

MICROMECHANICAL ANALYSIS OF YIELDING
FOR PARTICULATE FILLED COMPOSITES

by

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ABSTRACT

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Particulate filled composites are hybrid materials which consist of particulates held together by a matrix. In this thesis, a simulation method to obtain overall failure criterion for particulate filled composites based on the computer software “*MATHEMATICA*” will be presented. The composite is assumed to fail at the matrix just outside the inclusion. By Self-Consistent Method, the composite can be imaged as an inclusion embedded within effective matrix. The Equivalent Inclusion method is then applied to simulate the stress field within the inclusion. Stress just outside inclusion can be calculated because interfacial traction force on the boundary of inclusion is

continuous. The stress just outside inclusion can be applied to a suitable failure criterion. Therefore, local failure criteria can be obtained and transferred into global failure criteria.

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CHAPTER 1

INTRODUCTION

Particulate-filled composites are hybrid materials which consist of particulate reinforcements held together by a common matrix. Generally, depending on the requirement of special engineering needs, engineers try to make composites with desirable materials properties such as high strength, high stiffness, lower density, and low coefficient of thermal expansion. Therefore, it makes sense to enhance the materials' properties by adding a certain amount of reinforcements into the common structural materials which are generally referred to ceramics, polymers, or metals.

The development of particulate-filled composites is due to the reason that the commercial applications for fiber-reinforced composites are restricted in industries because of the high cost and difficult processes. Engineers try to cut down the expense for reinforcements by using short fibers, whiskers, fabric, and particulates. The manufacturing processes for particulate-filled composites are relatively easy and inexpensive, so particulate-filled composites are widely used in automobiles, electronic packages, and other consumer products.

In the late 1980s, readily available ceramic powders such as aluminum oxide, silicon nitride and silicon carbide, which are used for abrasives, made it possible to

make particulate-filled composites at a relatively lower cost and with easiness [1]. These ceramic powders combine high strength and stiffness with capability at high temperature, so they are widely used as particulate reinforcements to improve the material's overall properties of the structural materials.

The process of particulate-filled composites is simple and inexpensive; however, the strength of particulate-filled composites cannot be compared with the fiber-reinforced composites. The efficiency to enhance the strength by particulates is not as good as the one provided by continuous fibers. Therefore, in order to provide a range of sufficient strength at a relative low cost, engineers try to better understand how to use particulate-filled composites in specific engineering applications.

Among many mechanical and physical properties, failure criteria for particulate-filled composites are of significant importance. For a long time, an empirical failure criterion equation widely used is the Tsai-Wu equation [2], which is a macroscopically derived empirical equation. However, because of mathematical difficulty and complexity, microscopic modeling of particulate filled composites is a difficult task and no exact formulation is available to the author's best knowledge.

Micromechanical analysis of the stress field for an inclusion (a second phase with stress-free strains) embedded in a surrounding elastic medium was first derived by Eshelby [3] and is considered to be one of the most important results in applied mechanics for analyzing inclusion-dispersed composites. Eshelby's result shows that if an ellipsoidal inclusion in infinite elastic medium is subjected to a uniform strain (called the "stress-free strain," "unconstrained strain," "eigenstrain," or "transformation strain"),

uniform stress and strain states are induced in the constrained inclusion. This eigenstrain can be any kind of non-elastic strains such as thermal expansion, phase transformation, or a strain which involves no changes in the elastic constants of the inclusion. By choosing a proper eigenstrain, stress field within the ellipsoidal inclusion can be obtained.

By further extending Eshelby's original result, it is possible to analyze an inhomogeneous object embedded in the surrounding medium. This method is called the "equivalent inclusion method" which was studied and developed by many researchers (see [4] for example). The stress-free strain can be related with the eigenstrain by Eshelby's tensor which depends on the shape of a particulate and the elastic moduli of the surrounding medium. The stress field inside an inhomogeneity subjected to a uniform stress at the far field can be simulated by properly choosing the eigenstrain. By solving an equivalency problem for the stress field, the equivalent eigenstrain can be found. Consequently, stress field within the inclusion is obtained.

In real applications, particulate-filled composites contain many particulates with the elastic modulus differing from the surrounding matrix. The overall elastic behavior of composites with particulate volume fractions must explicitly account for the interaction between individual particulate. In order to approximate the reinforcement interaction effects as well as the concomitant perturbation of the stress and strain fields in the matrix, a Self-Consistent Method can be introduced to obtain the effective elastic modulus of matrix [5].

In this thesis, a failure condition for particulate-filled composites is studied and derived by a micromechanics approach with an assumption that yielding takes place in the matrix phase just outside the inclusion. A composite is modeled as an infinitely extended body that contains spherical inhomogeneities. The effective elastic modulus of matrix is derived from the Self-Consistent Method. The stress field in the matrix just outside inclusion is derived as a function of the far-field stress and thus the local failure condition at the interface of the matrix and an inclusion can be translated into the composite (global) failure criterion in terms of the far-field stress. The analytical expression of the stress field just outside inclusion is derived with the help of a computer algebra system which automates tedious algebra required for tensor manipulations.

The background of particulate filled composites will be discussed in Chapter 2. The fundamental theory will be described in Chapter 3. The analysis will be shown in Chapter 4. The results and discussions are presented in Chapter 5.

CHAPTER 2

BACKGROUND INFORMATION OF COMPOSITE MATERIALS

A composite material is a combination of two or more distinct phases, which are called the matrix and reinforcements. The purpose for making composites is to provide materials having low weight, high strength, and other desirable physic properties. Materials with low density and high strength are considered as reinforcements. In engineering applications, reinforcements are used to improve the characteristics of matrix which are common materials such as metals, ceramics, and polymers [6]. Matrix materials typically have two important functions: transfer the load to reinforcement materials and protect the reinforcement materials from corrosion, chemicals, and others. According to the type of matrix phases, composites can be classified as ceramic matrix composites (CMCs), polymer matrix composites (PMCs), and metal matrix composites (MMCs).

The need for reinforcement comes from the demand that engineers require light weight materials which can provide the same mechanical properties as common materials used in military and aerospace industries. Therefore, they tried to make light materials by putting some reinforcements into materials (see Figure 2.1) [6].

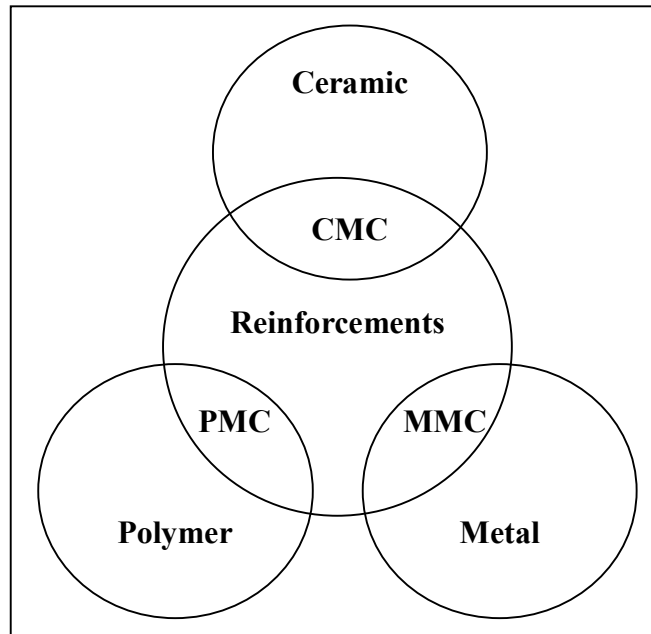


Figure 2.1 The relation for reinforcements and common materials.

Reinforcements are typically made in the forms of continuous fibers, short fibers, whiskers, or particulates. Continuous fibers provide the most effective improvement in strength for structural materials. On the other hand, short fibers, whiskers, and particulates give only a range of sufficient improvement.

The aspect ratio of the reinforcement is a factor used to determine the efficiency of load transfer from the matrix to the reinforcement. The larger the aspect ratio, the more efficient the load transfer. In most applications, the reinforcement phase can be idealized as an ellipsoidal inclusion. The aspect ratio is, therefore, defined as the ratio of the major axis to the minor axis or, alternatively, the effective length to the diameter of the inclusion (see Figure 2.2).

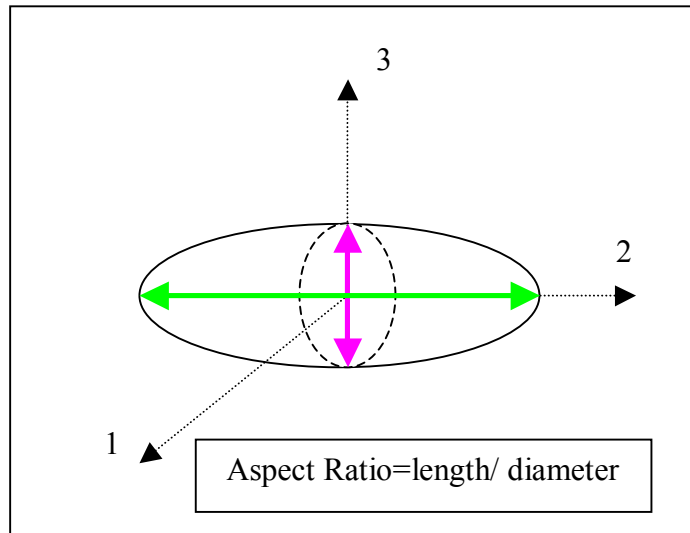


Figure 2.2 The length and diameter of ellipsoidal inclusion.

The ranges of aspect ratio for various reinforcements used in metal matrix composites are generally summarized below (see Table 2.1) [7].

Table 2.1 Ranges of aspect ratio for various reinforcements

| Type | Aspect Ratio | Diameter (μm) | |
|----------------------------|-------------------|----------------------------|---|
| Continuous fibers | 1000 $\sim\infty$ | 3 \sim 150 | SiC, Al ₂ O ₃ , C, B, W, Nb-Ti, Nb ₃ Sn |
| Short fibers (or whiskers) | 10 \sim 1000 | 1 \sim 5 | C, SiC, Al ₂ O ₃ , Al ₂ O ₃ +SiO ₂ |
| Particulates | 1 \sim 4 | 1 \sim 25 | SiC, Al ₂ O ₃ , BN, B ₄ C, WC |

Generally, the aspect ratio tends to be infinity for continuous fibers and around one for particulates. As a result, the efficiency to improve the strength of structural

materials is much better for continuous fibers than for short fibers, whiskers, or particulates.

Despite the fact that continuous fibers offer the dramatic improvement for structural materials, the method of preparation for continuous fibers is complicated and expensive [7]. It is an expensive investment on equipment, raw materials, and electrical energy. The usage of fiber reinforced composites is restricted and only accepted in the advanced materials for military and aerospace industries.

In order to use composite materials in commercial applications, engineers from industries tried to find solutions to decrease the expense for reinforcement materials. Short fibers, whiskers, fabric, or particulates are then used to provide a range of sufficient strength for the composites [7].

In the late 1980s, due to the wide range of readily available ceramic powders such as aluminum oxide and silicon carbide, which are used for abrasive and cutting media, the cost of particulate reinforcement became very low. This makes it much easier and cheaper to obtain the sources for particulate reinforcements. In order to use particulate filled composites properly, we'll evaluate the behavior of particulate filled composite based on the existing mathematic model in the following chapters.

CHAPTER 3

FUNDAMENTAL THEORY

The failure criterion of particulate filled composites is an important subject that has been studied for decades [3]. In this Chapter, a mathematical method to predict the failure condition for particulate filled composites is discussed in detail.

The mechanical behavior of materials can be obtained easily by a simple stress-strain test. A typical stress and strain curve from the load elongation measurement is shown as below (see Figure 3.1). The stress-strain curve by this method is a bulk mechanical behavior of materials. This method is limited in its application because it assumes that the sample is homogeneous. For materials containing inhomogeneities, a refined model is necessary to have a better understanding for the stress field in the neighborhood of inhomogeneities.

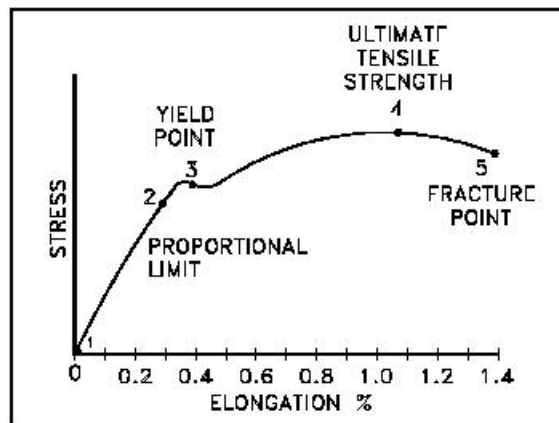


Figure 3.1 The engineering stress and strain curve

3.1 Eigenstrain Problem

First, let's consider a finite domain, Ω , that contains an eigenstrain (see Figure 3.2). The eigenstrain is defined as non-elastic strain caused by non-elastic actions such as thermal stress, residue stresses or plastic deformation.

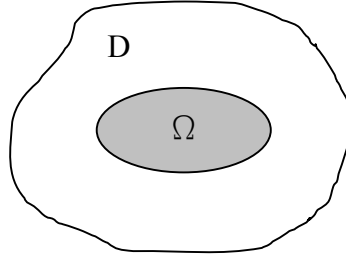


Figure 3.2 Inclusion contains an eigenstrain.

When slight elastic deformation takes place, the total strain within the domain, Ω , is computed as the summation of the elastic strain, e_{ij} , and the eigenstrain, ε_{ij}^* , as

$$\boldsymbol{\varepsilon}_{ij} = \boldsymbol{e}_{ij} + \boldsymbol{\varepsilon}_{ij}^* \quad (3.1)$$

According to Hooke's law, the stress σ_{ij} related to the elastic strain e_{ij} within the domain Ω can be described as

$$\boldsymbol{\sigma}_{ij} = C_{ijkl} \boldsymbol{e}_{kl} = C_{ijkl} (\boldsymbol{\varepsilon}_{kl} - \boldsymbol{\varepsilon}_{kl}^*) \quad (3.2)$$

where C_{ijkl} is the elastic modulus of domain Ω .

The inverse expression of (3.2) is

$$\boldsymbol{\varepsilon}_{ij} - \boldsymbol{\varepsilon}_{ij}^* = C_{ijkl}^{-1} \boldsymbol{\sigma}_{kl} \quad (3.3)$$

where C_{ijkl}^{-1} is the elastic compliance of domain Ω .

For isotropic materials, (3.2) and (3.3) can be further expressed as

$$\sigma_{ij} = 2\mu(\epsilon_{ij} - \epsilon_{ij}^*) + \lambda \delta_{ij}(\epsilon_{kk} - \epsilon_{kk}^*) \quad (3.4)$$

$$\epsilon_{ij} - \epsilon_{ij}^* = \frac{1}{2\mu} \left\{ \sigma_{ij} - \frac{\nu}{1+\nu} \delta_{ij} \sigma_{kk} \right\} \quad (3.5)$$

where λ and μ are the Lamé constants, and ν is Poisson's ratio. Young's modulus E , the shear modulus, μ , and the bulk modulus, K , are related by $2\mu = E/(1+\nu)$, $K = E/3(1-2\nu)$, and $\lambda = 2\mu\nu/(1-2\nu)$.

3.1.1. Inclusion

The term “inclusion” refers to a domain whose elastic constants are the same as those in the surrounding matrix but having an eigenstrain inside. Let's consider a subdomain, Ω , as an inclusion embedded in domain D (see Figure 3.2). The domain, $D - \Omega$, has the same elastic modulus with sub-domain, Ω , and domain, D . The domain, Ω , has an eigenstrain ϵ_{ij}^* that is defined as a non-elastic strain.

The displacement, u_i , and the interfacial traction across $D - \Omega$, t_i , must be continuous. That is

$$[u_i] = u_i(out) - u_i(in) = 0 \quad (3.6)$$

$$[\sigma_{ij}]n_j \equiv \{\sigma_{ij}(out) - \sigma_{ij}(in)\}n_j = 0 \quad (3.7)$$

where n_j is the unit normal to the boundary of Ω and $[.]$ denotes the difference in the quantity across the interface. When an inclusion or inhomogeneity can slide on the interfacial surface, the first condition does not hold. According to the discussions by Hill (1963) and by Walpole (1967), the strain ϵ_{ij} is continuous inside D and Ω but discontinuous at the interface between the two domains. The jump across the interface can be written as

$$[\boldsymbol{\varepsilon}_{ij}] = \boldsymbol{\varepsilon}_{ij}(out) - \boldsymbol{\varepsilon}_{ij}(in) = \lambda_i \mathbf{n}_j \quad (3.8)$$

where λ_i is the proportionality constant (the magnitude of the jump) to be determined.

By substituting (3.2) into (3.7), the continuity of the traction force can be written as

$$\begin{aligned} [\boldsymbol{\sigma}_{ij}] \mathbf{n}_j &\equiv \{\boldsymbol{\sigma}_{ij}(out) - \boldsymbol{\sigma}_{ij}(in)\} \mathbf{n}_j \\ &= C_{ijkl} \{[\boldsymbol{\varepsilon}_{kl}(out) - 0] - [\boldsymbol{\varepsilon}_{kl}(in) - \boldsymbol{\varepsilon}_{kl}^*]\} \mathbf{n}_j \\ &= C_{ijkl} \{\boldsymbol{\varepsilon}_{kl}(out) - \boldsymbol{\varepsilon}_{kl}(in) + \boldsymbol{\varepsilon}_{kl}^*\} \mathbf{n}_j \\ &= C_{ijkl} [\boldsymbol{\varepsilon}_{kl}(out) - \boldsymbol{\varepsilon}_{kl}(in)] \mathbf{n}_j + C_{ijkl} \boldsymbol{\varepsilon}_{kl}^* \mathbf{n}_j \end{aligned} \quad (3.9)$$

By substituting (3.8) into (3.9), this equation can be written as

$$C_{ijkl} \lambda_k \mathbf{n}_l \mathbf{n}_j = -C_{ijkl} \boldsymbol{\varepsilon}_{kl}^* \mathbf{n}_j \quad (3.10)$$

Equation (3.10) is a system of equations to determine λ for given \mathbf{n} and $\boldsymbol{\varepsilon}_{ij}^*$.

3.1.2 Inhomogeneity

The term ‘‘inhomogeneity’’ refers to a domain whose elastic constants are different from those in the surrounding domain. Let’s consider a sub-domain Ω as an inhomogeneity embedded within the domain D . The elastic modulus is C_{ijkl}^* in the sub-domain Ω and C_{ijkl} in the domain D . In the condition of a uniform stress applied at the far field, the stress field near the inhomogeneity is not uniform. The stress disturbance due to the inhomogeneity can be simulated by a stress field caused by an inclusion when the eigenstrain is chosen properly.

The displacement and the interfacial traction across the boundary must be continuous. From Hooke's law, the stress fields just outside the boundary of Ω can be written as

$$\boldsymbol{\sigma}_{ij}(\text{out}) = \mathbf{C}_{ijkl} \boldsymbol{\varepsilon}_{kl}(\text{out}) \quad (3.11)$$

By substituting (3.11) into (3.7), the continuity of traction force is written as

$$\mathbf{C}_{ijkl} \boldsymbol{\varepsilon}_{kl}(\text{out}) \mathbf{n}_j = \mathbf{C}_{ijkl}^* \boldsymbol{\varepsilon}_{kl}(\text{in}) \mathbf{n}_j \quad (3.12)$$

The relation of (3.8), $\varepsilon_{ij}(\text{out}) = \lambda_i n_j + \varepsilon_{ij}(\text{in})$, is now substituted into (3.12). Then, we have the equation to determine the unknown vector λ as

$$\mathbf{C}_{ijkl} \lambda_k \mathbf{n}_l \mathbf{n}_j = (\mathbf{C}_{ijkl}^* - \mathbf{C}_{ijkl}) \boldsymbol{\varepsilon}_{kl}(\text{in}) \mathbf{n}_j \quad (3.13)$$

3.2 Equivalent Inclusion Method

First, let's consider an inhomogeneity with the elastic moduli \mathbf{C}_{ijkl}^* occupying the domain, Ω , in an infinitely extended matrix, $D-\Omega$, with the elastic constant, \mathbf{C}_{ijkl} (see Figure 3.3). Because of the presence of the inhomogeneity with the elastic modulus differing from the infinitely extended matrix, we denote the applied stress disturbance by σ_{ij}^d and the strain disturbance by ε_{kl}^d . The total stress is $\sigma_{ij}^a + \sigma_{ij}^d$, and the total strain is $\varepsilon_{kl}^a + \varepsilon_{kl}^d$. Hooke's law is written as

$$\boldsymbol{\sigma}_{ij}^a + \boldsymbol{\sigma}_{ij}^d = \mathbf{C}_{ijkl}^* (\boldsymbol{\varepsilon}_{kl}^a + \boldsymbol{\varepsilon}_{kl}^d) \quad (3.14)$$

$$\boldsymbol{\sigma}_{ij}^a + \boldsymbol{\sigma}_{ij}^d = \mathbf{C}_{ijkl} (\boldsymbol{\varepsilon}_{kl}^a + \boldsymbol{\varepsilon}_{kl}^d) \quad (3.15)$$

The equivalent inclusion method is used to simulate the stress disturbance resulted from the inhomogeneity occupying in the domain Ω by choosing a suitable

eigenstrain ϵ_{kl}^* . Consider an infinitely extended homogeneous material with the elastic constant C_{ijkl}^m everywhere, containing a sub-domain Ω having an eigenstrain ϵ_{kl}^* (see Figure 3.3). Then, Hooke's law yields

$$\sigma_{ij}^a + \sigma_{ij}^d = C_{ijkl}^* (\epsilon_{kl}^a + \epsilon_{kl}^d - \epsilon_{kl}^*) \quad (3.16)$$

$$\sigma_{ij}^a + \sigma_{ij}^d = C_{ijkl} (\epsilon_{kl}^a + \epsilon_{kl}^d) \quad (3.17)$$

The necessary and sufficient condition for the equivalency of the stresses and strains field is

$$C_{ijkl}^* (\epsilon_{kl}^a + \epsilon_{kl}^d) = C_{ijkl} (\epsilon_{kl}^a + \epsilon_{kl}^d - \epsilon_{kl}^*) \quad (3.18)$$

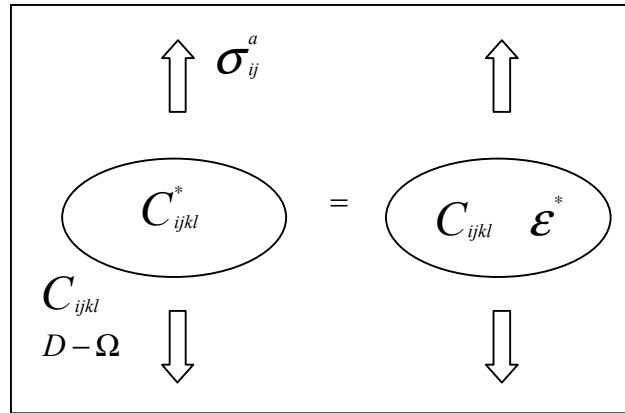


Figure 3.3 Schematic view of equivalent inclusion method (Stress σ_{ij}^a is the far-field stress applied at infinity)

The uniform strain in the constrained region, Ω , (the “constrained strain”), ϵ_{ij}^c , is related to the stress-free strain (eigenstrain), ϵ_{mn}^* , by the relation

$$\epsilon_{ij}^c = S_{ijmn} \epsilon_{mn}^* \quad (3.19)$$

where S_{ijmn} is called the Eshelby tensor [4, 3, 8] and ε_{mn}^* may be any kind of eigenstrain which is uniform over the inclusion.

Substitution of equation (3.19) into equation (3-18) leads to

$$C_{ijkl}^* (\varepsilon_{kl}^a + S \varepsilon_{mn}^*) = C_{ijkl} (\varepsilon_{kl}^a + S \varepsilon_{mn}^* - \varepsilon_{kl}^*) \quad (3.20)$$

The eigenstrain, ε_{kl}^* , is determined by solving equation (3-20). Therefore, the uniform stress field inside the ellipsoidal inhomogeneity can be described as

$$\begin{aligned} \sigma_{ij}(in) &= C_{ijkl}^* \varepsilon_{kl}(in) \\ &= C_{ijkl}^* (\varepsilon_{kl}^a + \varepsilon_{kl}^d) \\ &= C_{ijkl}^* (\varepsilon_{kl}^a + S \varepsilon_{mn}^*) \\ &= C_{ijkl}^* \left\{ I + S \left[(C_{ijkl} - C_{ijkl}^*) S - C_{ijkl} \right]^{-1} (C_{ijkl}^* - C_{ijkl}) \right\} \varepsilon_{kl}^a \end{aligned} \quad (3.21)$$

where $\varepsilon_{kl}(in)$ is the total strain within the ellipsoidal inhomogeneity.

$$\varepsilon_{kl}(in) = A_p \varepsilon_{kl}^a \quad (3.22)$$

A_p is the proportional factor defined for the relation between total strain within the inhomogeneity and strain in the far field.

$$A_p = C_{ijkl}^* \left\{ I + S \left[(C_{ijkl} - C_{ijkl}^*) S - C_{ijkl} \right]^{-1} (C_{ijkl}^* - C_{ijkl}) \right\} \quad (3.23)$$

3.3 Self-Consistent Method

Consider an elastic medium containing many particulate reinforcements (inhomogeneities) having the elastic modulus differing from the surrounding (matrix).

The disturbance in stress and strain from individual particulate (inhomogeneity) would

definitely affect the response strain and stress field in the far field. Therefore, the descriptions of the overall elastic behavior of composites with particulate volume fractions must explicitly account. A Self-Consistent Method is introduced in this section to evaluate the effective elastic modulus of the composite

The Self-Consistent Method is a mathematical approximation of a microstructure by imaging that a single particulate (inhomogeneity) embedded within the effective matrix having the elastic modulus of composite (see Figure 3.4).

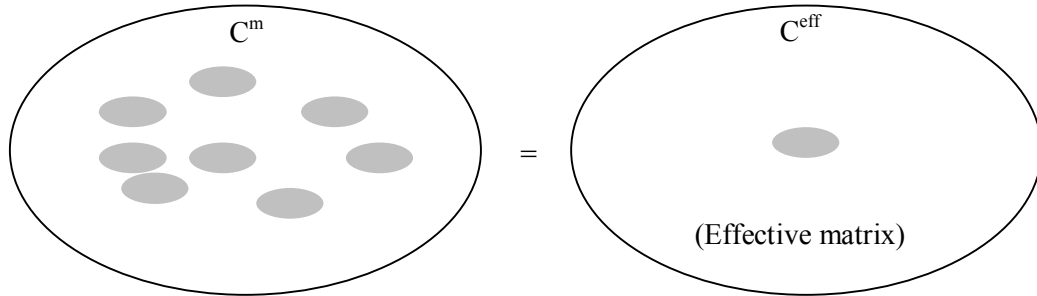


Figure 3.4 Schematic view of Self-Consistent Method.

For particulate filled composite in iso-stress state ($\sigma_{ij}^m = \sigma_{ij}^p = \sigma_{ij}^a$):

$$\begin{aligned}
 \boldsymbol{\varepsilon}_{kl}^a &= V_m \boldsymbol{\varepsilon}_{kl}^m + V_p \boldsymbol{\varepsilon}_{kl}^p \\
 &= V_m \boldsymbol{\varepsilon}_{kl}^a + V_p A_p \boldsymbol{\varepsilon}_{kl}^a \\
 &= (V_m + V_p A_p) \boldsymbol{\varepsilon}_{kl}^a
 \end{aligned} \tag{3.24}$$

Therefore, the proportional factor A_p related to the volume fraction factors, V_m and V_p , is obtained as

$$V_m + V_p A_p = 1 \quad (3.25)$$

For particulate filled composite in iso-strain state ($\boldsymbol{\varepsilon}_{kl}^m = \boldsymbol{\varepsilon}_{kl}^p = \boldsymbol{\varepsilon}_{kl}^a$):

$$\begin{aligned} \boldsymbol{\sigma}_{ij}^a &= \mathbf{C}_{ijkl} \boldsymbol{\varepsilon}_{kl}^a \\ &= \boldsymbol{\sigma}_{ij}^p V_p + \boldsymbol{\sigma}_{ij}^m V_m \\ &= (\mathbf{C}_{ijkl}^p A_p \boldsymbol{\varepsilon}_{kl}^a) V_p + \mathbf{C}_{ijkl}^m \boldsymbol{\varepsilon}_{kl}^a V_m \end{aligned} \quad (3.26)$$

Therefore, the overall elastic modulus of composite, \mathbf{C}_{ijkl}^c , is expressed as

$$\mathbf{C}_{ijkl}^c = \mathbf{C}_{ijkl}^m V_m + A_p \mathbf{C}_{ijkl}^p V_p \quad (3.27)$$

If the particulate filled composite display isotropy, it can be hold in the states of iso-stress and iso-strain simultaneously when a uniform stress applied in infinity. By substitution equation (3.25) into equation (3.27), it can be written as

$$\mathbf{C}_{ijkl}^c = \mathbf{C}_{ijkl}^m + A_p (\mathbf{C}_{ijkl}^p - \mathbf{C}_{ijkl}^m) V_p \quad (3.28)$$

To obtain the elastic modulus for the effective matrix, an equation revised from equation (3.28) is applied.

$$\mathbf{C}_{ijkl}^{eff} = \mathbf{C}_{ijkl}^m + V_p (\mathbf{C}_{ijkl}^p - \mathbf{C}_{ijkl}^m) A_p \quad (3.29)$$

The following is a procedure to apply the self-consistent approximation to obtain \mathbf{C}_{ijkl}^{eff} .

The expression for the effective elastic modulus for composite containing particulates is given by equation (3.29).

The term, A_p , is the proportionality factor of the strain field inside an inhomogeneity embedded in an infinitely extended matrix to the far field strain at infinity. In the Self-Consistent approximation, the (unknown) composite effective modulus is used as the elastic modulus of the matrix surrounding the inhomogeneity. Equation (3-23) is revised as

$$A_{p,S-C} = \left\{ I + S \left[(C_{ijkl}^{eff} - C_{ijkl}^p) S - C_{ijkl}^{eff} \right]^{-1} (C_{ijkl}^p - C_{ijkl}^{eff}) \right\} \quad (3.30)$$

Equation (3.29) thus becomes a set of non-linear simultaneous equations for C_{ijkl}^{eff} .

$$C_{ijkl}^{eff} = C_{ijkl}^m + V_p (C_{ijkl}^p - C_{ijkl}^m) \left\{ I + S \left[(C_{ijkl}^{eff} - C_{ijkl}^p) S - C_{ijkl}^{eff} \right]^{-1} (C_{ijkl}^p - C_{ijkl}^{eff}) \right\} \quad (3.31)$$

The set of simultaneous equations can be solved by successive iterations or the Newton-Raphson method until convergence is achieved.

3.4 von Mises Criterion

The stress tensor, σ_{ij} , can be uniquely decomposed into the shear part (deviatoric part) and the volume expansion part (hydrostatic part) as

$$\sigma_{ij} = \sigma'_{ij} + \frac{1}{3} \delta_{ij} \sigma_{kk} \quad (3.32)$$

The equation to determine the principal stress deviations can be expressed as

$$\left| \sigma'_{ij} - \sigma' \delta_{ij} \right| = 0 \quad (3.33)$$

The equation (3.33) can be expand as

$$\sigma'^3_{ij} - J_1 \sigma'^2_{ij} - J_2 \sigma'_{ij} - J_3 = 0 \quad (3.34)$$

$$J_1 = \sigma'_{ii} = 0 \quad (3.35)$$

$$J_2 = -\frac{1}{2} \sigma'_{ij} \sigma'_{ij} \quad (3.36)$$

$$J_3 = -\frac{1}{6} \sigma'_{ij} \sigma'_{jk} \sigma'_{kl} \quad (3.37)$$

where J1, J2 and J3 are defined as the stress invariants of the deviatoric part of the stress.

The von Mises criterion (1913), also known as the maximum distortion energy criterion, can be defined as

$$J_2 = -\frac{1}{2} \sigma'_{ij} \sigma'_{ij} = -k^2 \quad (3.38)$$

where the constant C is defined as a constant for the state of von Mises criterion.

For different material, the constant k can be computed if we put a value of yield stress in uniaxial direction (σ_y). Therefore, the constant k can be expressed as

$$k^2 = \frac{1}{2} \sigma'_y \sigma'_y \quad (3.39)$$

Substitute equation (3.32) into Equation (3.39), it can be simplified as

$$k^2 = \frac{1}{2} \left(\sigma_y - \frac{1}{3} \delta_{ij} \sigma_y \right) \left(\sigma_y - \frac{1}{3} \delta_{ij} \sigma_y \right) = \frac{1}{3} \sigma_y^2 \quad (3.40)$$

Therefore, the von Mises criterion can be further simplified as

$$\sigma_{ij} \sigma_{ij} = \frac{2}{3} \sigma_y^2 \quad (3.41)$$

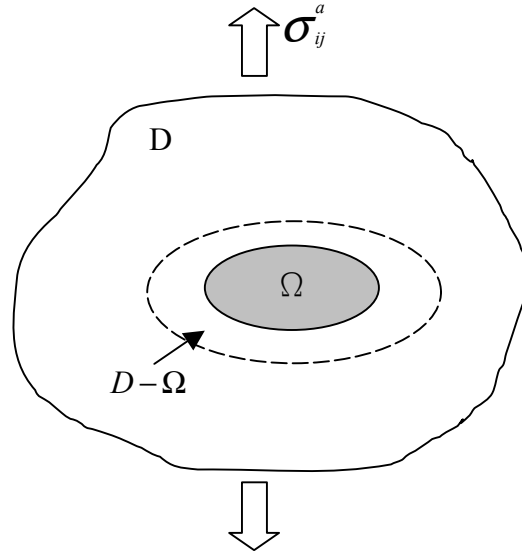
CHAPTER 4

ANALYSIS

In this Chapter, the fundamental theories that were developed in the previous Chapter will be consolidated together and applied to simulation of the yielding condition for particulate filled composites. The cases for an infinitely extended matrix reinforced by a single particulate and multiple particulates will be discussed in detail.

4.1 Infinitely Extended Matrix Reinforced by an Inhomogeneity

Consider an infinitely extended matrix reinforced by an ellipsoidal particulate that has the elastic modulus differing from the remainder (matrix). The stress disturbance due to the presence of this particulate will be induced while a uniform stress is applied at infinity (see Figure 4.1).



Ω : particulate
 $D-\Omega$: matrix affected by disturbance
 D : matrix in infinity

Figure 4.1 Infinitely extended matrix reinforced by a single particulate

Hooke's law according to equations (3.11), (3.14) and (3.15) can be written as:

$$\sigma_{ij}^a = C_{ijkl}^m \epsilon_{kl}^a \quad (4.1)$$

$$\sigma_{ij}^a + \sigma_{ij}^d = C_{ijkl}^m (\epsilon_{kl}^a + \epsilon_{kl}^d) \quad (4.2)$$

$$\sigma_{ij}^a + \sigma_{ij}^d = C_{ijkl}^p (\epsilon_{kl}^a + \epsilon_{kl}^d) \quad (4.3)$$

Let's choose a suitable eigenstrain, ϵ_{kl}^* , to simulate the stress disturbance by using the equivalent inclusion method. Hooke's law considering an inclusion having an eigenstrain can be written as:

$$\sigma_{ij}^a = C_{ijkl}^m \epsilon_{kl}^a \quad (4.4)$$

$$\sigma_{ij}^a + \sigma_{ij}^d = C_{ijkl}^m (\epsilon_{kl}^a + \epsilon_{kl}^d) \quad (4.5)$$

$$\sigma_{ij}^a + \sigma_{ij}^d = C_{ijkl}^p (\epsilon_{kl}^a + \epsilon_{kl}^d - \epsilon_{kl}^*) \quad (4.6)$$

The necessary and sufficient condition for the equivalency of the stress and strain is

$$C_{ijkl}^p (\epsilon_{kl}^a + \epsilon_{kl}^d) = C_{ijkl}^m (\epsilon_{kl}^a + \epsilon_{kl}^d - \epsilon_{kl}^*) \quad (4.7)$$

From equation (3.19), the relation between the strain disturbance (also called the constrained strain) and the eigenstrain (stress-free strain) in the inclusion can be written in the form

$$\epsilon_{kl}^d = S_{klmn} \epsilon_{mn}^* \quad (4.8)$$

Substitution of (4.8) into equation (4.7), the equation of equivalency can be written as

$$C_{ijkl}^p (\epsilon_{kl}^a + S_{klmn} \epsilon_{mn}^*) = C_{ijkl}^m (\epsilon_{kl}^a + S_{klmn} \epsilon_{mn}^* - \epsilon_{kl}^*) \quad (4.9)$$

By solving equation (4.9), the eigenstrain (stress-free strain) is determined as

$$\epsilon_{mn}^* = \left[(C_{ijkl}^m - C_{ijkl}^p) S_{klmn} - C_{ijkl}^m \right]^{-1} (C_{ijkl}^p - C_{ijkl}^m) \epsilon_{kl}^a \quad (4.10)$$

By substituting (4.10) into (4.8), the stress disturbance due to the presence of ellipsoidal particulate can be found as

$$\epsilon_{kl}^d = S_{klmn} \epsilon_{mn}^* = S_{klmn} \left[(C_{ijkl}^m - C_{ijkl}^p) S_{klmn} - C_{ijkl}^m \right]^{-1} (C_{ijkl}^p - C_{ijkl}^m) \epsilon_{kl}^a \quad (4.11)$$

Therefore, the total strain and total stress within the ellipsoidal particulate are

$$\boldsymbol{\varepsilon}_{kl}(in) = \left\{ I + S_{klmn} \left[(C_{ijkl}^m - C_{ijkl}^p) S_{klmn} - C_{ijkl}^m \right]^{-1} (C_{ijkl}^p - C_{ijkl}^m) \right\} \boldsymbol{\varepsilon}_{kl}^a \quad (4.12)$$

$$\boldsymbol{\sigma}_{ij}(in) = C_{ijkl}^p \left\{ I + S_{klmn} \left[(C_{ijkl}^m - C_{ijkl}^p) S_{klmn} - C_{ijkl}^m \right]^{-1} (C_{ijkl}^p - C_{ijkl}^m) \right\} \boldsymbol{\varepsilon}_{kl}^a \quad (4.13)$$

From equation (3.8), the total stress just outside the interface of domain Ω and D- Ω is

$$\boldsymbol{\varepsilon}_{kl}(out) = \boldsymbol{\varepsilon}_{kl}(in) + \lambda_k \mathbf{n}_l \quad (4.14)$$

The magnitude of the jump, λ_k , can be determined by equation (3.13).

$$C_{ijkl}^m \lambda_k \mathbf{n}_i \mathbf{n}_j = (C_{ijkl}^p - C_{ijkl}^m) \boldsymbol{\varepsilon}_{kl}(in) \mathbf{n}_j \quad (4.15)$$

As a result, the total stress field just outside the interface between Ω and D- Ω can be calculated by substitution equation (4.14) into equation (4.2). It can be rewritten as

$$\boldsymbol{\sigma}_{ij}(out) = C_{ijkl}^m \boldsymbol{\varepsilon}_{kl}(out) = C_{ijkl}^m (\boldsymbol{\varepsilon}_{kl}(in) + \lambda_k \mathbf{n}_l) \quad (4.16)$$

4.2 Infinitely Extended Matrix Reinforced by Multiple Inhomogeneities

Consider an infinitely extended matrix containing multiple ellipsoidal particulates having the elastic modulus differing from the remainder (matrix) (see Figure 4.2). While a uniform stress is applied at infinity, the stress disturbance induced from each ellipsoidal particulate is also affected by the presence of the neighboring particulates.

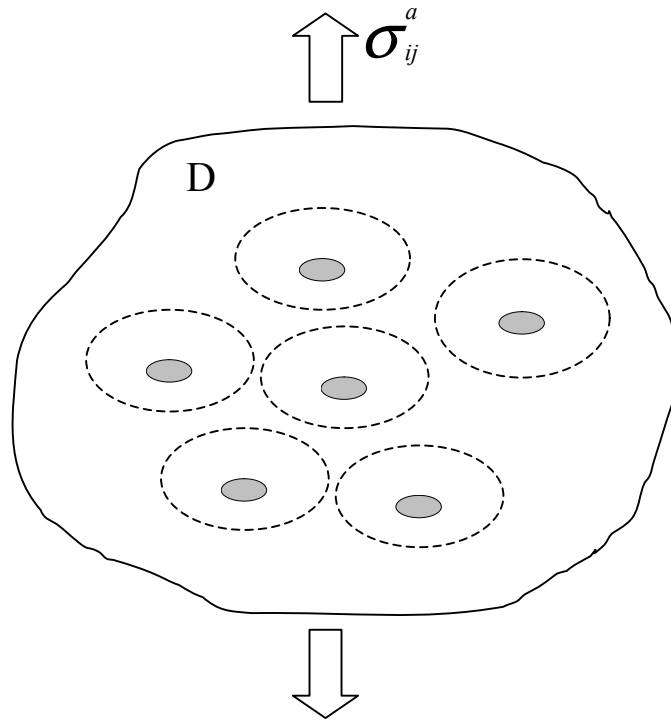


Figure 4.2 Infinitely extended matrix reinforced by multiple particulates

In order to simulate the effect due to the presence of the neighboring particulates, the method used is so-called the “Self-Consistent Method” by assuming a composite containing a particulate embedded within an effective matrix that has the elastic modulus of the composite. Once the effective elastic modulus of the matrix is derived, the stress field just outside the particulate can be obtained based on the concept mentioned above for the infinitely extended matrix containing a single ellipsoidal particulate.

For example, let's consider a composite containing multiple spherical particulates. Because the aspect ratio of spherical particulates is around one, the composite displays isotropy. Therefore, we can apply both the iso-stress and iso-strain condition for the composites when a uniform stress is applied at infinity.

The effective elastic modulus for the composite containing multiple spherical particulates can be expressed by the equation:

$$C_{ijkl}^{eff} = C_{ijkl}^m + V_p (C_{ijkl}^p - C_{ijkl}^m) A_p \quad (4.17)$$

To derive the effective elastic modulus for composites containing multiple spherical particulates, the proportionality factor, A_p , is modified as

$$A_{p,S-C} = \left\{ I + S \left[(C_{ijkl}^{eff} - C_{ijkl}^p) S - C_{ijkl}^{eff} \right]^{-1} (C_{ijkl}^p - C_{ijkl}^{eff}) \right\} \quad (4.18)$$

Then, equation (3.31) becomes a set of non-linear simultaneous equations for C_{ijkl}^{eff} . This set of simultaneous equations can be solved by successive iterations or the Newton-Raphson method until convergence is achieved.

CHAPTER 5

RESULTS AND DISCUSSIONS

In this Chapter, the global yielding criterion for composites reinforced by particulates is derived. The matrix being considered is metals such as aluminum, magnesium, and titanium. The evaluated reinforcements are ceramic particulates such as aluminum oxide (Al_2O_3), silicon carbide (SiC) and titanium carbide (TiC). By taking advantage of the existing empirical material's properties such as the elastic modulus, Poisson's ratio and yield strength, simulation of the yielding condition for particulate filled composites can be carried out.

Consider an aluminum matrix composite reinforced by ceramic particulates. General information for ceramics and aluminum at room temperature are presented as below (see Table 5.1 and Table 5.2).

Table 5.1 Properties for typical Ceramics

| Ceramic Powder | Density [g/cm ³] | Elastic Modulus [GPa] | Poisson's Ratio | Yielding Strength [MPa] |
|---|------------------------------|-----------------------|-----------------|-------------------------|
| Al_2O_3 99.9% | 3.98 | 380 | 0.22 | - |
| Al_2O_3 96% | 3.72 | 303 | 0.21 | - |
| Al_2O_3 90% | 3.6 | 275 | 0.22 | - |
| SiC ¹ | 3.3 | 483 | 0.17 | - |
| SiC ² | 3.2 | 483 | 0.16 | - |
| Si ₃ N ₄ ¹ | 3.3 | 304 | 0.3 | - |
| Si ₃ N ₄ ³ | 2.7 | 304 | 0.22 | - |
| Si ₃ N ₄ ² | 3.3 | 304 | 0.28 | - |

¹Hot Pressed

²Sintered

³Reaction Bonded

Table 5.2 Properties for Aluminum alloys

| Type | Density [g/cm ³] | Elastic Modulus [GPa] | Poisson's Ratio | Yielding Strength [MPa] |
|-------------|------------------------------|-----------------------|-----------------|-------------------------|
| Al | 2.71 | 69 | 0.33 | 35 |
| Alloy 1100 | 2.71 | 69 | 0.33 | 34 |
| Alloy 2024 | 2.77 | 72.4 | 0.33 | 75 |
| Alloy 6061 | 2.7 | 69 | 0.33 | 55 |
| Alloy 7075 | 2.8 | 71 | 0.33 | 103 |
| Alloy 356.0 | 2.69 | 72.4 | 0.33 | 124 |

For example, consider an aluminum alloy (1100) containing a certain amount of Al₂O₃ (99.9%) in the shape of particulate subject to the externally applied simple shear stress at infinity as:

$$\sigma_{ij} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The aluminum oxide in the form of particulates can be thought as randomly distributed spherical inhomogeneities embedded in the aluminum matrix. The self-consistent method in Chapter 3 is then applied to compute the effective elastic modulus (see the APPENDIX-B). The Self-Consistent Method converged to yield the effective elastic moduli of the composite after several iterations. For an aluminum alloy (1100) that contains 10 vol % of Al₂O₃ (99.9%), the simulation result shows that the effective elastic modulus was obtained after seven iterations in the self-consistent method (see Figure 5.1).

```

In[1]:=eff[122.37,155.74,50.35,25.94,0.1,20]
i= 1  50.35  25.94
i= 2  53.9747 29.8261
i= 3  54.224 30.2897
i= 4  54.2428 30.3429
i= 5  54.2444 30.3489
i= 6  54.2446 30.3496
i= 7  54.2446 30.3497
i= 8  54.2446 30.3497
i= 9  54.2446 30.3497
i= 10 54.2446 30.3497
i= 11 54.2446 30.3497
i= 12 54.2446 30.3497
i= 13 54.2446 30.3497
i= 14 54.2446 30.3497
i= 15 54.2446 30.3497
i= 16 54.2446 30.3497
i= 17 54.2446 30.3497
i= 18 54.2446 30.3497
i= 19 54.2446 30.3497
i= 20 54.2446 30.3497
Out[2]:= {54.2446,30.3497}

```

Figure 5.1 The effective elastic modulus of composite expressed in the form of Lamé constants converged after the seventh iteration.

The simulation results by the self-consistent method are expressed by the Lamé constants at various particulate volume fractions as shown below (see Table 5.3):

Table 5.3 The simulation result for the Self-Consistent Method in various particulate fractions.

| Content [vol%] | λ_{eff} | μ_{eff} |
|-------------------|------------------------|--------------------|
| 0 | 50.4 | 25.9 |
| 10 | 54.2 | 30.3 |
| 20 | 58.7 | 36.0 |
| 30 | 64.0 | 43.4 |
| 40 | 70.0 | 52.7 |
| 50 | 77.0 | 64.5 |
| 60 | 84.8 | 78.7 |

The values of the effective elastic modulus are then used to calculate the stress just outside the inclusion (see the APPENDIX-C). Therefore, we can apply this local stress field into the von Mises failure criterion (see the APPENDIX-D). The local von Mises criterion at the matrix just outside the inclusion is shown below (see Table 5.4).

Table 5.4 The simulation result for local von Mises criterion at matrix just outside inclusion.

| Content [vol%] | -J2 |
|-------------------|---------------|
| 0 | $5.73*\tau^2$ |
| 10 | $5.16*\tau^2$ |
| 20 | $4.56*\tau^2$ |
| 30 | $3.94*\tau^2$ |
| 40 | $3.34*\tau^2$ |
| 50 | $2.77*\tau^2$ |
| 60 | $2.26*\tau^2$ |

The values shown in Table 5.4 are the maximum von Mises stress (stress invariant) at the interface of the inclusion and the matrix phase in terms of the applied simple shear, τ . These values are equated with the yield strength of the matrix phase from which the composite (global) yield strength is obtained. The plots for global yielding strength at various particulate volume fractions are presented in Figure 5.2.

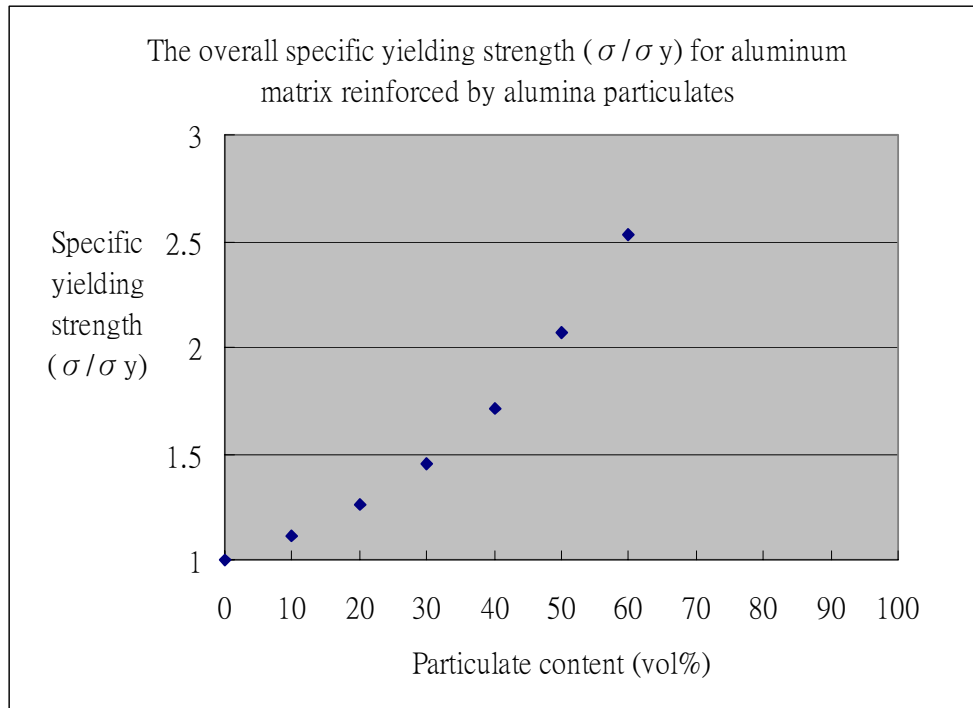


Figure 5.2 The global yielding strength v.s. Particulate content

Using this approach developed in this thesis, it is possible to derive the composite yield strength under different loading patterns such as biaxial loadings. It is recommended that result and the approach in this thesis be extended and further explored.

APPENDIX A

MATHEMATICA PROGRAM FOR FIBER EFFICIENCY PARAMETER


```

(*fep[lambda1_,mu1_,lambda2_,mu2_] is defined as the fiber
efficiency parameter*)
fep[lambda1_,mu1_,lambda2_,mu2_] :=
Module[ {delta,id,ci,cm,pr,S} , delta[i_,j_] := If[i==j,1,0] ;
id=Table[(1/2)*(delta[i,k]*delta[j,l]+delta[i,l]*delta[j,k]
),{i,3},{j,3},{k,3},{l,3}] ;
g[x1_,x2_] := Table[x1*(delta[i,j]*delta[k,l])+x2*(delta[i,k]
*delta[j,l]+delta[i,l]*delta[j,k]),{i,3},{j,3},{k,3},{l,3}];
ci=g[lambda1,mu1] ;
cm=g[lambda2,mu2] ;
pr=lambda2/(2*(lambda2+mu2)) ;
S=g[((5*pr-1)/(15*(1-pr))), ((4-5*pr)/(15*(1-pr)))] ;
add[a_,b_] := Table[a[[i,j,k,l]]+b[[i,j,k,l]],{i,3},{j,3},{k,
3},{l,3}] ;
mul[c_,d_] := Table[Sum[c[[i,j,k,l]]
d[[k,l,m,n]],{k,3},{l,3},{i,3},{j,3},{m,3},{n,3}] ;
inv[X_] := Table[1/4/X[[1,2,1,2]] (delta[i,k]
delta[j,l]+delta[i,l] delta[j,k])-
X[[1,1,2,2]]/2/X[[1,2,1,2]]/(3X[[1,1,2,2]]+2X[[1,2,1,2]])
delta[i,j] delta[k,l],{i,3},{j,3},{k,3},{l,3}] ;
add[id,mul[S,mul[inv[add[add[mul[cm,S],-mul[ci,S]],-cm]],
add[ci,-cm]]]]]

```

APPENDIX B

MATHEMATICA PROGRAM FOR SELF-CONSISTENT METHOD

```

(*fep[lambda1_,mu1_,lambda2_,mu2_]is defined as the fiber
efficiency parameter*)
delta[i_,j_]:=If[i==j,1,0];
id=Table[(1/2)*(delta[i,k]*delta[j,l]+delta[i,l]*delta[j,k]
),{i,3},{j,3},{k,3},{l,3}];
g[lambda_,mu_]:=Table[lambda delta[i,j]
delta[k,l]+mu(delta[i,k]
delta[j,l]+delta[i,l]delta[j,k]),{i,3},{j,3},{k,3},{l,3}];
mul[c_,d_]:=Table[Sum[c[[i,j,k,l]]
d[[k,l,m,n]],{k,3},{l,3},{i,3},{j,3},{m,3},{n,3}];
inv[X_]:=Table[1/4/X[[1,2,1,2]] (delta[i,k]
delta[j,l]+delta[i,l] delta[j,k])-
X[[1,1,2,2]]/2/X[[1,2,1,2]]/(3X[[1,1,2,2]]+2X[[1,2,1,2]])
delta[i,j] delta[k,l],{i,3},{j,3},{k,3},{l,3}];

f[lambda1_,mu1_,lambda2_,mu2_]:=Module[{ci,cm,pr,S},ci=g[la
mbda1,mu1];cm=g[lambda2,mu2];
pr=lambda2/(2(lambda2+mu2));
S=g[(5pr-1)/(15(1-pr)),(4-5pr)/(15(1-pr))];
id+mul[mul[S,inv[mul[cm-ci,S]-cm]],ci-cm]

(*eff[lambda1,mu1,lambda2,mu2,vf,imax]returns (lambda,mu)*)
eff[lambda1_,mu1_,lambda2_,mu2_,vf_,imax_]:=Module[{lambda,
mu,i},lambda=lambda2;mu=mu2;cm=g[lambda2,mu2];ci=g[lambda1,
mu1]; Do[Print["i=",i, " ",lambda, " ", mu](c=cm+vf
mul[ci-cm,f[lambda1,mu1,lambda,mu]]; lambda=c[[1,1,2,2]];
mu=c[[1,2,1,2]]);,{i,imax}];{lambda,mu}];

```

APPENDIX C

MATHEMATICA PROGRAM FOR STRESS JUST OUTSIDE INCLUSION

```

Off[General::spell1]
Off[General::spell]

SetAttributes[delta, Orderless];
SetAttributes[epsiloni, Orderless];
SetAttributes[epsilono, Orderless];
SetAttributes[sigmao, Orderless];
SetAttributes[sigmaint, Orderless];
delta[i_Integer, j_Integer]:=If[i==j, 1, 0];
delta[i_Symbol, i_Symbol]:=3;

Unprotect[Times];
Times[delta[i_Symbol, j_], delta[i_Symbol, k_ ]]:=delta[j,k];
Times[n_[j_Symbol], delta[i_, j_Symbol]]:=n[i];
Times[epsiloni[i_Symbol, j_], delta[i_Symbol, k_]]:=epsiloni[k, j];
Times[epsilono[i_Symbol, j_], delta[i_Symbol, k_]]:=epsilono[k, j];
Times[sigmao[i_Symbol, j_], delta[i_Symbol, k_]]:=sigmao[k, j];
Times[sigmaint[i_Symbol, j_], delta[i_Symbol, k_]]:=sigmaint[k, j];
Protect[Times];

Unprotect[Power];
Power[n[i_Symbol], 2]:=1;
Power[delta[i_Symbol, j_Symbol], 2]:=3;
Protect[Power];

sumSimplify[f_]:=Simplify[(Expand[f]/.n[i_Symbol]n[j_Symbol]
]epsiloni[i_Symbol, j_Symbol]-
>n[p]n[q]epsiloni[p, q])/ .n[i_Symbol]epsiloni[i_Symbol, j_]-
>n[p]epsiloni[p, j]]

ci[i_, j_, k_, l_]:=lambdai
delta[i, j]delta[k, l]+mui(delta[i, k]delta[j, l]+delta[i, l]de
lta[j, k])

co[i_, j_, k_, l_]:=lambdao
delta[i, j]delta[k, l]+muo(delta[i, k]delta[j, l]+delta[i, l]del
ta[j, k])

j1=(ci[i, j, k, l]-co[i, j, k, l])epsiloni[k, l]n[j]//Expand;
j2=co[i, j, k, l]jump[k]n[l]n[j]//Expand;

```

```

ainv[i_,j_]:=1/muo delta[i,j]-
(lambdao+muo)/muo/(2muo+lambdao) n[i]n[j]
sumSimplify[ainv[m,i] j1]/.epsilon[1,1]->epsilon[p,p]

jump[m_] := (1/(muo (lambdao+2 muo)))*((lambdai-lambdao)
muo epsilon[p,p] n[m]+2 (mui-muo) n[p] ((lambdao+2 muo)
epsilon[m,p]-(lambdao+muo) epsilon[p,q] n[m] n[q]))

epsilono[i_,j_]:=sigmao[i,j]/(2*muo)-
(lambdao*delta[i,j]*sigmao[z,z])/(2*muo*(3*lambdao+2*muo))

epsilon[i_,j_]:= (3 (lambdao+2 muo) (5 muo (3 lambdai+2
mui+4 muo) epsilono[i,j]+((-5 lambdai+2 mui-2 muo)
muo+lambdao (2 mui+3 muo)) delta[i,j] epsilono[p,p]))/((3
lambdai+2 mui+4 muo) (2 muo (8 mui+7 muo)+lambdao (6 mui+9
muo)))

Simplify[Expand[co[i, j, k, l]*(epsilono[k, l] -
jump[k]*n[l])] /. {sigmao[l, l] -> sigmao[p, p], sigmao[z,
z] -> sigmao[p, p], n[i_]n[j_]*sigmao[i_, j_] ->
n[p]*n[q]*sigmao[p, q]}]

sigmaint[i_, j_] := (muo*(3*lambdao + 2*muo)*(3*lambdai +
2*mui + 4*muo)*(2*muo*(8*mui + 7*muo) + lambdao*(6*mui +
9*muo))* sigmao[i, j] - 3*((mui - muo)*(3*lambdao^2 +
8*muo*lambdao + 4*muo^2)*(lambdao*(2*mui + 3*muo) +
2*muo*(5*lambdai + 6*mui + 9*muo))*n[i]*n[p]*sigmao[j, p] +
lambdao*delta[i, j]* ((mui - muo)*(3*lambdao +
2*muo)*(lambdao*(2*mui + 3*muo) + 2*muo*(5*lambdai + 6*mui
+ 9*muo))*n[p]*n[q]*sigmao[p, q] - (lambdao*mui -
lambdai*muo)*(2*muo*(8*mui + 7*muo) + lambdao*(6*mui +
9*muo))*sigmao[p, p]) + n[j]*(2*(lambdai*muo -
lambdao*mui)*muo*(2*muo*(8*mui + 7*muo) + lambdao*(6*mui +
9*muo))*n[i]*sigmao[p, p] + (mui - muo)*(3*lambdao +
2*muo)*(lambdao*(2*mui + 3*muo) + 2*muo*(5*lambdai + 6*mui
+ 9*muo))*n[p]* ((lambdao + 2*muo)*sigmao[i, p] -
2*(lambdao + muo)*n[i]*n[q]*sigmao[p, q])))/
(muo*(3*lambdao + 2*muo)*(3*lambdai + 2*mui +
4*muo)*(2*muo*(8*mui + 7*muo) + lambdao*(6*mui + 9*muo)))

```

APPENDIX D

MATHEMATICA PROGRAM FOR VON MISES CRITERION

```

Off[General::spell1]
Off[General::spell]

SetAttributes[delta, Orderless];
SetAttributes[epsiloni, Orderless];
SetAttributes[epsilono, Orderless];
SetAttributes[sigmao, Orderless];
SetAttributes[sigmaint, Orderless];
delta[i_Integer, j_Integer]:=If[i==j, 1, 0];
delta[i_Symbol, i_Symbol]:=3;

Unprotect[Times];
Times[delta[i_Symbol, j_], delta[i_Symbol, k_ ]]:=delta[j,k];
Times[n_[j_Symbol], delta[i_, j_Symbol]]:=n[i];
Times[epsiloni[i_Symbol, j_], delta[i_Symbol, k_]]:=epsiloni[k, j];
Times[epsilono[i_Symbol, j_], delta[i_Symbol, k_]]:=epsilono[k, j];
Times[sigmao[i_Symbol, j_], delta[i_Symbol, k_]]:=sigmao[k, j];
Times[sigmaint[i_Symbol, j_], delta[i_Symbol, k_]]:=sigmaint[k, j];
Protect[Times];

Unprotect[Power];
Power[n[i_Symbol], 2]:=1;
Power[delta[i_Symbol, j_Symbol], 2]:=3;
Protect[Power];

sigmaint[i_, j_]:=
(mu*o*(3*lambdao+2*muo)*(3*lambdai+2*mu+4*muo)*
(2*muo*(8*mu+7*muo)+lambdao*(6*mu+9*muo))*sigmao[i, j]-
3*(mu-
muo)*(3*lambdao^2+8*muo*lambdao+4*muo^2)*(lambdao*(2*mu+3*muo)+
2*muo*(5*lambdai+6*mu+9*muo))*n[i]*n[p]*sigmao[j, p]+
lambdao*delta[i, j]*((mu-
muo)*(3*lambdao+2*muo)*(lambdao*(2*mu+3*muo)+2*muo*(5*lambdai+
6*mu+9*muo))*n[p]*n[q]*sigmao[p, q]-
(lambdao*mu-
lambdai*muo)*(2*muo*(8*mu+7*muo)+lambdao*(6*mu+9*muo))*sigmao[p, p]+
n[j]*(2*(lambdai*muo-
lambdao*mu)*muo*(2*muo*(8*mu+7*muo)+lambdao*(6*mu+9*muo))*n[i]*sigmao[p, p]+
(mu-
muo)*(3*lambdao+2*muo)*(lambdao*(2*mu+3*muo)+2*muo*(5*lambdai+6*mu+9*muo))*n[p]*
((lambdao+2*muo)*sigmao[i, p]-
2*(lambdao+muo)*n[i]*n[q]*sigmao[p, q])))/
(mu*o*(3*lambdao+2

```



```
*muo)*(3*lambdai+2*mui+4*muo)*(2*muo*(8*mui+7*muo)+lambdao*
(6*mui+9*muo))
```

```
(*n[i] is the normal in 3-D*)
```

```
normal = {n[1] -> Sin[phi]*Cos[theta], n[2] ->
Sin[phi]*Sin[theta], n[3] -> Cos[phi]};
```

```
(*Enter values here from the s-c model.*)
```

```
mat={lambdai -> 122.37,mui -> 155.74,lambdao -> 54.2446,
muo -> 30.3497};
```

```
(*Define far-field stress sigmao*)
```

```
junk1 = {{0, tau, 0}, {tau, 0, 0}, {0, 0, 0}};
sigmaoValue = Flatten[Table[sigmao[i, j] -> junk1[[i,j]],
{i, 3}, {j, 3}]];
```

```
(*Summation expansion*)
```

```
sumApply[f_] := Expand[f] /.
n[p_Symbol]*n[q_Symbol]*sigmao[p_Symbol, q_Symbol] ->
Sum[n[p]*n[q]*sigmao[p, q], {p, 1, 3}, {q, 1, 3}] /.
sigmao[p_Symbol, p_Symbol] -> Sum[sigmao[p, p], {p, 1, 3}]
/. n[p_Symbol]*sigmao[i_, p_Symbol] -> Sum[n[p]*sigmao[i,
p], {p, 1, 3}]
```

```
(*All components*)
```

```
stress = Table[sumApply[Expand[sigmaint[i, j]]] /.
sigmaoValue /. mat, {i, 3}, {j, 3}];
```

```
(*hydrostatic stress*)
```

```
hydro = Sum[stress[[i,i]], {i, 3}];
```

```
(*von Mises stress*)
```

```
vonmises = (1/2)*Expand[Sum[stress[[i,j]]^2, {i, 3}, {j, 3}]
- (1/3)*hydro^2] /. normal;
```

```
(*Find Maximum von Mises stress*)
```

```
-NMinimize[-vonmises/tau^2, {phi, theta}]
```

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BIOGRAPHICAL INFORMATION

Chih-Ta Chen was born in Taipei, Taiwan on September 28, 1976. He received a B.S. degree in Chemical Engineering from Tatung University, Taipei, Taiwan, in 1998. He is currently pursuing an M.S. degree at the University of Texas at Arlington, Texas, USA. His research interests include continuum mechanics and micromechanics.