatrix}.$$

Equating the second row, first column entries give

$$x(a'd + c'd) + y(ca' - cc') + z(ca' + dc') = 0, \text{ which means}$$

$$d(a' + c') = c(a' - c') = ca' + dc' = 0.$$

Consider the first term, $d(a' + c') = 0$. If $d = 0$, then either $c = 0$ or $a' = 0$. If $c = 0$ then the first matrix has a row of zeros. But if $a' = 0$, then $c' = 0$, and now the second matrix has a column of zeros. Therefore, $d \neq 0$. That forces $a' = -c'$, which gives $c = 0$, and again, $d = 0$. Thus, we have a singular matrix, and the isomorphism does not exist.

Now, view the effects of removing the Conca condition. As given by the next example, we find two non-isomorphic matrices that have the same free resolution, making the Conca condition a necessary condition for the theorem.

Example 8.3.2. *Let* $R = k[x, y, z, w]/(x^2, y^2, z^2, w^2, xw, xz, yw)$. *Then* R *has Hilbert series* $H_R(t) = 1+4t+3t^2$ *with* k*-basis* $\{xy, yz, zw\}$. *Let* $A = \begin{pmatrix} 0 & z & zw & yz & 0 & 0 \\ w & 0 & 0 & 0 & yz & xy \end{pmatrix}$. *Determine those matrices* B *for which* A *is the syzygy matrix.*

From the monomial basis, we see there is no element that serves as a Conca generator. The unique free resolution is given by

$$\cdots \to R^{22} \xrightarrow{\begin{pmatrix} w & y & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & z & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & z & y & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & z & 0 & y & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & z & 0 & y & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & z & y & x \end{pmatrix}} R^6 \xrightarrow{\begin{pmatrix} 0 & z & zw & yz & 0 & 0 \\ w & 0 & 0 & 0 & yz & xy \end{pmatrix}} R^2.$$

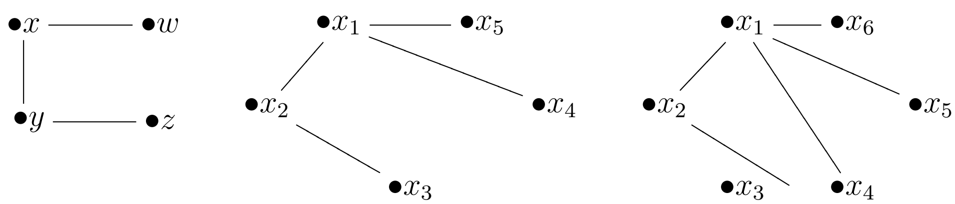
Using methods similar to the previous example, we push forward to find two non-isomorpic choices for the next step:

$$B = \begin{pmatrix} z & x \\ w & y \end{pmatrix} \text{ and } B' = \begin{pmatrix} z & x \\ w & y + w \end{pmatrix}.$$

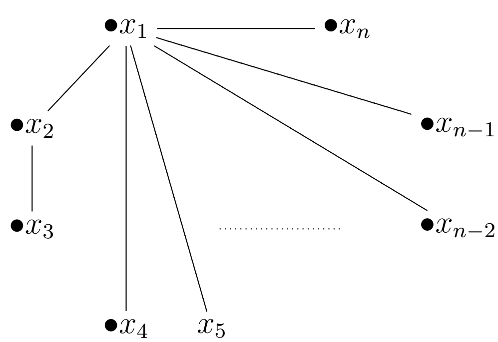
CHAPTER 9

SEMI-CONCA CASE

Notice that for each $e \geq 4$, there exists one ring where one vertex is connected to exactly $e-2$ vertices, and another vertex is connected to only one of the $e-2$ vertices:



or, for a general picture:



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9.1 Semi-Conca

Definition 9.1.1. A monomial algebra is **semi-conca** if socle $R \subseteq \mathfrak{m}^2$ and there exists an indeterminate x that appears in the R_2 basis exactly $e-2$ times.

We examine the linear transformations $R_1 \to R_2$,

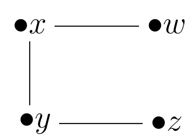
$$t_{x_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, t_{x_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$t_{x_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, t_{x_4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$t_{x_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \cdots, t_{x_n} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

9.2 The e = 4 case

Up to this point, we have determined that for monomial algebras with Hilbert series $H_R(t) = 1 + et + (e-1)t^2$, $e = 2$ and $e = 3$ there is no branching. For $e = 4$, we have only two non-isomorphic rings to consider. The first is the Conca case, which we discovered does not branch. To continue our examination, consider the other ring, which is semi-conca,



$$R = k[x, y, z, w]/(x^2, y^2, z^2, w^2, xz, yw, zw).$$

Once again, examine the linear transformations $R_1 \to R_2$ where R_1 is with respect to the basis $\{x, y, z, w\}$ and R_2 is respect to the basis $\{xy, xw, yz\}$, having the form

$$t_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, t_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, t_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, t_w = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Starting with the 1×1 case, consider the general matrix $A = \begin{pmatrix} ax + by + cz + dw \end{pmatrix}$, with $T_A = \begin{pmatrix} b & a & 0 & 0 \\ d & 0 & 0 & a \\ 0 & c & b & 0 \end{pmatrix}$. Since $A = A^t$, we have Syz $(A) =$ Syz $(A^t) =$ $\begin{pmatrix} by + cz - dw - x \end{pmatrix}$.

Recall that for a minimal acyclic complex to branch, we start with a matrix A, then attempt to push forward to two non-isomorphic choices for B. For this to occur, we need an A whose matrix of linear transformations has full rank, but the matrix of linear transformations for A^t does not.

In the 1×1 case we obtain exact pairs, where each element is paired with its annihilator. Since there is one syzygy, we find conclusively there is no branching for the 1×1 case. In fact, the 1×1 matrix A cannot even be pushed forward to two choices for B. However, this is not true for matrix A in general.

To discover possibilities for an A in the 2×3 case that can be pushed forward, we look at general linear entries of the form

$$A = \begin{pmatrix} ax+by+cz+dw & ex+fy+gz+hw & jx+ky+mz+nw \\ \alpha x+\beta y+\gamma z+\delta w & \epsilon x+\zeta y+\eta z+\theta w & \iota x+\kappa y+\mu z+\nu w \end{pmatrix}.$$

This gives

$$T_A = \left(\begin{array}{cccc|cccc|cccc} b & a & 0 & 0 & f & e & 0 & 0 & k & j & 0 & 0 \\ d & 0 & 0 & a & h & 0 & 0 & e & n & 0 & 0 & j \\ 0 & c & b & 0 & 0 & g & f & 0 & 0 & m & n & 0 \\ \hline \beta & \alpha & 0 & 0 & \zeta & \epsilon & 0 & 0 & \kappa & \iota & 0 & 0 \\ \delta & 0 & 0 & \alpha & \theta & 0 & 0 & \epsilon & \nu & 0 & 0 & \iota \\ 0 & \gamma & \beta & 0 & 0 & \eta & \zeta & 0 & 0 & \mu & \nu & 0 \end{array}\right)$$

and

$$T_{A^t} = \left(\begin{array}{cccc|cccc} b & a & 0 & 0 & \beta & \alpha & 0 & 0 \\ d & 0 & 0 & a & \delta & 0 & 0 & \alpha \\ 0 & c & b & 0 & 0 & \gamma & \beta & 0 \\ \hline f & e & 0 & 0 & \zeta & \epsilon & 0 & 0 \\ h & 0 & 0 & e & \theta & 0 & 0 & \epsilon \\ 0 & g & f & 0 & 0 & \eta & \zeta & 0 \\ \hline k & j & 0 & 0 & \kappa & \iota & 0 & 0 \\ n & 0 & 0 & j & \nu & 0 & 0 & \iota \\ 0 & m & n & 0 & 0 & \mu & \nu & 0 \end{array}\right)$$

By determining the 6×6 minors of both T_A and T_{A^t}, we are given a set of 924 polynomials. We want to choose values of the coefficients $\{a, b, \cdots, n, \alpha, \beta, \cdots, \nu\}$ such that the minors for T_{A^t} are equal to 0, but the minors of T_A are not, implying less than full rank for T_{A^t} with full rank for T_A.

Example 9.2.1. *For the $e = 4$ ring $R = k[x, y, z, w]/(x^2, y^2, z^2, w^2, xz, yw, zw)$, determine a matrix A that can be pushed forward to two non-isomorphic choices for B.*

By examining the minors of T_A and T_{A^t}, we find that T_A has full rank and T_{A^t} does not when $d = f = g = \iota = \kappa = 1$ and all other coefficients are 0. This gives

$$A = \begin{pmatrix} w & y+z & 0 \\ 0 & 0 & x+y \end{pmatrix},$$

with the syzygies of A^t given by $\begin{pmatrix} w \\ 0 \end{pmatrix}$, $\begin{pmatrix} y-z \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ x-y \end{pmatrix}$. Use these syzygies to build the general matrix for B^t, and find the 2×2 B :

$$B = \begin{pmatrix} \chi w + \rho y - \rho z & \sigma x - \sigma y \\ \tau w + \phi y - \phi z & \zeta x - \zeta y \end{pmatrix}.$$

Two possible non-isomorphic choices for B are given, along with their resolutions:

$$\cdots \longrightarrow R^6 \xrightarrow{\begin{pmatrix} w & z & y & 0 & 0 & 0 \\ 0 & 0 & 0 & w & y-z & 0 \\ 0 & 0 & 0 & 0 & 0 & x-y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} w & y+z & 0 \\ 0 & 0 & x+y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} w & x-y \\ y-z & 0 \end{pmatrix}} R^2$$

and

$$\cdots \longrightarrow R^6 \xrightarrow{\begin{pmatrix} w & z & y & 0 & 0 & 0 \\ 0 & 0 & 0 & w & y-z & 0 \\ 0 & 0 & 0 & 0 & 0 & x-y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} w & y+z & 0 \\ 0 & 0 & x+y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} w & x-y \\ w+y-z & 0 \end{pmatrix}} R^2.$$

For each of these two choices for B, the syzygies of B^t are given by $\begin{pmatrix} 0 \\ w \end{pmatrix}$, $\begin{pmatrix} 0 \\ y+z \end{pmatrix}$, $\begin{pmatrix} xw \\ 0 \end{pmatrix}$, $\begin{pmatrix} yz \\ 0 \end{pmatrix}$, and $\begin{pmatrix} xy \\ 0 \end{pmatrix}$. Since there are only the two linear syzygies, B cannot be pushed forward to two non-isomorphic matrices.

Although we do not have an example of this ring branching, we are able to push forward one step, which is more than the Conca case. So $e = 4$ is the first monomial algebra case of this Hilbert series where branching is, in terms of the push forward method above, a possibility.

CHAPTER 10

SESQUI-ACYCLIC COMPLEXES

In [7] it is shown that not every minimal acyclic complex is sesqui-acyclic. In this chapter we study whether branching occurs over sesqui-acyclic complexes. We first look at an example of a sesqui-acyclic complex that is not totally acyclic.

10.1 Syzygies of Complete Duals

Question 10.1.1. Does every infinite syzygy M arise in the following way? There exists an R-module N such that $Ext^i_R(N,R)=0$ for all $i>0$ and $M\cong\Omega_n(N^*)$ for some n.

To rephrase, is there an N where M is the nth syzygy module of N^*. The answer is no, in general. In this section we show that the Question 10.1.1 has a negative answer. To show this, we construct an example over a ring R with $\operatorname{codim} R=5$ and $\mathfrak{m}^3=0$.

10.1.2. *Let k be a field and $\alpha\in k$. Consider the polynomial ring $Q=k[x_1,x_2,x_3,x_4,x_5]$ in five variables (each of degree one) and set*

$$R_\alpha=Q/I,$$

where I is the ideal generated by the following 11 quadratic relations:

$$x_1^2,\ x_4^2,\ x_2x_3,\ \alpha x_1x_2\ +x_2x_4,\ x_1x_3+x_3x_4,$$

$$x_2^2,\ x_2x_5-x_1x_3,\ x_3^2-x_1x_5,\ x_4x_5,\ x_5^2, x_3x_5.$$

As a vector space over k, R_α has a basis consisting of the following 10 elements:

$$1,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5.$$

In particular, R_α has Hilbert series $1+5t+4t^2$.

For each integer $i \in \mathrm{B}Z$ we let $d_i \colon R_\alpha^2 \to R_\alpha^2$ denote the map given with respect to the standard basis of R_α^2 by the matrix

$$\begin{pmatrix} x_1 & \alpha^i x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Consider the sequence of homomorphisms:

$$\mathbf{A}_\alpha : \quad \cdots \to R_\alpha^2 \xrightarrow{d_i+1} R_\alpha^2 \xrightarrow{d_i} R_\alpha^2 \xrightarrow{d_i-1} R_\alpha^2 \to \cdots .$$

From the given ring, we support the following theorem, which establishes a negative answer to Question 10.1.1:

Theorem 10.1.3. *For every nonzero $\alpha \in k$, the sequence $\mathbf{A}_\alpha$ is a minimal acyclic complex of free modules with $H_i(\mathbf{A}_\alpha^*) \neq 0$ for all $i \in \mathbb{Z}$.*

Corollary 10.1.4. *The right side $(\mathbf{A}_\alpha)_{\leq i}$ of the acyclic complex $\mathbf{A}_\alpha$ is the dual of no acyclic complex of free modules, for all i.*

Proof. Fix $\alpha \in k$ and set $R = R_\alpha$ and $\mathbf{A} = \mathbf{A}_\alpha$. Using the defining relation of R, one can easily show that $d_i \circ d_{i+1} = 0$ for all i, hence $\mathbf{A}$ is a complex. We let (a, b) denote an element of R^2 written in the standard basis of R^2 as a free R-module. For each i, the k-vector space image d_i is generated by the elements:

$$
\begin{aligned}
d_i(1,0) &= (x_1, x_3) & \qquad d_i(x_5,0) &= (x_1x_5, 0) \\
d_i(0,1) &= (\alpha^i x_2, x_4) & \qquad d_i(0,x_1) &= (\alpha^i x_1x_2, x_2x_4) \\
d_i(x_1,0) &= (0, x_1x_3) & \qquad d_i(0,x_2) &= (0, -\alpha x_1x_2) \\
d_i(x_2,0) &= (x_1x_2, 0) & \qquad d_i(0,x_3) &= (0, -x_1x_3) \\
d_i(x_3,0) &= (x_1x_3, x_1x_5) & \qquad d_i(0,x_4) &= (-\alpha^{i+1} x_1x_2, 0) \\
d_i(x_4,0) &= (x_1x_4, -x_1x_3) & \qquad d_i(0,x_5) &= (\alpha^i x_1x_3, 0)
\end{aligned}
$$

Excluding $d_i(0,x_3)$ and $d_i(0,x_4)$, the above equations provide 10 linearly independent elements in image d_i. Thus $\dim_k(\text{image}\, d_i) = 10$. Since

$$\dim_k \operatorname{Ker} d_{i+1} + \dim_k \text{image}\, d_i = \dim_k R^2 = 20$$

we have $\dim \operatorname{Ker} d_i = 10$. Thus, image $d_{i+1} = \operatorname{Ker} d_i$, so $\mathbf{A}$ is acyclic. To prove $\mathrm{H}_i(\mathbf{A}^*) \neq 0$, let $d_i^* \colon R^2 \to R^2$ denote the map given with respect to the standard basis of R^2 by the matrix

$$\begin{pmatrix} x_1 & x_3 \\ \alpha^i x_2 & x_4 \end{pmatrix}.$$

For each i, the vector space image d_i^* is generated by the following elements

$$d_i^*(1,0) = (x_1, \alpha^i x_2) \qquad d_i^*(x_5,0) = (x_1x_5, \alpha^i x_1x_3)$$

$$d_i^*(0,1) = (x_3, x_4) \qquad d_i^*(0,x_1) = (x_1x_3, x_1x_4)$$

$$d_i^*(x_1,0) = (0, \alpha^i x_1x_2) \qquad d_i^*(0,x_2) = (0, -\alpha x_1x_2)$$

$$d_i^*(x_2,0) = (x_1x_2, 0) \qquad d_i^*(0,x_3) = (x_1x_5, -x_1x_3)$$

$$d_i^*(x_3,0) = (x_1x_3, 0) \qquad d_i^*(0,x_4) = (-x_1x_3, 0)$$

$$d_i^*(x_4,0) = (x_1x_4, -\alpha^(i+1)x_1, x_2) \qquad d_i^*(0,x_5) = (0,0)$$

The map d_i^* represents the i^{th} map in $\mathbf{A}^*$. Excluding $d_i^*(0,x_2)$, $d_i^*(0,x_4)$, and $d_i^*(0,x_5)$ which are redundant, we have only 9 linearly independent elements in image d_i^*, hence $\dim_k \text{image}\, d_i^* = 9$ for every i. It follows that $\dim_k(\operatorname{Ker} d_i^*) = 11$, hence $\mathrm{H}_i(\mathbf{A}^*) \neq \mathbf{0}$. $\square$

We actually have a stronger result than that of 10.1.4:

Proposition 10.1.5. *There exists no module N such that $\operatorname{Ext}_R^i(N,R) = 0$ for all $i > 0$ and $\Omega(N^*) \cong \text{image}\, d_j$, some $n, j \in \mathbb{Z}$.*

Proof. To answer Question 10.1.1 negatively, in general, consider this example 10.1.2 over a finite field k. Let M be an infinite syzygy. Assume there exists an N such that $\mathrm{Ext}^i_R(N, R) = 0$ for all $i > 0$ and $\Omega_n(N^*) \cong \mathrm{image}\, d_j$ for some n. Then we have a complex

$$\cdots \to A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \to \cdots .$$

Let M be the module from the previous example. Dualizing we have the complex B, the right side is the minimal acyclic complex containing N^*. Since M is a syzygy module for N^*, and M has a unique resolution to the left, we have $\mathbf{A_{\geq 0}} \cong \mathbf{B_{\geq 0}}$, where M is the $0th$ syzygy module.

$$\begin{array}{lccccccccc}
\mathbf{A} & \cdots \longrightarrow & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)} & R^2 & \longrightarrow \cdots \\
 & & \| & & \| & & & & & \\
\mathbf{B} & \cdots \longrightarrow & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)} & R^2 & \xrightarrow{\partial_1^*} & R^2 & \xrightarrow{\partial_2^*} & R^2 & \longrightarrow \cdots
\end{array}$$

From the previous chapter, since k is a finite field, we have $\mathbf{A} \cong \mathbf{B}$ by Theorem 3.2.1. Then $\mathbf{A}^* \cong \mathbf{B}^*$.

$$\begin{array}{lccccccccc}
\mathbf{A}^* & \cdots \longrightarrow & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_3 \\ x_2 & x_4 \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_3 \\ x_2 & x_4 \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_3 \\ x_2 & x_4 \end{smallmatrix}\right)} & R^2 & \longrightarrow \cdots \\
 & & \downarrow f_2 & & \downarrow f_1 & & \| & & \| & \\
\mathbf{B}^* & \cdots \longrightarrow & R^2 & \xrightarrow{\partial_2} & R^2 & \xrightarrow{\partial_1} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x_1 & x_3 \\ x_2 & x_4 \end{smallmatrix}\right)} & R^2 & \longrightarrow \cdots
\end{array}$$

But $\mathrm{H}_i(\mathbf{A}^*) \neq \mathbf{0}$ for all i and $\mathrm{H}_i(\mathbf{B}^*) = \mathbf{0}$ for all $i \gg 0$. □

10.2 Uniqueness of Sesqui-Acyclic Complexes

The following theorem also provides a negative answer to the question for totally acyclic complexes, which have point of duality infinity.

Theorem 10.2.1. *Let A be a sesqui-acyclic complex with point of duality p, and B a sesqui-acyclic complex with point of duality p such that $A_{\geq r} \cong B_{\geq r}$, and $r < p \leq q$. Then $A \cong B$.*

Proof. By assumption we have a commutative diagram

$$\begin{array}{lcccccccccccccccccc}
A: & \cdots \longrightarrow & A_p & \longrightarrow \cdots \longrightarrow & A_{r+1} & \longrightarrow & A_r & \longrightarrow & A_{r-1} & \longrightarrow & A_{r-2} & \longrightarrow \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & & & & \\
B: & \cdots \longrightarrow & B_p & \longrightarrow \cdots \longrightarrow & B_{r+1} & \longrightarrow & B_r & \longrightarrow & B_{r-1} & \longrightarrow & B_{r-2} & \longrightarrow \cdots
\end{array}$$

both A and B sesqui-acyclic. Thus

$$A^* : \cdots \to A^*_{r-1} \to A^*_r \to A^*_{r+1} \to \cdots \to A^*_{p-1} \to A^*_p \to A^*_{p+1} \to \cdots$$

has $H(A^*_j) = 0$ for $j \leq p$, and

$$B^* : \cdots \to B^*_{r-1} \to B^*_r \to B^*_{r+1} \to \cdots \to B^*_{q-1} \to B^*_q \to B^*_{q+1} \to \cdots$$

has $H(B^*_j) = 0$ for $j \leq q$. Consider the diagram

$$\begin{array}{lcccccccccccccc}
A^*: & \cdots \longrightarrow & A^*_{r-2} & \longrightarrow & A^*_{r-1} & \longrightarrow & A^*_r & \longrightarrow & A^*_{r+1} & \longrightarrow \cdots \longrightarrow & A^*_{p-1} & \longrightarrow & A^*_p & \longrightarrow \cdots \\
 & & \downarrow ? & & \downarrow ? & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
B^*: & \cdots \longrightarrow & B^*_{r-2} & \longrightarrow & B^*_{r-1} & \longrightarrow & B^*_r & \longrightarrow & B^*_{r+1} & \longrightarrow \cdots \longrightarrow & B^*_{p-1} & \longrightarrow & B^*_p & \longrightarrow \cdots .
\end{array}$$

Since $r < p \leq q$ we have exactness at A^*_r. Therefore we can complete the diagram to obtain $A^*_{<r} \cong B^*_{<r}$:

$$\begin{array}{lcccccccccccccc}
A^*: & \cdots \longrightarrow & A^*_{r-2} & \longrightarrow & A^*_{r-1} & \longrightarrow & A^*_r & \longrightarrow & A^*_{r+1} & \longrightarrow \cdots \longrightarrow & A^*_{p-1} & \longrightarrow & A^*_p & \longrightarrow \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
B^*: & \cdots \longrightarrow & B^*_{r-2} & \longrightarrow & B^*_{r-1} & \longrightarrow & B^*_r & \longrightarrow & B^*_{r+1} & \longrightarrow \cdots \longrightarrow & B^*_{p-1} & \longrightarrow & B^*_p & \longrightarrow \cdots .
\end{array}$$

Dualize back to get $A \cong B$. □

Corollary 10.2.2. *Let A and B be totally acyclic complexes such that $A_{\geq r} \cong B_{\geq r}$. Then $A \cong B$.*

Proof. Since totally acyclic complexes are a special case of sesqui-acyclic complexes, by Theorem 10.2.1, $A \cong B$. □

Avramov and Martsinkovsky found in [2] that minimal acyclic complexes over Gorenstein rings are unique. However, the previous corollary provides alternative evidence to this fact, as all minimal acyclic complexes are totally acyclic in the Gorenstein case.

CHAPTER 11

CONCLUSION

Although we have not determined that branching of a minimal acyclic complex is possible, we have found certain scenarios where these complexes are decidedly unique. Periodic minimal acyclic complexes do not branch. From this result, we additionally conclude that rings over finite residue fields with $m^3 = 0$, rings of codepth ≤ 3, and rings that are one link from a complete intersection are unique to the right as well. We have learned conclusively that conca rings have no branching, and that branching cannot occur over sesqui-acyclic complexes in general.

In addition to determining circumstances where branching cannot happen, we have found possible affirmative scenarios via the push-forward method. Specifically, this occurs with the semi-conca $e = 4$ monomial algebra case as well as with conca-generated non-monomial algebras.

Possible future results include:

- Pushing a syzygy module forward an infinite number of times. (Achieving branching of a minimal acyclic complex.)
- Examining uniqueness of minimal acyclic complexes for monomial algebras for $e \geq 5$.
- Answering the question: If M is an nth syzygy module for all n, is M an infinite syzygy?

REFERENCES

[1] L. L. Avramov, *Homological asymptotics of modules over local rings,* Commutative Algebra, Berkeley, 1987 (M. Hochster, C. Huneke, J. Sally, eds.) MSRI Publ **15**, Springer, New York, 1989 Birkhuser, pp. 33-62.

[2] L. L. Avramov, A. Martsinkovsky, *Absolute, relative, and Tate Cohomology of modules of finite Gorenstein dimension,* Proc. London Math. Soc. **85** (2002), 393-440.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay rings,* Cambridge studies in advanced mathematics, Vol. 39, Revised edition, Cambridge University Press.

[4] L. W. Christensen, O. Veliche, *Acyclicity over local rings with radical cube zero,* Illinois J. Math. **54** (2007), 1439-1454.

[5] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry,* Springer -Verlag, New York, 1995.

[6] P. J. Hilton, U. Stammbach, *A Course in Homological Algebra, 2nd Ed.* Springer -Verlag, New York, 1997.

[7] D. A. Jorgensen, L. M. Sega, *Independence of the total reflexivity conditions for modules,* Algebr. Represent. Theory **9** (2006), 217-226.

[8] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, Inc., Orlando, 1979.

BIOGRAPHICAL STATEMENT

Meri Trema Hughes was born in Dallas, Texas, on July 30, 1971, to Dudley and Trema Hughes, moved to Combine, Texas, as a small child, and graduated as valedictorian from Crandall High School in 1989. She then attended East Texas Baptist University in Marshall, Texas, earning a Bachelor of Science in Education, Summa Cum Laude, in 1993, specializing in mathematics and theatre arts. Meri completed a Masters of Science from Baylor University in Waco in 1995, writing a thesis on Reed-Solomon Codes under the direction of David Arnold. She began her teaching career in the math department of Dallas Baptist University as a temporary full-time instructor. After two years there, she spent the next nine years honing her skills while on the faculty at Weatherford College. In 2006, with the assistance of a GAANN Fellowship funded by the U.S. Department of Education, she was fortunate to return to the University of Texas at Arlington to complete her education. Under the tutelage of David Jorgensen, Meri studied homological algebra and began her research in the area of minimal acyclic complexes. While there, she received the Outstanding Graduate Teaching Award for 2008 and the Outstanding Graduate Student Award for 2009. Her greatest accomplishment by far is being mom to Marshall and Ranger.