

APPLICATION OF ELLIPSE FOR HORIZONTAL ALIGNMENT

by

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Presented to the Faculty of the Graduate School of
The University of Texas at Arlington in Partial Fulfillment
of the Requirements
for the Degree of

MASTER OF SCIENCE IN CIVIL ENGINEERING

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2012

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ACKNOWLEDGEMENTS

First, I want to express my sincere appreciation to my advisor, Dr. Siamak Ardekani, for his guidance in this research and many opportunities he provided for me to learn. I would also like to thank my other committee members, Dr. James C. Williams and Dr. Stephen P. Mattingly for their valuable comments and guidance. I am also grateful to Dr. Ali Abolmaali for his advice and support during my master's program. I would like to thank Dr. Mostafa Ghandehari for his endless help he have generously offered me.

I am deeply indebted to my father Farrokh, my mother Sholeh, and my brothers Shervin and Ramin for the tremendous support, encouragement, and love they have given me. I would like to express my gratitude to my aunt Shayan Hatam and my cousin Lamie for being my family at Arlington. Also, I am truly grateful to my aunts Farah Maniei, Fereshteh Maniei, and Farzaneh Maniei for their support, kindness, and hospitality. Finally, I want to thank my UTA friends Babak H. M. H. Mamaqani, Bahareh Bakhtiari, Mohammad Razavi, Roozbeh Kiamanesh, Reza Bayat, Sajjad Moradi, Soheil Shafiee, Hamed Janzadeh, Farzad Daneshgar, Sina Kahrobaei, Hossein Hashemi, Arezoo Memarian, Hossein Tavakoli, Hoda Vaziri, Azadeh Nazy and Peyman Fakour.

April 6, 2012

ABSTRACT

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In highway design, horizontal alignments provide directional transition of roadways. Three categories of horizontal transition curves are simple circular curves, compound circular curves, and spiral circular curves. Compound and spiral curves, as alternatives to a simple circular curve, are often more costly since they are longer in length and require additional right-of-way. These costs differences are amplified at higher design speed. This study presents calculations associated with using a single elliptical arc in lieu of compound or spiral curves in situations where the use of simple circular curves is not prudent due to driver safety and comfort considerations. The study presents an approach to

analytically determine the most suitable substitute elliptical curve for a given design speed and intersection angle.

Computational algorithms are also provided to stake out the elliptical curve. These include algorithms to determine the best fit elliptical arc with the minimum arc length and minimum right-of-way; and algorithms to compute chord lengths and deflection angles and the associated station numbers for points along the elliptical curve.

These algorithms are next applied to an example problem in which elliptical curve results are compared to the equivalent circular curve and spiral-circular curve results. According to this comparison, the elliptical curve not only provides a smoother and safer transition, but also shortens the length of the roadway. However, the right-of-way requirement for the elliptical curve for this specific example is slightly higher than the right-of-way for the equivalent circular and spiral-circular curves. An added advantage of using an elliptical horizontal curve is found to be a smoother transition in cross-section from the normal crown to full superelevation, as this transition can be achieved more gradually through the entire length of the elliptical arc. The transition is likely to be more aesthetically pleasing as well.

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CHAPTER 1

INTRODUCTION OF HORIZONTAL CURVES

1.1 Introduction and Problem Definition

First, in this chapter, horizontal alignment is defined. The circular curve, as the simplest horizontal curve, is introduced and equations relating to circular curve calculations are tabulated. Transition curves are also discussed. Transition curves are used to provide a smooth transition between the tangent section and horizontal curves when the rate of change of radial acceleration is too great, as specified by the American Association of State Highway and Transportation Officials (AASHTO). Finally, the origin of applying an ellipse as a horizontal highway curve is discussed.

Using compound curves, as horizontal curves, could be problematic in terms of right-of-way cost and calculations. Horizontal alignments which avoid these complexities are always preferred, provided that driver safety is not compromised. While this favors the use of simple circular curves, the use of such curves is not always possible due to driver safety and comfort requirements. A key question, therefore, is whether or not there are other single curves that may be used in lieu of compound curves or spiral-circular combinations. This thesis examines the potential applicability of an elliptical arc as an alternative to the spiral-circular or compound circular curves. The use of a

single elliptical arc in situations where a single circular curve is not feasible may offer advantages in terms of smoother transitions and possible ROW savings.

Sometimes, looking at nature offers clues to deal with problems. In astronomy, planetary motion reminds us of some similarities. Considering planetary orbits, both centrifugal and centripetal forces are subjected to planets. For example, the planet Earth experiences an outward force called centrifugal force which represents the effect of inertia that arises in connection with the rotation. On the other hand, our planet is subjected to gravitational forces, as given by the relation below (Grant and Phillips, 2001, p. 134):

$$F = (G) \frac{(m_1)(m_2)}{r^2} \quad (1.1)$$

in which

F = Gravitational force between m_1 and m_2

m_i = magnitude of mass i .

G = Gravitational constant

r = Distance between m_1 and m_2 .

For the Earth, the centripetal force is supplied by this gravitational force. As we know, the Earth's orbit is not a perfect circle. **Johannes Kepler** discovered that the orbits of planets around the Sun are elliptical (Grant et al., 2001, p. 145). Considering the Earth as a dynamic object on an elliptical orbit around the Sun raises the question of whether vehicles could also have an elliptical trajectory from PC to PT. If so, would there be inherent advantages to

elliptical curves in terms of safe and smooth angular transitions as well as right-of-way advantages?

1.2 Horizontal Alignments

In highway design, horizontal alignment, which provides for a directional transition of roadway is typically a circular curve connecting two straight sections of the roadway, known as tangents. The angle between these two tangents is called the intersection angle and the point of intersection is abbreviated to PI. Mostly, the basic form of a horizontal alignment consists of a circular curve and two transition curves forming a curve which joins two straight lines. In some cases, the transition curve has zero length (i.e. it is not needed) and the horizontal curve is a single circular curve.

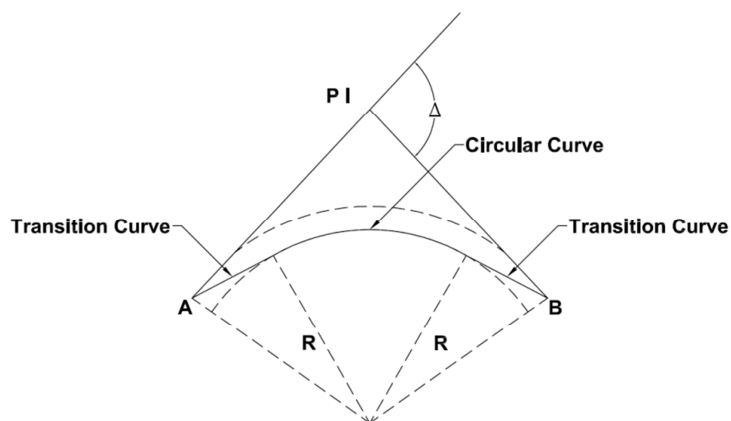


Figure 1.1 A schematic diagram of a horizontal curve with and without transition curves

The most important factor in designing roadways is the design speed. Let us suppose that such a horizontal curve is already provided. When a vehicle goes around a curve like this, it experiences a lateral force known as the

centrifugal force. This centrifugal force, caused by the change in the direction of the velocity vector, pushes the vehicle outward from the center of curvature of the curve. The vehicle is also subjected to an inward radial force which is called the centripetal force. In fact, the centripetal force is always directed orthogonally to the velocity vector of the vehicle, toward the instantaneous center of curvature of the curve. At high speeds, the centripetal force acting inward may be inadequate to balance the centrifugal force acting outward without any assistance. To handle this problem, the angle of incline of roadway, known as the superelevation angle e (or bank angle) is provided (Garber and Hoel, 2002, p. 70). As shown in Figure 1.2, F_r is the side friction force between the vehicle and the surface of the roadway, and N is the reaction to the weight of the vehicle normal to the surface of the roadway. F_c is the centrifugal force acting horizontally on the vehicle and has a magnitude of $((m)(v^2)/R)$, where m is the mass of the vehicle. On the other hand, all forces shown in Figure 1.2 should be in equilibrium. They can be resolved along the angle of incline of the road (Garber et al., 2002, p. 70):

$$\frac{(m)(v^2)}{R} = (m)(g)(\sin e) + (m)(g)(f_{\text{side}})(\cos e) \quad (1.2)$$

Since e is a small angle, we have:

$$\sin e \approx e \text{ (in rad.) and } \cos e \approx 1. \quad (1.3)$$

Then:

$$\frac{(m)(v^2)}{R} = (m)(g)(e) + (m)(g)(f_{\text{side}}), \quad (1.4)$$

and

$$\frac{v^2}{R} = (g)(e + f_{\text{side}}). \quad (1.5)$$

Therefore (Garber et al., 2002, p. 71):

$$R = \frac{v^2}{(g)(e + f_{\text{side}})}. \quad (1.6)$$

This R is the minimum radius that should be provided for a roadway with a design speed of v and a superelevation of e .

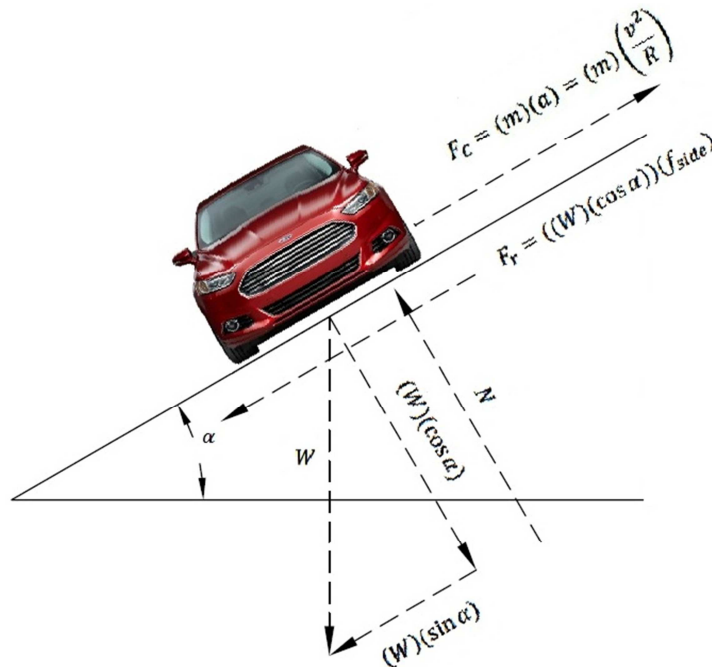


Figure 1.2 Centrifugal and centripetal forces

1.3. Circular Curves

One type of horizontal alignment is the circular curve, which is the simplest curve to connect two straight lines. As the name implies, the curve is a segment of a circle with radius R . This radius should satisfy the equation below:

$$R_{min} = \frac{v^2}{(g)(e + f_{side})} \quad (1.7)$$

As shown in Figure 3, the point at which the circular curve begins is known as the point of curvature, abbreviated as PC. As stated earlier, the intersection of the two tangents is called the PI. The point at which the circular curve ends is known as the point of tangency, abbreviated as PT.

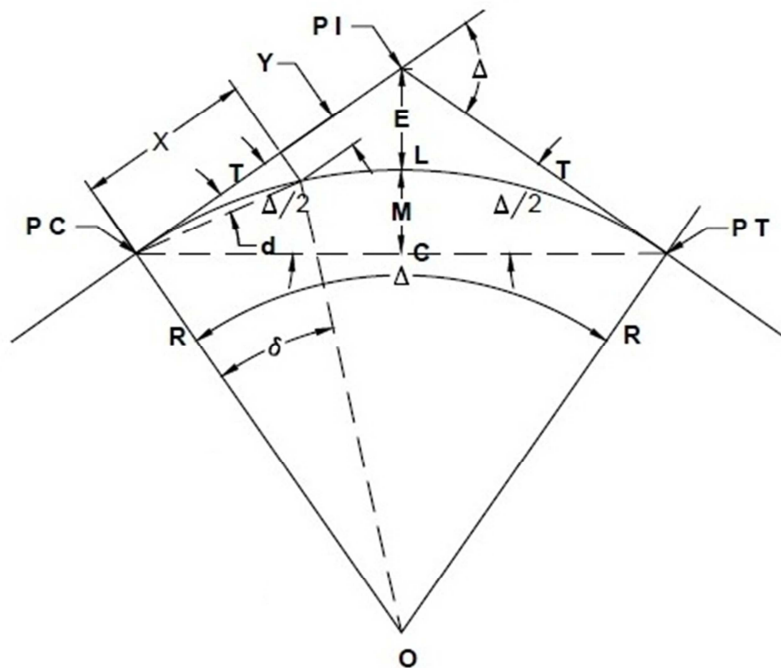


Figure 1.3 A circular curve and its elements.

To define a circular curve fitted to the tangents, we first need to know the degree of the curve. Here, this simple circular curve is described either by its radius (for example, 300-ft-radius curve) or by the degree of the curve. If Δ is the angle in radians subtended at the center by an arc of a circle, then the length of that arc would be (Garber et al., 2002, p. 708):

$$L = (\Delta)(R) . \quad (1.8)$$

Assume that D is the degree of the curve which represents a 100-ft circular arc. If D is the angle in degrees, then (Garber et al., 2002, p. 708-709):

$$\frac{D^\circ}{360^\circ} = \frac{100}{(2)(\pi)(R)} . \quad (1.9)$$

Therefore,

$$D^\circ = \frac{5729.6}{R} , \quad (1.10)$$

where R is in feet.

Thus, the radius of the curve can be determined if the degree of the curve is known. As mentioned above, we can also define the degree of curve using the chord length, which is the length of a line segment connecting the PC to another point on the circular curve. The chord length defines the degree of the curve in terms of the angle subtended at the center by a chord of 100 ft. in length (Figure 1.3). In this case (Garber et al., 2002, p. 708), “the radius is given as:

$$R = \frac{50}{\sin\left(D_c/2\right)} \quad (1.11)$$

where D_c is the angle in degrees subtended at the center by a 100-ft chord. The arc definition is commonly used for highway work, and the chord definition is commonly used for railway work.”

Referring to Figure 1.3 and using the properties of a circle, the relations in Table 1.1 can be derived based on the arc definition of D .

Table 1.1 Circular Curve Relations (Garber et al., 2002, pp. 708-712)

Equation	Comments
$T = (R) \left(\tan \frac{\Delta}{2} \right)$	Tangent
$D = \frac{18000}{\pi R} \left(D = \frac{5729.6}{R} \right)$	Degree of Curve – Central angle which subtends a 100 ft. arc
$L = \frac{\pi}{180} (\Delta)(R)$	Length of Curve
$L = 100 \left(\frac{\Delta}{D} \right)$	
$E = (T) \left(\tan \frac{\Delta}{4} \right) = (R) \left(\text{exsec} \frac{\Delta}{2} \right)$	External
$M = R \left(1 - \cos \frac{\Delta}{2} \right) = R \left(\text{vers} \frac{\Delta}{2} \right)$	Mid-Ordinate
$X = (R)(\sin \delta)$	
$Y = (R)(1 - \cos \delta)$	
$C = (2)(R) \left(\sin \frac{\Delta}{2} \right)$	Long Chord
$c = (2)(R) \left(\sin \frac{\delta}{2} \right)$	Chord
$d = \frac{\delta}{2}$	Deflection Angle
$\delta = \left(\frac{L_{arc}}{R} \right) \left(\frac{180}{\pi} \right)$	The angle between any two points chosen on a curve
$Area_{Sector} = \left(\frac{L}{2} \right) (R) = (\pi)(R^2) \left(\frac{\Delta}{360^\circ} \right)$	Sector is bounded by two radii and the included arc of the circle.
$Area_{Segment} = \left(\frac{R^2}{2} \right) \left(\frac{(\pi)(\Delta)}{180^\circ} - \sin \Delta \right)$	Segment is between the chord and arc of a circle.
$Area_{curve/Tang} = \left(T - \frac{L}{2} \right) (R)$	Area between the arc and the tangents.

1.4. Transition Curve

Another type of horizontal alignment is the transition curve, used to connect the curved and the tangent sections of the roadway. Transition curves can also be used to connect two circular curves where the difference in radius is large. The radial acceleration experienced by vehicles travelling at a given velocity of v changes the centrifugal force from zero on the tangent to v^2/R on the circular curve. The form of transition curve should be such that the rate of change of the radial acceleration along the transition curve is constant to provide a smooth maneuver for vehicles (Garber et al., 2002, p. 719-920).

In fact, the radius of curvature of a transition curve gradually decreases from infinity at the intersection of the tangent and the transition curve to the designated radius R at the intersection of the transition curve with the circular curve. The minimum length of spiral recommended by AASHTO for a horizontal curve of radius R is given by (Garber et al., 2002, p. 720):

$$l_s = \frac{(3.15)(V^3)}{(R)(C)} \quad (1.7)$$

where

l_s = minimum length of transition spiral (ft)

V = design speed (mph)

R = radius of curvature (ft)

C = rate of change of centripetal acceleration (ft/sec³).

C is an important factor indicating the level of comfort and safety involved. A higher value of l_s provides a smoother and easier transition from the tangent to the circular curve. Therefore, the smaller the value of C , the smoother the transition. The most common values of C are between 1 to 3 ft/sec³. According to AASHTO, a desirable value of C for railroad design and high design speed highways is 1 ft/sec³. According to AASHTO (2004), under operational conditions, the most desirable length of a spiral curve is approximately the length of the natural spiral path used by drivers as they traverse the curve. Based on this, AASHTO recommends lengths of spiral curves shown in Table 1.2.

As discussed earlier, the highway surface on circular curves needs to be superelevated to deal with the effect of centrifugal force. The length of highway section required to achieve a full superelevated section from a section with adverse crown removed or vice versa, is known as the **superelevation runoff**. Depending on the design speed, the rate of superelevation, and the width of pavement, the length of superelevation runoff varies. In those design cases where spiral curves are needed, AASHTO recommends that the superelevation runoff be achieved over the length of the spiral curve. Based on this recommendation, the length of the spiral curve should be the length of the superelevation runoff, as shown in Table 1.3. Another possible advantage of using a single elliptical arc is ample length to develop the full superelevation, as

the full superelevation will only be needed at the middle of the arc where the radius is the tightest.

Table 1.2 Spiral lengths corresponding to design speed (AASHTO, 2004).

Design Speed (mph)	Spiral Length (ft.)
15	44
20	59
25	74
30	88
35	103
40	117
45	132
50	147
55	161
60	176
65	191
70	205
75	220
80	235

1.5. Problem with Compound Curves

A compound curve is a combination of two or more simple curves. These curves will be used when simple circular curves cannot be used. In some cases, transition curves should be used to provide smoother and easier transition. As described above, more consideration and calculations are necessary to fit appropriate curves to provide a smooth and safe transition.

Another problem with the compound curves relates to the ***Right-of-Way***. Right-of-Way is the strip of land upon which a roadway or railroad will be constructed. Right-of-Way can also include additional land purchased for the purpose of future expansion. For roadways with high design speed, exclusive use of compound curves is often impractical due to a large amount of right-of-way, which results in excessive cost. The most high-speed turning roadways include a combination of tangents and curves. Applying this approach, the design should consider the right-of-way acquisition costs in addition to the driver comfort and safety. The next chapter will discuss the use of single elliptical arcs in lieu of spiral-circular or compound circular curves.

Table 1.3. Tangent runout length corresponding to design speed for different superelevation rates (AASHTO, 2004, p. 192)

	Tangent Runout Length (ft.)				
	Superelevation Rates				
Design Speed (mph)	2%	4%	6%	8%	10%
15	44	-	-	-	-
20	59	30	-	-	-
25	74	37	25	-	-
30	88	44	29	-	-
35	103	52	34	26	-
40	117	59	39	29	-
45	132	66	44	33	-
50	147	74	49	37	-
55	161	81	54	40	-
60	176	88	59	44	-
65	191	96	64	48	38
70	205	103	68	51	41
75	220	110	73	55	44
80	235	118	78	59	47

CHAPTER 2

USE OF ELLIPTICAL CURVES FOR HORIZONTAL ALIGNMENT

2.1 Introduction

In this chapter, we will discuss use of ellipses as horizontal alignment curves. This includes a general discussion of properties of ellipse and how to find an appropriate elliptical curve that provides smooth and safe transition from the PC to PT. Also, discussions of chord length and deflection angle calculations for an elliptical arc are presented.

2.2 Ellipse and Its Properties

Mathematically speaking (Larson and Edwards, 2010, p. 700), an ellipse is the set of points in a plane the sum of whose distances from two points F_1 and F_2 is constant (see Figure 2.1). Each of these two fixed points is called the **focus**. One of the Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

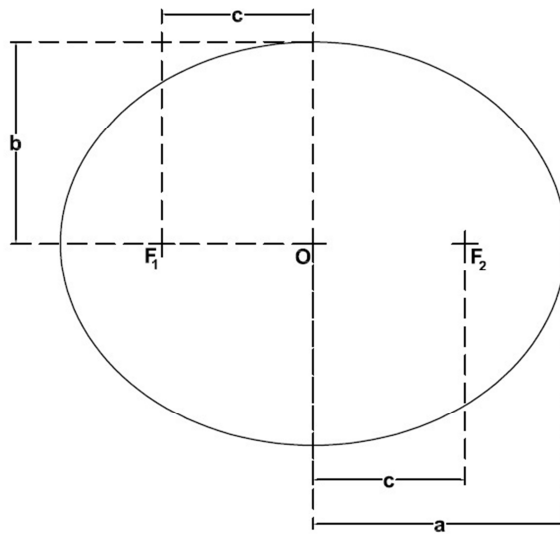


Figure 2.1. An ellipse.

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 2.2 so that the origin, which is called **center** of ellipse, is halfway between F_1 and F_2 . Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Let us suppose that $P(x, y)$ is any point on the ellipse. According to the definition of the ellipse, we will have:

$$|PF_1| + |PF_2| = 2a \quad (2.1)$$

that is,

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a \quad (2.2)$$

or

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}. \quad (2.3)$$

Squaring both sides, we get:

$$\begin{aligned}
x^2 - (2)(c)(x) + c^2 + y^2 \\
= (4)(a^2) - (4)(a) \left(\sqrt{(x+c)^2 + y^2} \right) (x^2)
\end{aligned} \tag{2.4}$$

We square again to obtain:

$$\begin{aligned}
(a^2)(x^2 + (2)(c)(x) + c^2 + y^2) = \\
a^4 + (2)(a^2)(c)(x) + (c^2)(x^2)
\end{aligned} \tag{2.5}$$

or

$$(a^2 - c^2)(x^2) + (a^2)(y^2) = (a^2)(a^2 - c^2). \tag{2.6}$$

From triangle F_1F_2P in Figure 2.2, we see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes

$$(b^2)(x^2) + (a^2)(y^2) = (a^2)(b^2). \tag{2.7}$$

Dividing both sides by a^2b^2 , we have:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.8}$$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the vertices of the ellipse and the line segment joining the **vertices** is called the **major axis**. To find the y-intercepts, we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. Equation (2.8) is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$ and $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

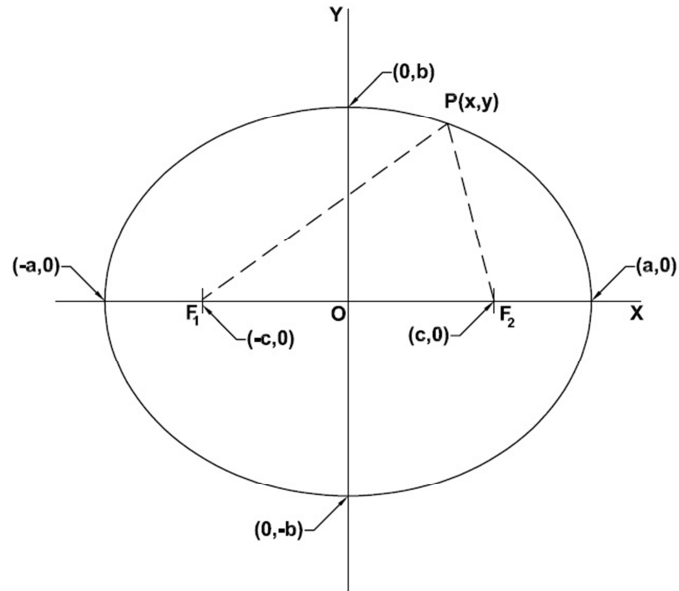


Figure 2.2 Cartesian components of a point on ellipse with center at the origin

In mathematics, there is a parameter for every conic section called **eccentricity** (Larson et al., 2010, p. 701). Eccentricity shows how much the conic section deviates from being a circle. An ellipse as a conic section has its own eccentricity τ , which is calculated using the formula below (Larson et al., 2010, p. 701):

$$\tau = \frac{c}{a}, \quad (2.9)$$

in which:

τ = eccentricity of ellipse,

a = length of major axis,

$c = \sqrt{a^2 - b^2}$.

In mathematics texts, the eccentricity is denoted by e or ϵ . In this text, we use τ to denote the eccentricity of an ellipse in order to avoid confusion between eccentricity and superelevation.

2.3 Circular Curve, Design Speed, and Superelevation

As shown in Chapter 1, the relation between the radius of circular curve, the design speed, and the superelevation is expressed by the equation below:

$$R = \frac{v^2}{(g)(e + f_{\text{side}})}. \quad (2.10)$$

According to this relation, AASHTO tabulates values of the minimum radius of circular curves required for each combination of superelevation, side friction factor, and design speed.

Therefore, the desired elliptical curve should, as a minimum, satisfy the minimum radius curve required by AASHTO. This brings us to one of our constraints to find an appropriate elliptical curve. Before considering this and other constraints, we should determine what distance should be considered as the “radius” of an ellipse. To do this, we would use the polar coordinate system.

In the polar coordinate system, there are two common equations to describe an ellipse depending on where the origin of polar coordinates is assumed to be. As shown in Figure 2.3, if the origin of the polar coordinates is located at the center of the ellipse and the angular coordinate θ is measured from the major axis, then the ellipse’s equation will be:

$$r(\theta) = \frac{(a)(b)}{\sqrt{(b \cos \theta)^2 + (a \sin \theta)^2}}. \quad (2.11)$$

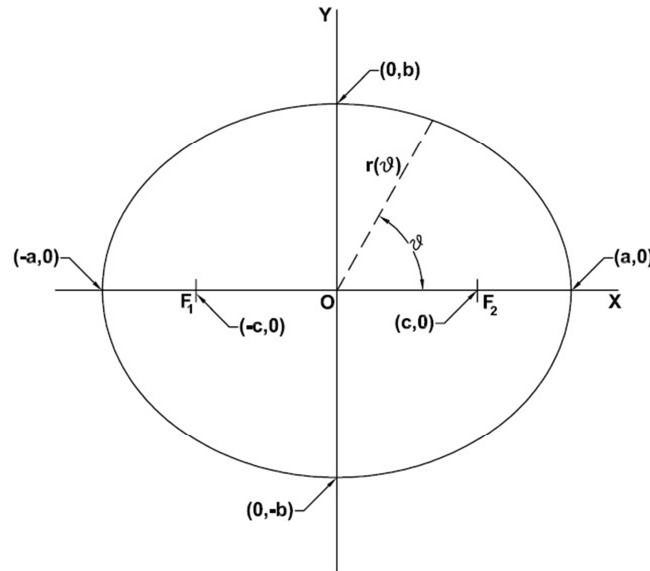


Figure 2.3 Polar coordinates system with origin at center of the ellipse

On the other hand, if the origin of polar coordinates is located at one focus and the angular coordinate θ is still measured from the major axis, then the ellipse's equation will be:

$$r(\theta) = \frac{(a)(1 - \tau^2)}{1 \pm (\tau)(\cos \theta)} \quad (2.12)$$

where the sign in the denominator will be negative if the reference direction is from $\theta = 0$ towards the center. The sign in the denominator is positive if the reference direction points away from the center.

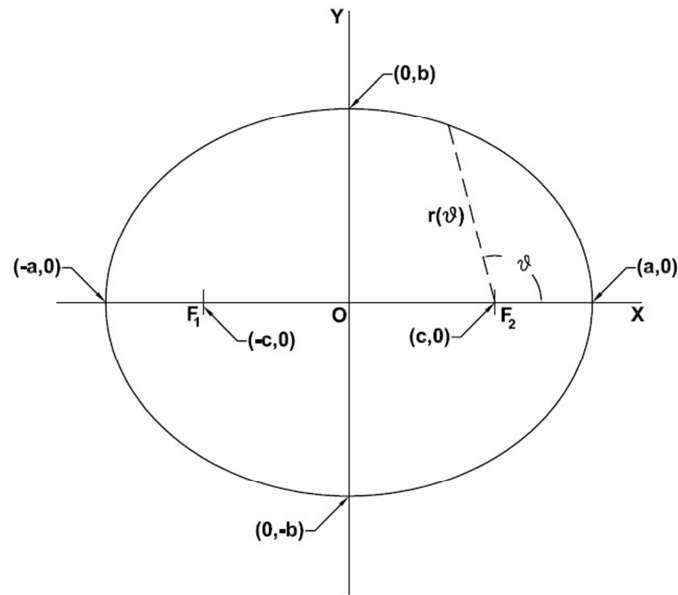


Figure 2.4 Polar coordinates system with origin at foci of the ellipse

From the astronomical point of view, Kepler's laws claimed that the orbits of planets in a solar system are ellipses with a sun at one focus. Thus, the ellipse's polar equation (2.6), where the origin of the polar coordinates is assumed at one focus, will be helpful to describe the desired elliptical arc.

Figure 2.4 shows that the minimum radius of the desired ellipse with respect to the foci F_2 is $a - c$. Since $r = (a)(\tau)$, then we have: $a - c = a - (a)(\tau) = (a)(1 - \tau)$. On the other hand, the minimum radius should not be smaller than the minimum radius R_{min} recommended by AASHTO. Thus, we have:

$$(a)(1 - \tau) \geq R_{min} \quad (2.13)$$

Actually, we are looking for an elliptical curve to connect the PC to PT which would be an arc of an ellipse that satisfies the inequality below as a constraint:

$$(a)(1 - \tau) = R_{min} . \quad (2.14)$$

Therefore, we should first find an appropriate ellipse and then identify the desired arc to be used as a highway curve. In our design problem, the known parameters are location of PI, the angle Δ , and the design speed. Based on the known design speed, we can read a value for R_{min} recommended from the design tables provided by AASHTO. With R_{min} known, we now need to identify an equivalent elliptical arc. In equation (2.14), we have two unknown variables relating to the ellipse. Using numeric methods, we can find all pairs of (a, τ) which satisfy our constraint by inserting acceptable values for τ and solving the equation for a . The eccentricity of ellipse, τ ranges from 0 to 1. To make a finite set of value for τ , we should consider only one or two decimal points for eccentricity depending on the level of accuracy required.

By having the major axis a and the eccentricity τ , the equivalent ellipse can be easily identified. We know that

$$c = \sqrt{a^2 - b^2} . \quad (2.15)$$

But $c = (a)(\tau)$, then

$$b = (a) \left(\sqrt{1 - \tau^2} \right) . \quad (2.16)$$

One of the constraints indicates that the arc of the ellipse should be tangent to the PT and the PC. Therefore, the slope of the tangent line on the ellipse at the PC and PT has to be the same as the slope of the tangent line passing through the PI.

To identify an arc, just two points are needed. We first start with finding a point on the ellipse where the slope of the tangent line is the same as the slope of the line passing through PI and PT or PI and PC, called m_1 . Then, the location of the other point can be identified by drawing a line starting at the end of the first line and ending at a point on ellipse so that the angle between these two lines provides the desired intersection angle Δ .

Another constraint is an aesthetic factor required by AASHTO. According to AASHTO, symmetric design enhances the aesthetics of highway curves. Therefore, a symmetric arc of the ellipse is desirable to meet the aesthetics requirement.

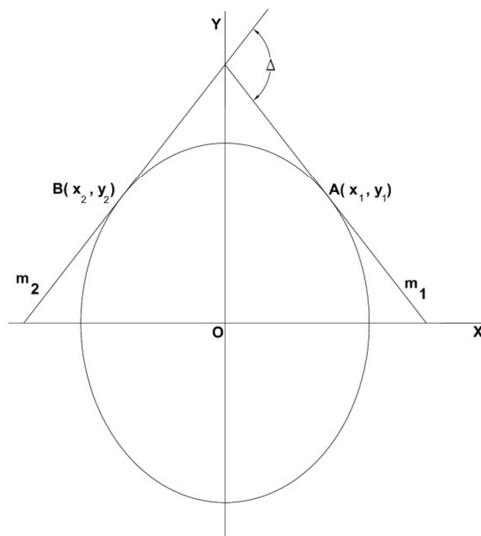


Figure 2.5 A schematic diagram of the ellipse desired

Since the desired arc should be symmetrical, the arc must be symmetric with respect to either the major axis or the minor axis of the ellipse. In addition, the desired arc should have the shortest length among all possible arcs of the

same ellipse. Thus, the desired arc is symmetric with respect to the major axis of the ellipse because the length of the elliptical arc with respect to the major axis would be of minimum length. Let us assume a hypothetical ellipse in the Cartesian coordinates system with the center at the origin and the foci on the y-axis. Suppose that the desired arc is the smallest arc between points $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Since the arc is symmetric with respect to the major axis, which lies on the y-axis, we have $x_1 = -x_2$, and $y_1 = y_2$.

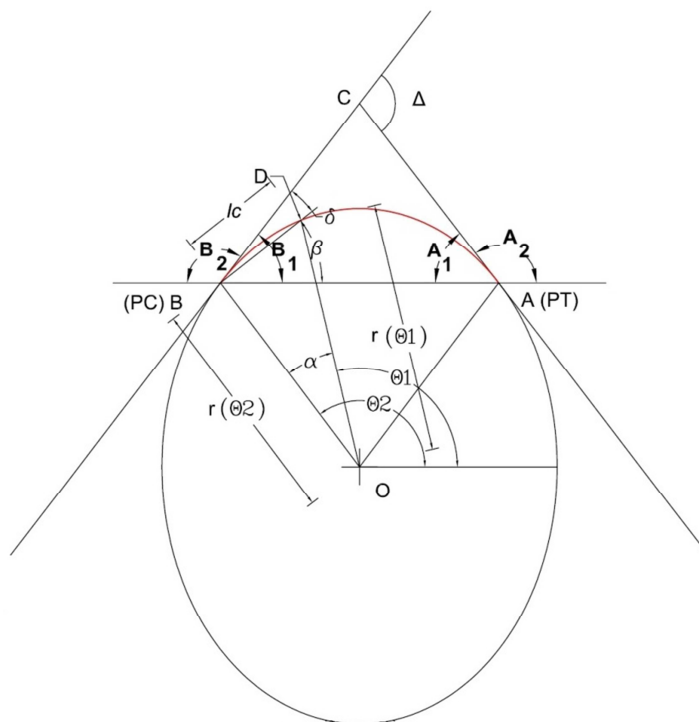


Figure 2.6 An arc of ellipse needed to connect PC to PT

Let us also assume that the slope of the tangent line at points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are m_1 and m_2 , respectively. So, $m_1 = -m_2$. As shown in Figure 2.6, the long chord for the desired arc of the ellipse and the tangent lines form an isosceles triangle since:

$$\widehat{A}_1 = \widehat{B}_1 = \frac{\Delta}{2}. \quad (2.17)$$

Then,

$$\widehat{A}_2 = \widehat{B}_2 = 180 - \frac{\Delta}{2}. \quad (2.18)$$

Therefore,

$$m_1 = \tan\left(180 - \frac{\Delta}{2}\right), \quad (2.19)$$

and

$$m_2 = -\tan\left(180 - \frac{\Delta}{2}\right). \quad (2.20)$$

On the other hand, the equation of the ellipse in the Cartesian coordinate system is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \quad (2.21)$$

By taking the derivative of the ellipse equation, the slope of the tangent line at any arbitrary point of the ellipse is:

$$\frac{dy}{dx} = \frac{-(a^2)(x)}{(b^2)(y)}. \quad (2.22)$$

From the ellipse equation, y-coordinate of any point on the ellipse is:

$$y = \pm \left(\frac{a}{b}\right) \left(\sqrt{b^2 - x^2}\right). \quad (2.23)$$

By substituting equation (2.23) in the equation (2.22), we will have:

$$\frac{dy}{dx} = \frac{-(a^2)(x)}{(b^2)(y)} = \frac{-(a^2)(x)}{(b^2)\left(\frac{a}{b}\right)(\sqrt{b^2 - x^2})} = \frac{-(a)(x)}{(b)(\sqrt{b^2 - x^2})}. \quad (2.24)$$

Therefore,

$$m_1 = \frac{-(a)(x_1)}{(b)(\sqrt{b^2 - x_1^2})}. \quad (2.25)$$

Thus,

$$x_1 = \frac{(m_1)(b^2)}{\sqrt{a^2 + (m_1^2)(b^2)}}, \quad (2.26)$$

and

$$y_1 = \left(\frac{a}{b}\right) \left(\sqrt{b^2 - x_1^2}\right). \quad (2.27)$$

As a result, the location of point $B = (x_2, y_2)$ will be determined to be:

$$x_2 = -x_1 = \frac{-(m_1)(b^2)}{\sqrt{a^2 + (m_1^2)(b^2)}} \quad (2.28)$$

and

$$y_1 = y_2. \quad (2.29)$$

Consequently, any desired arc of the ellipse can be found by having the slope (direction) of the tangent lines and the intersection angle between them.

2.4 Length of the Arc of an Ellipse

To minimize the ROW, the minimum length arc is desired that meets all requirements. In the previous section, a method was introduced to identify an arc that only satisfies the tangent lines constraint. After this, the resulting arc should be checked to ensure it is the minimum-length arc.

To find the length of an ellipse arc, the polar coordinate system is again useful. It is proven (Larson et al., 2010, p. 704) that the length of an ellipse arc between 0 and θ can be found from the following integration:

$$E(\theta, \tau) = a \int_0^\theta \sqrt{1 - \tau^2 \sin^2 t} dt \quad (2.30)$$

in which

a : is the length of the major axis of the ellipse;

τ : is the eccentricity of the ellipse; and

$E(\theta, \tau)$: is the length of the arc of an ellipse with eccentricity of τ , between 0 and θ .

To be able to use this integration, we need to know the coordinates of points A and B in the polar coordinate system. Since $A = (x_1, y_1)$ and $B = (x_2, y_2)$, the polar coordinates of points A and B will be easily gained by applying trigonometry, namely,

$$\theta_1 = \tan^{-1} \left(\frac{y_1}{x_1} \right), r_1 = \sqrt{x_1^2 + y_1^2}; \quad (2.31)$$

and

$$\theta_2 = \tan^{-1} \left(\frac{y_2}{x_2} \right), r_2 = \sqrt{x_2^2 + y_2^2}. \quad (2.32)$$

Now, let us define the length of the arc as:

$$l_{a,\tau}(\theta_1, \theta_2) = E(\theta_2, \tau) - E(\theta_1, \tau) \quad (2.33)$$

in which

θ_1 = the angle at which the ellipse arc starts;

θ_2 = the angle at which the ellipse arc ends;

$l_{a,\tau}(\theta_1, \theta_2)$ = is the length of the arc starting at angle θ_1 and ending at θ_2 on an ellipse with major axis a and the eccentricity of τ .

2.5 Area of an Ellipse Sector and the Right-Of-Way

As described, there are two ellipse sectors commonly used: one is defined with respect to the center of the ellipse and the other can be defined with respect to each focus. The ellipse equation in the polar coordinate system with respect to the center is:

$$r(\theta) = \frac{(a)(b)}{\sqrt{(b \cos \theta)^2 + (a \sin \theta)^2}}. \quad (2.34)$$

Also, the ellipse equation can be written with respect to each focus as:

$$r(\theta) = \frac{(a)(1 - \tau^2)}{1 \pm \tau \cos \theta} \quad (2.35)$$

where the sign in the denominator will be negative if the reference direction is from $\theta = 0$ towards the center. The sign in the denominator is positive if the reference direction points away from the center.

Therefore, the areas of the ellipse sector with respect to the center and foci, respectively, are

$$\begin{aligned}
 A_{E,\tau}(\theta_1, \theta_2) &= \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2(\theta) \cdot d\theta \\
 &= \int_{\theta_1}^{\theta_2} \frac{((a)(b))^2}{2[(b \cos \theta)^2 + (a \sin \theta)^2]} \cdot d\theta
 \end{aligned}
 \tag{2.36}$$

and

$$\begin{aligned}
 A_{E,\tau}(\theta_1, \theta_2) &= \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2(\theta) \cdot d\theta \\
 &= \int_{\theta_1}^{\theta_2} \frac{a^2(1 - \tau^2)^2}{2[1 \pm \tau \cos \theta]^2} \cdot d\theta .
 \end{aligned}
 \tag{2.37}$$

Figures 2.7 and 2.8 show each case:

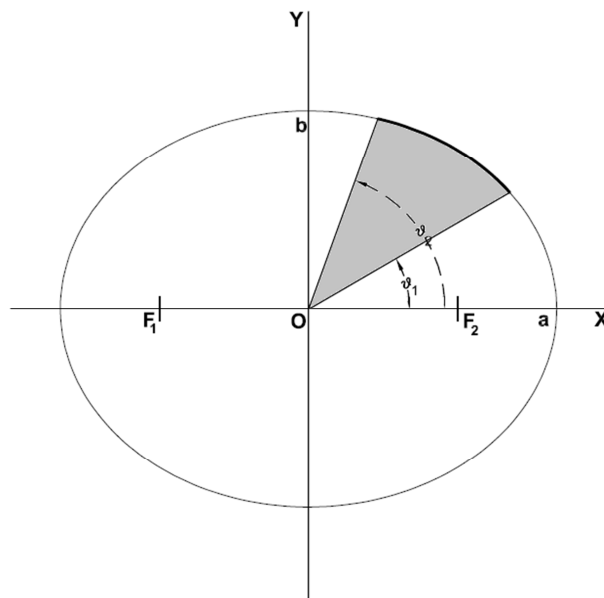


Figure 2.7 Sector of ellipse with respect to the center

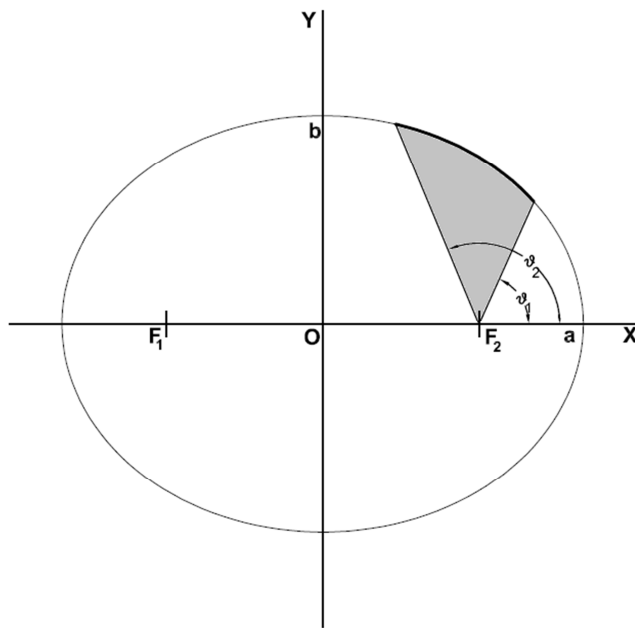


Figure 2.8 Sector of ellipse with respect to the foci

Let us assume the width of right-of-way is 100 ft., 50 ft. on each side of the centerline. Since the ellipse arc is supposed to be found with respect to one of the foci, the ellipse's equation with respect to the same focus should be used to calculate the area of the sector. In Figure 2.9, the shaded strip shows the right-of-way for an arbitrary ellipse arc. The right-of-way can be calculated as:

$$\begin{aligned}
 ROW &= \int_{\theta_1}^{\theta_2} \frac{1}{2} (r(\theta) + 50)^2 d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2} (r(\theta) - 50)^2 d\theta \\
 &= \frac{1}{2} \int_{\theta_1}^{\theta_2} \left[\left(\frac{(a)(1 - \tau^2)}{1 \pm \tau \cos \theta} + 50 \right)^2 - \left(\frac{(a)(1 - \tau^2)}{1 \pm \tau \cos \theta} - 50 \right)^2 \right] d\theta \quad (2.38)
 \end{aligned}$$

2.6 Chord Length and Deflection Angle Calculations

Assuming that the desired arc of an ellipse connecting the PC to PT is identified, the next step is the calculation of the chord length and deflection angle. In Figure 2.10, the chord length, l_c , and the deflection angle, δ , are schematically shown. Using the polar equation of the ellipse with respect to the center of the ellipse, we know that:

$$r(\theta) = \frac{(a)(b)}{\sqrt{(b \cos \theta)^2 + (a \sin \theta)^2}} \quad (2.39)$$

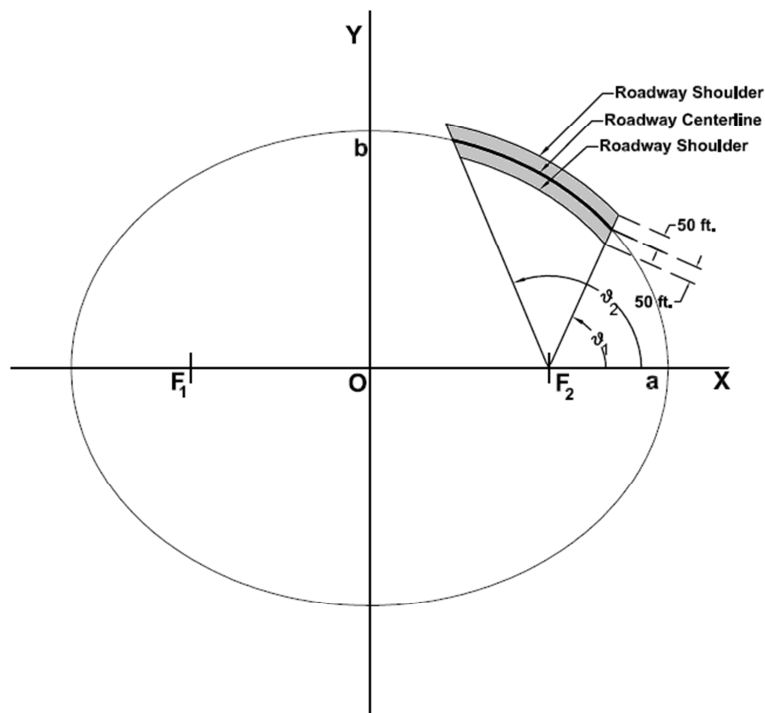


Figure 2.9 Right-Of-Way

As shown in Figure 2.10, OB , BD , and DO form the triangle OBD . The length of OB and DO can be calculated by inserting θ_1 and θ_2 in the polar

equation of the ellipse. Let us suppose that the deflection angles need to be calculated in decrement of α from θ_2 to θ_1 . Therefore, the angle between OB and DO is α , as shown in Figure 2.10. Thus, the length of chord l_c can be obtained by applying the Law of Cosines:

$$l_c^2 = r(\theta_2)^2 + r(\theta_1)^2 - 2 r(\theta_1) r(\theta_2) \cos(\alpha), \quad (2.40)$$

Now, we need to find the deflection angle δ . According to the Law of Sines, in the triangle OBD we have:

$$\frac{\sin(\gamma)}{r(\theta_2)} = \frac{\sin(\alpha)}{l_c}. \quad (2.41)$$

Then:

$$\gamma = \sin^{-1} \left(\frac{r(\theta_2) \sin \alpha}{l_c} \right) \quad (2.42)$$

In Figure 2.10, we also have:

$$\varphi = 180^\circ - \theta_1 \quad (2.43)$$

On the other hand, in triangle BDE :

$$\begin{aligned} \beta &= 180^\circ - \gamma - \varphi \\ &= 180^\circ - \gamma - (180^\circ - \theta_1) = \theta_1 - \gamma. \end{aligned} \quad (2.44)$$

Then,

$$\delta = B_1 - \beta = \frac{\Delta}{2} - (\theta_1 - \gamma) = \frac{\Delta}{2} - \theta_1 + \gamma. \quad (2.45)$$

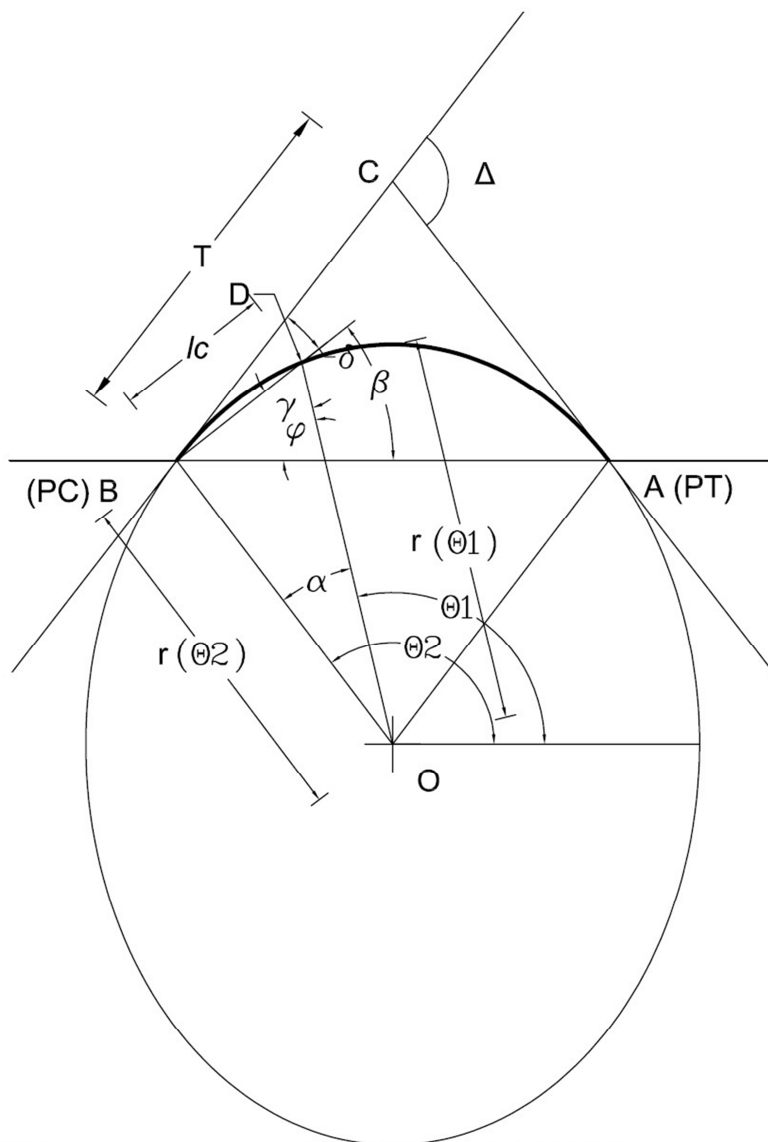


Figure 2.10 Diagram of an ellipse arc

2.7 Locations of PC_{elp} , and PT_{elp}

Referring to Figure 2.10, the locations of points A and B are known. Based on definition, the intersection of the lines tangent at points A and B will be the location of the PI . The equations of tangent lines are:

$$\text{Tangent Line at A: } y = y_1 + (m_1)(x - x_1), \quad (2.46)$$

and

$$\text{Tangent Line at B: } y = y_2 + (m_2)(x - x_2). \quad (2.47)$$

By solving the system of equations below,

$$\begin{cases} y = y_1 + (m_1)(x - x_1) \\ y = y_2 + (m_2)(x - x_2) \end{cases} \quad (2.48)$$

the location of PI can be determined. Note that $x_1, x_2, y_1, y_2, m_1,$ and m_2 are known. Let us suppose that point (x^*, y^*) is the intersection of tangent lines.

Then:

$$x^* = \frac{y_1 - y_2 + (m_1)(x_1) - (m_2)(x_2)}{m_1 - m_2} \quad (2.49)$$

Since $m_1 = -m_2$:

$$x^* = \frac{y_1 - y_2 + (m_1)(x_1 + x_2)}{2m_1}, \quad (2.50)$$

and

$$y^* = y_1 + (m_1)(x^* - x_1). \quad (2.51)$$

Therefore, the length of tangent line T is:

$$T = \sqrt{(y^* - y_1)^2 + (x^* - x_1)^2}. \quad (2.52)$$

2.8 Station Numbers Calculations for PC_{elp} and PT_{elp}

It is assumed that the station number of PI is given. As shown, the length of tangent, T , and the length of the ellipse arc, $l_{a_{elp}}$, can be calculated as:

$$T = \sqrt{(y^* - y_1)^2 + (x^* - x_1)^2};$$
$$l_{a_{elp}} = E(\theta_2, \tau) - E(\theta_1, \tau)$$
$$= a \left(\int_0^{\theta_2} \sqrt{1 - \tau^2 \sin^2 t} dt - \int_0^{\theta_1} \sqrt{1 - \tau^2 \sin^2 t} dt \right). \quad (2.53)$$

Therefore,

$$Sta. \# @ PC_{elp} = Sta. \# @ PI - T \quad (2.54)$$

and

$$Sta. \# @ PT_{elp} = Sta. \# @ PC_{elp} + l_{a_{elp}}. \quad (2.55)$$

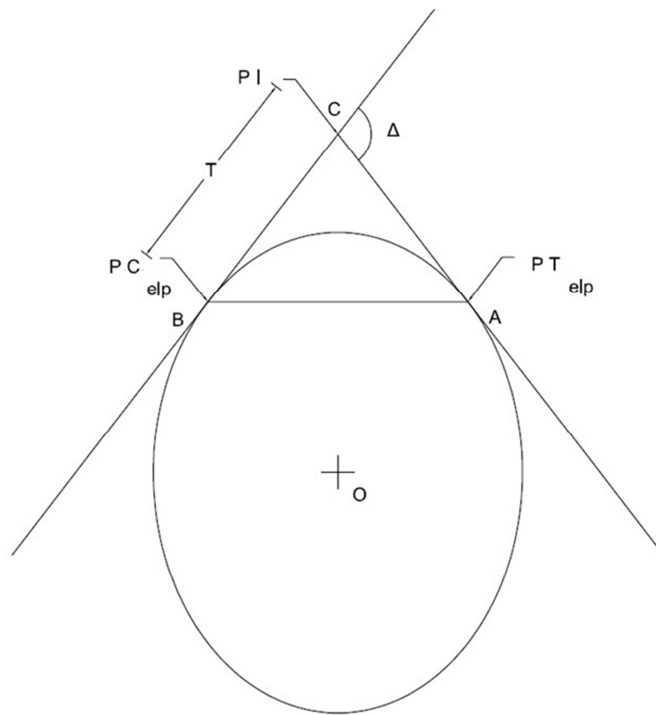


Figure 2.11 Location of PC and PT, and Length of Tangent Line

The next chapter will present the procedure for staking out the elliptical arc.

CHAPTER 3

STAKE-OUT PROCEDURE AND APPLICATION EXAMPLES

3.1 Algorithm to Find the Minimum Arc of Ellipse Connecting PC and PT

As discussed in Chapter 2, the angle Δ , speed design V_d and the location (station number) of PI . are typically given. According to the equations derived, the following algorithm results in the desired arc of ellipse connecting PC to PT

Algorithm (A):

0. Angle Δ , design speed, V_d , and location of PI are given.
1. According to the design speed V_d and minimum radius recommended by AASHTO, the value of R_{min} for V_d is known.
2. Start with eccentricity τ of 0.1.
3. Find the major axis, a , by plugging τ into

$$a = \frac{R_{min}}{1 - \tau}. \quad (3.1)$$

4. By applying equation below, calculate the minor axis, b :

$$b = (a) \left(\sqrt{1 - \tau^2} \right). \quad (3.2)$$

5. Find the slope of the tangent line at point A, m_1 :

$$m_1 = \tan \left(180 - \frac{\Delta}{2} \right) \quad (3.3)$$

6. The slope of the tangent line at point B, m_2 is $-m_1$:

$$m_1 = -m_2. \quad (3.4)$$

7. Find the x-coordinate of point A, x_1 :

$$x_1 = \frac{(m_1)(b^2)}{\sqrt{a^2 + (m_1^2)(b^2)}} \quad (3.5)$$

8. Find the y-coordinate of point A, y_1 :

$$y_1 = \left(\frac{a}{b}\right) \sqrt{b^2 - x_1^2}. \quad (3.6)$$

9. Determine $\theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right)$.

10. Determine $\theta_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right)$.

11. Calculate the length of the elliptical arc,

$$l_{a,\tau}(\theta_1, \theta_2) = E(\theta_2, \tau) - E(\theta_1, \tau) \quad (3.7)$$

in which

$$E(\theta, \tau) = a \int_0^\theta \sqrt{1 - \tau^2 \sin^2 t} \, dt \quad (3.8)$$

12. Calculate the area of piece of ellipse, $A_{E,\tau}(\theta_1, \theta_2)$:

$$ROW = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left[\left(\frac{(a)(1 - \tau^2)}{1 \pm \tau \cos \theta} + 50 \right)^2 - \left(\frac{(a)(1 - \tau^2)}{1 \pm \tau \cos \theta} - 50 \right)^2 \right] d\theta. \quad (3.9)$$

13. Repeat the preceding steps for new eccentricity τ with increments of 0.1 unit the current τ is less than 1.0.
14. Compare the length of the arc and the area of the piece of ellipse gained for each value of eccentricity τ , and pick the eccentricity τ with the minimum length of arc and area. This is the desired minimum length elliptical arc to be used.
15. It is the elliptical arc we desire.

3.2 Algorithm to Calculate Chords Length and Deflection Angles

After the desired elliptical arc is acquired, another algorithm is needed to stakeout the elliptical curve. To achieve this, chord lengths and deflection angles should be determined using the algorithm B, as follows:

Algorithm (B):

0. θ_1 and θ_2 are gained by algorithm (A).
1. Degree of curvature is:

$$D = \frac{(\theta_2 - \theta_1)(100)}{l_{a,\tau}(\theta_1, \theta_2)}. \quad (3.10)$$

2. If $\frac{(\theta_2 - \theta_1)}{D}$ is integer, then

$$N = \frac{(\theta_2 - \theta_1)}{D}; \quad (3.11)$$

Else

$$N = \left\lceil \frac{(\theta_2 - \theta_1)}{D} \right\rceil. \quad (3.12)$$

3. Get $i = 1$.
4. $\alpha = i \times D$
5. Get $\theta^* = \theta_2 - \alpha$.
6. Find the length of chord by applying the equation below:

$$(l_c)_i = \sqrt{r(\theta_2)^2 + r(\theta^*)^2 - 2 r(\theta_2) r(\theta^*) \cos(\alpha)} \quad (3.13)$$

in which

$$r(\theta) = \frac{(a)(b)}{\sqrt{(a \cos \theta)^2 + (b \sin \theta)^2}}. \quad (3.14)$$

7. Find γ :

$$\gamma = \sin^{-1} \left(\frac{r(\theta_2) \sin \alpha}{l_c} \right). \quad (3.15)$$

8. Find deflection angle, δ_i :

$$\delta_i = \frac{\Delta}{2} - \theta^* + \gamma. \quad (3.16)$$

9. $i = i + 1$.
10. Repeat step 4 and follow the algorithm until $i \leq N$.
11. Now, we have the deflection angles and their corresponding chord lengths.

3.3 Algorithm to Stake-out Station Numbers

As an output of Algorithm (A), the Cartesian coordinates of points A and B , respectively (x_1, y_1) and (x_2, y_2) , will be acquired. The respective slope tangent lines at points A and B , m_1 and m_2 , are determined. The following is the algorithm to find the station numbers:

Algorithm (C):

0. From Algorithm (A), we have coordinates of point A and B . We also know slopes m_1 , and m_2 .

1. Find x^* :

$$x^* = \frac{y_1 - y_2 + (m_1)(x_1 + x_2)}{2(m_1)}. \quad (3.17)$$

2. Find y^* :

$$y^* = y_1 + (m_1)(x^* - x_1). \quad (3.18)$$

3. Calculate length of tangent, T :

$$T = \sqrt{(y^* - y_1)^2 + (x^* - x_1)^2}. \quad (3.19)$$

4. Station number of \mathbf{PC}_{elp} :

$$Sta. \# @ \mathbf{PC}_{elp} = sta. \# @ \mathbf{PI} - T. \quad (3.20)$$

5. Station number of \mathbf{PT}_{elp} :

$$Sta. \# @ \mathbf{PT}_{elp} = Sta. \# @ \mathbf{PC}_{elp} + l_{a_{elp}}. \quad (3.21)$$

6. Station numbers along elliptical arc at any angle θ^* :

$$\text{Sta. \# @ angle } \theta^* = \text{Sta. \# @ PC}_{\text{elp}} + l_{a,\tau}(\theta^*, \theta_2) \quad (3.22)$$

where $l_{a,\tau}(\theta^*, \theta_2)$ is the length of arc between angle θ^* , and θ_2 .

Take this step, for each θ^* to find the corresponding station number.

7. At this step, all station numbers will be obtained.

3.4 An Application Example

Let us assume that it is desired to connect the PC to PT through an elliptical arc such that $\Delta = 120^\circ$, $R_{min} = 1000$ (ft), and $Sta. \# @ PI = 40 + 40$. First, the arc of ellipse should be found so that it satisfies the initial constraints. Applying the algorithm (A) yields with the results tabulated in Table 3.1. Comparing the length of the arc and the right-of-way area, the ellipse with $\tau = 0.1$ provides the minimum length and the minimum right-of-way. Therefore, the desired ellipse is an ellipse with major axis a of 1111.1 ft. and minor axis b of 1105.5 ft. Using the algorithm (B), chord lengths and deflection angles are then obtained, as shown in Table 3.2. Finally, the algorithm (C) yields the station numbers to stakeout the elliptical curve, as shown in Tables 3.3.a. and 3.3.b. In Table 3.4, a summary of all three algorithms is provided.

Table 3.1 Results of Algorithm (A).

Eccentricity τ	a (ft.)	b (ft.)	Δ (deg.)	Slope m_1	Point A		Point B		θ_1 (deg.)	θ_2 (deg.)	$I_{a,\tau}(\theta_1, \theta_2)$ (ft.)	ROW) (Sq.ft.)
					x_1 (ft.)	y_1 (ft.)	x_2 (ft.)	y_2 (ft.)				
					0.1	1111.1	1105.5	120				
0.2	1250.0	1224.7	120	-1.734	1055	635.2	-1055	635.2	31.06	148.94	2570.2	308,832
0.3	1428.6	1362.8	120	-1.734	1166	739.3	-1166	739.3	32.37	147.63	2869.8	385,238
0.4	1666.7	1527.5	120	-1.734	1293	887.3	-1293	887.3	34.46	145.54	3219.2	492,565
0.5	2000.0	1732.1	120	-1.734	1442	1108	-1442	1108	37.54	142.46	3636.5	652,790
0.6	2500.0	2000.0	120	-1.734	1622	1462	-1622	1462	42.04	137.96	4131.2	909,085
0.7	3333.3	2380.5	120	-1.734	1852	2094	-1852	2094	48.52	131.48	4734.7	1,369,860
0.8	5000.0	3000.0	120	-1.734	2163	3466	-2163	3466	58.04	121.96	5467.8	2,366,885
0.9	10000.0	4358.9	120	-1.734	2628	7976	-2628	7976	71.8	108.20	6283.4	5,593,074

Table 3.2 Results of Algorithm (B).

θ^* (deg.)	$r(\theta^*)$ (ft.)	γ (deg.)	β (deg.)	Deflection Angles δ_i (deg.)	Chord Length $(l_c)_i$ (ft.)
149.80	1106.9	0.00	149.80	0	0.0
144.64	1107.4	87.16	57.48	2.52	99.6
139.49	1107.9	84.57	54.91	5.09	199.1
134.33	1108.4	81.99	52.33	7.66	298.2
129.17	1108.9	79.41	49.76	10.24	396.7
124.02	1109.4	76.84	47.18	12.82	494.5
118.86	1109.8	74.26	44.59	15.41	591.3
113.70	1110.2	71.69	42.01	17.99	686.9
108.55	1110.5	69.13	39.42	20.58	781.2
103.39	1110.8	66.56	36.82	23.18	873.8
98.23	1111.0	64.00	34.23	25.77	964.7
93.08	1111.1	61.44	31.63	28.37	1053.6
87.92	1111.1	58.88	29.03	30.96	1140.4
82.76	1111.0	56.32	26.44	33.56	1224.8
77.60	1110.9	53.76	23.84	36.16	1306.6
72.45	1110.6	51.21	21.24	38.76	1385.7
67.29	1110.3	48.65	18.64	41.36	1462.0
62.13	1109.9	46.09	16.04	43.96	1535.2
56.98	1109.4	43.53	13.45	46.55	1605.3
51.82	1109.0	40.97	10.85	49.15	1672.1
46.66	1108.5	38.40	8.26	51.74	1735.4
41.51	1108.0	35.84	5.67	54.33	1795.2
36.35	1107.5	33.27	3.08	56.92	1851.3
31.19	1107.0	30.70	0.50	59.50	1903.7
30.22	1106.9	30.21	0.01	59.99	1913.2

Table 3.3 Results of algorithm (C) - initial part.

θ_1 (deg.)	θ_2 (deg.)	Intersection of Tangent Lines		T (ft.)	Dereee of Curvature, D (deg.)	N
		x^* (ft.)	y^* (ft.)			
30.22	149.78	0	2216.0	1914.9	5.16	24

Table 3.4 Results of algorithm (C) - second part.

Sta. # @ PI	T (ft.)	Sta. # @ PC_{elp}	$I_{a_{elp}}$ (ft.)	Sta. # @ PT_{elp}
40+40.0	1914.9	21+25.1	2318.9	44+44.0

Table 3.5 Stakeout Table.

	i	Station Numbers	θ^* (deg.)	$r(\theta^*)$ (ft.)	Deflection Angles (deg.) δ_i	Chord Length (ft.) $(l_c)_i$	Length of Arc (ft.) $l_{a,\tau}(\theta^*, \theta_2)$
Sta. # @ PC_{elp}	1	21+25.1	149.80	1106.9	0	0.0	0.0
	2	22+25.1	144.64	1107.4	2.52	99.6	100.0
	3	23+25.1	139.49	1107.9	5.09	199.1	200.0
	4	24+25.1	134.33	1108.4	7.66	298.2	300.0
	5	25+25.1	129.17	1108.9	10.24	396.7	400.0
	6	26+25.1	124.02	1109.4	12.82	494.5	500.0
	7	27+25.1	118.86	1109.8	15.41	591.3	600.0
	8	28+25.1	113.70	1110.2	17.99	686.9	700.0
	9	29+25.1	108.55	1110.5	20.58	781.2	800.0
	10	30+25.1	103.39	1110.8	23.18	873.8	900.0
	11	31+25.1	98.23	1111.0	25.77	964.7	1000.0
	12	32+25.1	93.08	1111.1	28.37	1053.6	1100.0
	13	33+25.1	87.92	1111.1	30.96	1140.4	1200.0
	14	34+25.1	82.76	1111.0	33.56	1224.8	1300.0
	15	35+25.1	77.60	1110.9	36.16	1306.6	1400.0
	16	36+25.0	72.45	1110.6	38.76	1385.7	1499.9
	17	37+25.0	67.29	1110.3	41.36	1462.0	1599.9
	18	38+25.0	62.13	1109.9	43.96	1535.2	1699.9
	19	39+25.0	56.98	1109.4	46.55	1605.3	1799.9
	20	40+25.0	51.82	1109.0	49.15	1672.1	1899.9
	21	41+25.0	46.66	1108.5	51.74	1735.4	1999.9
	22	42+25.0	41.51	1108.0	54.33	1795.2	2099.9
	23	43+25.0	36.35	1107.5	56.92	1851.3	2199.9
	24	44+25.0	31.19	1107.0	59.50	1903.7	2299.9
Sta. # @ PT_{elp}	25	44+44.0	30.22	1106.9	59.99	1913.2	2318.9

3.5 The Circular Curve Solution

To provide highway horizontal alignment, using a circular curve is the most typical solution. Again, let us assume that it is desired to connect PC to PT, but this time through a circular curve such that $\Delta = 120^\circ$, $R_{min} = 1000$ (ft), and $Sta. \# @ PI. = 40 + 40.0$. According to the Table 1.1, the length of tangent line T is:

$$T = (R) \tan\left(\frac{\Delta}{2}\right) = (1000) \tan\left(\frac{120^\circ}{2}\right) = 1732.1 \text{ ft.} \quad (3.23)$$

The length of arc l_a is:

$$\begin{aligned} l_a &= \left(\frac{\pi}{180}\right) (\Delta)(R) = \left(\frac{\pi}{180}\right) (120)(1000) \\ &= 2094.4 \text{ ft.} \end{aligned} \quad (3.24)$$

Therefore, the locations of PC and PT are as shown in Table 3.5 below:

Table 3.6 Locations of PC and PT for circular curve.

Sta. # @ PI	T	Sta. # @ PC	l_a	Sta. # @ PT
40+40.4	1732.1	23+07.9	2094.4	44+02.3

Then,

$$\text{Degree of Curvature } D = \frac{5729.6}{R} = \frac{5729.6}{1000} = 5.73^\circ. \quad (3.25)$$

Applying the formulas provided in Table 1.3, the resulting stakeout table is shown in Table 3.6.

Table 3.7 Stake-out Table for Circular Curve.

	<i>i</i>	Station Numbers	Deflection Angles (deg.) δ_i	D_i (deg.)	Chord Length (ft.) $(l_c)_i$	Length of Arc (ft.) l_a
Sta. # @ PC	1	23+07.9	0.00	0.00	0.0	0.0
	2	24+07.9	2.86	5.73	100.0	100.0
	3	25+07.9	5.72	11.46	199.7	200.0
	4	26+07.9	8.58	17.19	298.9	300.0
	5	27+07.9	11.44	22.92	397.4	400.0
	6	28+07.9	14.30	28.65	494.8	500.0
	7	29+07.9	17.16	34.38	591.1	600.0
	8	30+07.9	20.02	40.11	685.8	700.0
	9	31+07.9	22.88	45.84	778.9	800.0
	10	32+07.9	25.74	51.57	870.0	900.0
	11	33+07.9	28.60	57.30	958.9	1000.0
	12	34+07.9	31.46	63.03	1045.4	1100.0
	13	35+08.0	34.32	68.76	1129.4	1200.1
	14	36+08.0	37.18	74.49	1210.4	1300.1
	15	37+08.0	40.04	80.22	1288.5	1400.1
	16	38+08.0	42.90	85.95	1363.4	1500.1
	17	39+08.0	45.76	91.68	1434.8	1600.1
	18	40+08.0	48.62	97.41	1502.6	1700.1
	19	41+08.0	51.48	103.14	1566.7	1800.1
	20	42+08.0	54.34	108.87	1626.9	1900.1
	21	43+08.0	57.20	114.60	1683.0	2000.1
Sta. # @ PT	22	44+02.2	60.00	120.00	1732.1	2094.3

3.6 Comparing Circular Curve Results with Elliptical Curve Results

In highway design terms, the length of horizontal alignment and right-of-way are two significant factors in evaluating the alternative design proposed. In Figure 3.1, both the elliptical curve and the equivalent circular curve are shown. The blue curve is circular curve. The elliptical curve is shown in red. Point A is the start point of the elliptical curve, PC_{elp} . Point B is the start point of the circular curve, PC . Point C is the end point of the circular curve. Point D is the end point of the elliptical curve, PT . According to the results gained for the same problem through the circular and elliptical approaches, the right-of-way for circular curve connecting A to D, shown in Figure 3.1, is:

$$ROW_{Circular} = (\overline{AB})(100) + \frac{(\Delta)(\pi)}{360} [(r_{\widehat{BC}} + 50)^2 - (r_{\widehat{BC}} - 50)^2] + (\overline{CD})(100) \quad (3.24)$$

As shown in Figure 3.1,

$$\begin{aligned} \overline{AB} = \overline{CD} &= \left(\frac{LC_E - LC_C}{2} \right) \left(\frac{1}{\cos\left(\frac{\Delta}{2}\right)} \right) \\ &= \left(\frac{1913 - 1732}{2} \right) \left(\frac{1}{\cos\left(\frac{120}{2}\right)} \right) = \left(\frac{181}{2} \right) (2) = 181 \text{ ft.} \end{aligned} \quad (3.25)$$

Then,

$$\begin{aligned} ROW_{Circular} &= (181)(100) \\ &+ \left(\frac{\pi}{3} \right) [(1000 + 50)^2 - (1000 - 50)^2] \\ &+ (181)(100) = 245,633 \text{ sq. ft.} \end{aligned} \quad (3.26)$$

Length of roadway from A to D through circular curve is:

$$\begin{aligned}
 L_{ABCD} &= L_{AB} + L_{BC} + L_{CD} = \overline{AB} + l_{a_{Circular}} + \overline{CD} \\
 &= 181 + 2094.3 + 181 = 2456.3 \text{ ft.}
 \end{aligned}
 \tag{3.27}$$

Comparing circular results with the elliptical result, we have:

Table 3.8 Circular Curve vs. Elliptical Curve

	Circular Curve	Elliptical Curve
Length	2456 ft.	2319 ft.
Right-Of-Way	5.64 acres	5.81 acres

Therefore, the elliptical curve provided a smoother horizontal transition in a length which is 137 ft. shorter than the alternative circular curve. Another possible advantage of the elliptical alternative is that the transition from the normal crown to the superelevated cross-section can be achieved more gradually through the entire length of the elliptical arc. This also provides for a smoother cross-section transition. However, the circular curve needs smaller right of way, 0.17 acre less than the elliptical curve in this example; resulting in a somewhat smaller ROW purchase cost. Figure 3.1 depicts both the circular and the elliptical curve to provide a visual comparison.

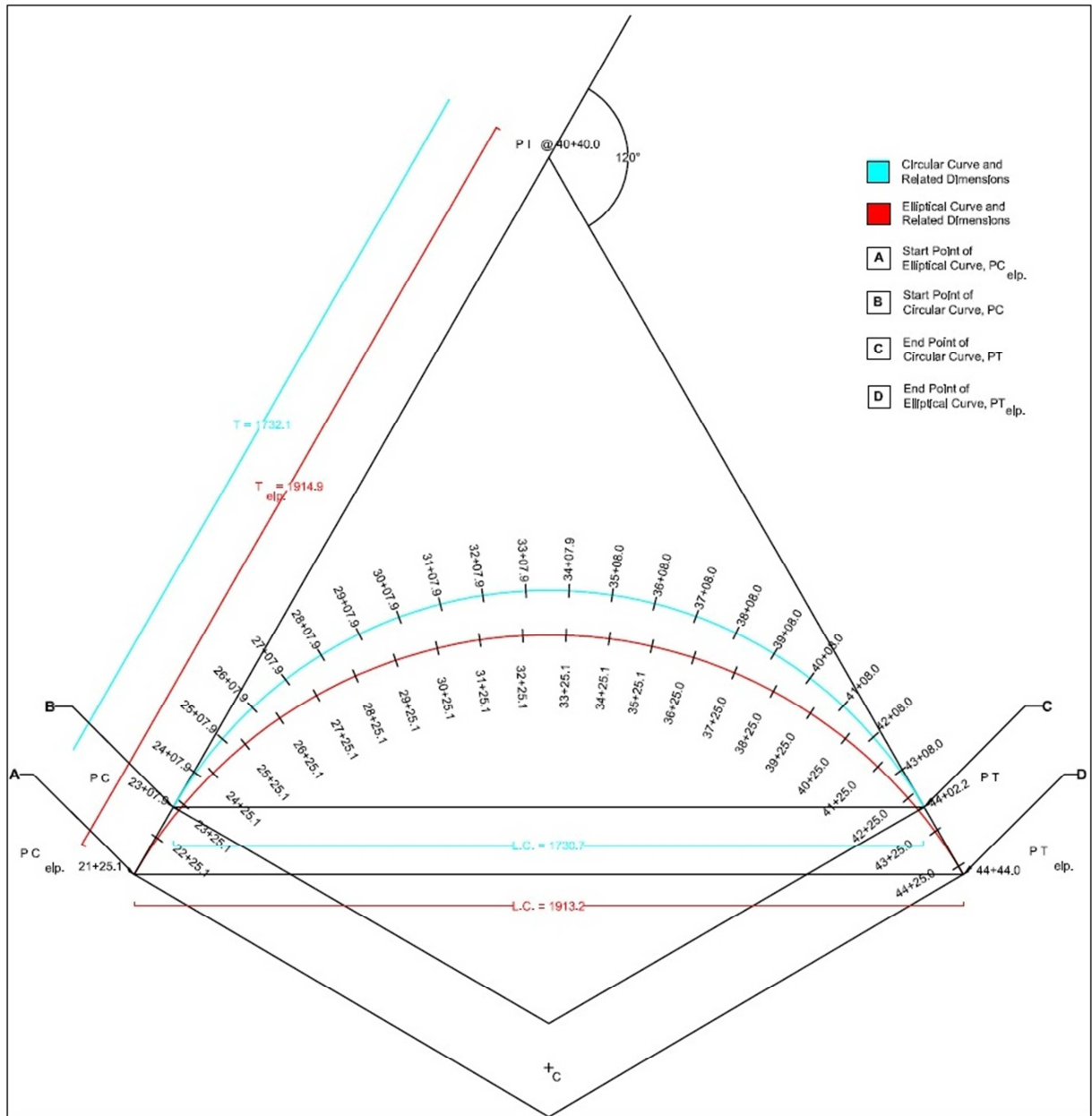


Figure 3.1 Final Profile: Elliptical curve vs. circular curve.

3.7 The Spiral-Circular Curve Solution

An alternative for horizontal alignment is spiral-circular curve, which consists of one circular curve at the middle and two spiral curves on sides. As discussed earlier in detail, the length of the spiral curves can be calculated from equation 1.7, as follows:

$$l_s = \frac{(3.15)(V^3)}{(R_s)(C)} .$$

The radius of spiral curves varies from infinity at TS to the radius of the circular curve at SC. Therefore, the degree of curvature varies from 0 to D_c . In fact, the average degree of curvature for spiral curves is $D_c/2$. As a result, it can be assumed that the radius of the spiral curve is on the average twice of the radius of the circular curve, $[R_{spiral}]_{max} = (2)(R_{circular})$. Referring to Figure 3.2 and applying geometric properties, the relations in Table 3.8 can be derived.

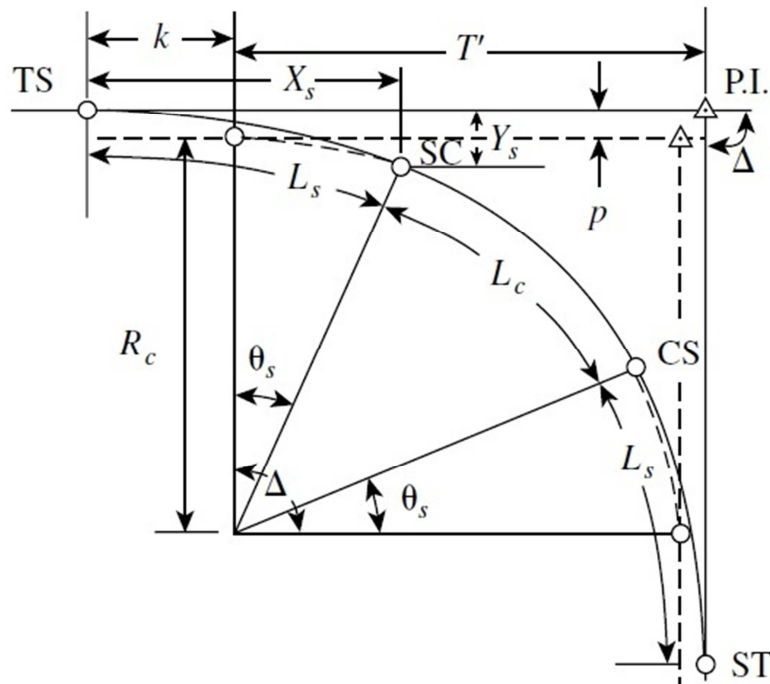


Figure 3.2 A schematic diagram of spiral-circular curve (Banks, 2002, p. 81).

In Figure 3.2, we have:

TS: the point of change from the tangent to the spiral curve.

SC: the point of change from the spiral curve to the circular curve.

CS: the point of change from the circular curve to the spiral curve.

ST: the point of change from the spiral curve to the tangent.

Table 3.9 Spiral-Circular Curve Relations (Banks, 2002, p. 81).

Equation	Comments
$l_s = \frac{(3.15)(v^3)}{(R_s)(C)}$	Length of Spiral Curve
$\theta_s = \frac{(L_s)(D_c)}{200}$	Spiral Angle
$\Delta_c = \Delta - (2)(\theta_s)$	Circular Angle
$D_c = \frac{5729.6}{R_c}$	Circular Degree of Curvature
$\Phi_c = \frac{\theta_s}{3}$	Spiral Degree of Curvature
$\Phi_i = \left(\frac{i}{5}\right)^2 (\Phi_c)$	Deflection Angles (5-Chord Method)
$T' = (R_c + p) \left(\tan \frac{\Delta}{2}\right)$	
$X_s = \left(1.0 - \frac{\theta_s^2}{10}\right) (L_s)$	θ_s is in radians
$Y_s = \left(\frac{\theta_s}{3} - \frac{\theta_s^3}{42}\right) (L_s)$	θ_s is in radians
$X = L - \frac{L^5}{(40)(A^4)} + \frac{L^9}{(3456)(A^8)}$	X coordinate of any point on the spiral. ($A^2 = (R_c)(L_s)$)
$Y = \frac{L^3}{(6)(A^2)} - \frac{L^7}{(336)(A^6)} + \frac{L^{11}}{(42240)(A^{10})}$	Y coordinate of any point on the spiral. ($A^2 = (R_c)(L_s)$)
$p = Y_s - (R_c)(1 - \cos \theta_s)$	Throw Distance (θ_s is in Radian)
$k = X_s - (R_c)(\sin \theta_s)$	

Let us suppose that an spiral-circular curve is used for the same problem with $\Delta = 120^\circ$, $R_{\min} = 1000$ ft., and Sta. # @ PI=40+40.0. To find the length of the spiral curve, design speed, V_d , and the rate of change of centripetal acceleration, C , must be specified. According to AASHTO, (2004, p. 147), $R_{\min} = 1000$ ft. satisfies a speed of 50 mph for all superelevation angles provided. Therefore, let us assume $V_d = 50$ mph. The roadways with design speed, $V_d = 50$ mph, are classified as high speed roadways. Thus, $C = 1$ ft./sec³. Then, we have:

$$R_s = (2)(R_c) = (2)(1000) = 2000 \text{ ft.} \quad (3.28)$$

Consequently, based on the Table 3.8, we have:

$$l_s = \frac{(3.15)(v^3)}{(R_s)(C)} = \frac{(3.15)(50^3)}{(2000)(1)} = 196.9 \approx 197 \text{ ft.}, \quad (3.29)$$

$$D_c = \frac{5729.6}{R_c} = \frac{5729.6}{1000} = 5.73^\circ, \quad (3.30)$$

$$\theta_s = \frac{(197)(5.73)}{200} = 5.644^\circ \approx 5.64^\circ = 0.0984 \text{ rad.} \quad (3.31)$$

$$\Phi_c = \frac{1}{3}(\theta_s) = \left(\frac{1}{3}\right)(5.64^\circ) = 1.88^\circ \quad (3.32)$$

$$\Delta_c = \Delta - (2)(\theta_s) = 120^\circ - (2)(5.64^\circ) = 108.72^\circ \quad (3.33)$$

$$X_s = \left(1 - \frac{\theta_s^2}{10}\right) L_s = \left(1 - \frac{0.0984^2}{10}\right) (197) = 196.8 \text{ (ft.)} \quad (3.34)$$

$$\begin{aligned} Y_s &= \left(\frac{\theta_s}{3} - \frac{\theta_s^2}{42}\right) (L_s) \\ &= \left(\frac{0.0984}{3} - \frac{0.0984^2}{42}\right) (197) = 6.42 \text{ (ft.)} \end{aligned} \quad (3.35)$$

$$\begin{aligned} k &= X_s - (R_c) (\sin \theta_s) = 196.8 - 1000 \times \sin(5.64^\circ) \\ &= 98.5 \text{ (ft.)} \end{aligned} \quad (3.36)$$

$$\begin{aligned} p &= Y_s - (R_c)(1 - \cos(\theta_s)) \\ &= 6.42 - (1000)(1 - \cos(5.64^\circ)) = 1.58 \end{aligned} \quad (3.37)$$

$$\begin{aligned} T' &= (R_c + p) \tan\left(\frac{\Delta}{2}\right) \\ &= (1000 + 1.58) \left(\tan\left(\frac{120^\circ}{2}\right)\right) = 1734.8 \text{ ft.} \end{aligned} \quad (3.38)$$

To stakeout the spiral-circular curve, the station number of TS needs to be determined, as follows:

$$\begin{aligned} \text{Sta. \# @ TS} &= \text{Sta. \# @ PI} - T' - k \\ &= (4040.0) - (1734.8) - (98.5) = 22 + 06.7. \end{aligned} \quad (3.39)$$

Tables 3.9, 3.10, and 3.11 provide the resulting station numbers and deflection angles for the spiral and the circular curves.

Table 3.10 Stake-out table for spiral-circular curve from TS to SC

<i>i</i>	Station Numbers	Length of Arc, <i>l</i> (5-Chord Method) (ft.)	$\Phi = \left(\frac{l_i}{L_s}\right)^2 (\Phi_s)$ (deg.)	$A^2 = \frac{(R_c)(L_s)}{(ft.^2)}$	<i>X</i> (ft.)	<i>Y</i> (ft.)	Chord Length, <i>l_c</i>	
TS	0	22+06.7	0	0.00	197000	0.00	0.00	0.00
	1	22+46.1	39.4	0.08	197000	39.40	0.05	39.40
	2	22+85.5	78.8	0.30	197000	78.80	0.41	78.80
	3	23+24.9	118.2	0.68	197000	118.19	1.40	118.19
	4	23+64.3	157.6	1.20	197000	157.54	3.31	157.57
SC	5	24+03.7	197	1.88	197000	196.81	6.46	196.92

Table 3.11 Stakeout table for spiral-circular curve from SC to CS

	<i>i</i>	Station Numbers	Deflection Angles (deg.) δ_i	Degree of Curvature (deg.) D_i	Chord Length (ft.) l_{c_i}	Length of Arc (ft.) l_{a_i}
SC	0	24+03.7	0	0	0.0	0.0
	1	25+03.7	2.86	5.73	99.8	100.0
	2	26+03.7	5.72	11.46	199.3	200.0
	3	27+03.7	8.58	17.19	298.4	300.0
	4	28+03.7	11.44	22.92	396.7	400.0
	5	29+03.7	14.3	28.65	494.0	500.0
	6	30+03.7	17.16	34.38	590.1	600.0
	7	31+03.7	20.02	40.11	684.7	700.0
	8	32+03.7	22.88	45.84	777.6	800.0
	9	33+03.7	25.74	51.57	868.6	900.0
	10	34+03.7	28.6	57.3	957.4	1000.0
	11	35+03.7	31.46	63.03	1043.8	1100.0
	12	36+03.7	34.32	68.76	1127.6	1200.1
	13	37+03.7	37.18	74.49	1208.6	1300.1
	14	38+03.7	40.04	80.22	1286.6	1400.1
	15	39+03.7	42.9	85.95	1361.4	1500.1
	16	40+03.7	45.76	91.68	1432.8	1600.1
	17	41+03.7	48.62	97.41	1500.7	1700.1
	18	42+03.7	51.48	103.14	1564.8	1800.1
CS	19	4301.1	54.36	108.72	1625.4	1897.5

Table 3.12 Stake-out table for spiral-circular curve from CS to ST

1	2	3	4	5	6	7	8	
	i	Station Numbers	Length of Arc, l (5-Chord Method) (ft.)	$\Phi = \left(\frac{l_i}{L_s}\right)^2 (\Phi_s)$ (deg.)	$A^2 = \frac{(R_c)(L_s)}{(\text{ft.}^2)}$	X (ft.)	Y (ft.)	Chord Length, l_c
CS	0	43+01.1	0	0.00	197000	0.00	0.00	0.00
	1	43+40.5	39.4	0.08	197000	39.40	0.05	39.40
	2	43+79.9	78.8	0.30	197000	78.80	0.41	78.80
	3	44+19.3	118.2	0.68	197000	118.19	1.40	118.19
	4	44+58.7	157.6	1.20	197000	157.54	3.31	157.57
ST	5	44+98.1	197	1.88	197000	196.81	6.46	196.92

3.8 Comparing Spiral-Circular Curve Results with Elliptical Curve Results

In Figure 3.3, both the elliptical curve and the equivalent spiral-circular curve are shown. The green curves are the spiral curves and the middle blue curve is circular curve. The elliptical curve is shown in red. Point A is the start point of the elliptical curve, PC_{elp} . Point B is the start point of the spiral-circular curve, TS. Point C is the end point of the spiral-circular curve. Point D is the end point of the elliptical curve, PT_{elp} . The right-of-way and the length of the spiral-circular curve are as follows:

$$\begin{aligned}
 ROW_{Spiral} &= (\overline{AB})(100) \\
 &+ \left(\frac{(\Delta_c)(\pi)}{360}\right) [(r_{\widehat{BC}} + 50)^2 - (r_{\widehat{BC}} - 50)^2] \\
 &+ \left(\frac{(2)(\theta_s)(\pi)}{360}\right) [(r_{\widehat{B-S.C.}} + 50)^2 - (r_{\widehat{B-S.C.}} - 50)^2] \\
 &+ (\overline{CD})(100).
 \end{aligned} \tag{3.40}$$

As shown in Figure 3.3,

$$\begin{aligned}
 \overline{AB} = \overline{CD} &= \left(\frac{LC_{Ellipse} - LC_{Spiral-Circular}}{2}\right) \left(\frac{1}{\cos \frac{\Delta}{2}}\right) \\
 &= \left(\frac{1913.2 - 1831.6}{2}\right) \left(\frac{1}{\cos \left(\frac{120}{2}\right)}\right) = \left(\frac{81.6}{2}\right) (2) \\
 &= 81.6 \text{ ft.}
 \end{aligned} \tag{3.41}$$

Then,

$$\begin{aligned}
 ROW_{Spiral} &= (81.6)(100) \\
 &+ \left(\frac{(108.72)(\pi)}{360} \right) [(1000 + 50)^2 \\
 &- (1000 - 50)^2] \\
 &+ \left(\frac{(2)(5.64)(\pi)}{360} \right) [(2000 + 50)^2 \\
 &- (2000 - 50)^2] + (81.6)(100) \\
 &= 8,160 + 189,752 + 39,375 \\
 &+ 8,160 = 245,447 \text{ sq. ft.} \\
 &= 5.63 \text{ acres.}
 \end{aligned} \tag{3.42}$$

Length of roadway from A to D through spiral-circular curve is:

$$\begin{aligned}
 L_{ABCD} &= L_{AB} + L_{B-SC} + L_{BC} + L_{SC-C} + L_{CD} \\
 &= \overline{AB} + l_s + l_{a_{Circular}} + l_s + \overline{CD} \\
 &= 81.6 + 197 + 1,897.5 + 197 \\
 &+ 81.6 = 2,454.7 \text{ ft.}
 \end{aligned} \tag{3.43}$$

Comparing the circular results with the elliptical result, we have:

Table 3.13 Spiral-Circular Curve vs. Elliptical Curve.

	Spiral-Circular Curve	Elliptical Curve
Length	2455 ft.	2319 ft.
Right-Of-Way	5.63 acres	5.81 acres

Therefore, the elliptical curve provided a smoother horizontal transition in a length which is 136 ft. shorter than spiral-circular curve. However, the spiral-circular curve needs a slightly smaller right of way, 0.18 acre less than the elliptical curve and 0.01 acre less than the circular curve in this example. In other word, the right of way for the spiral-circular is nearly the same as the right of way for the circular curve.

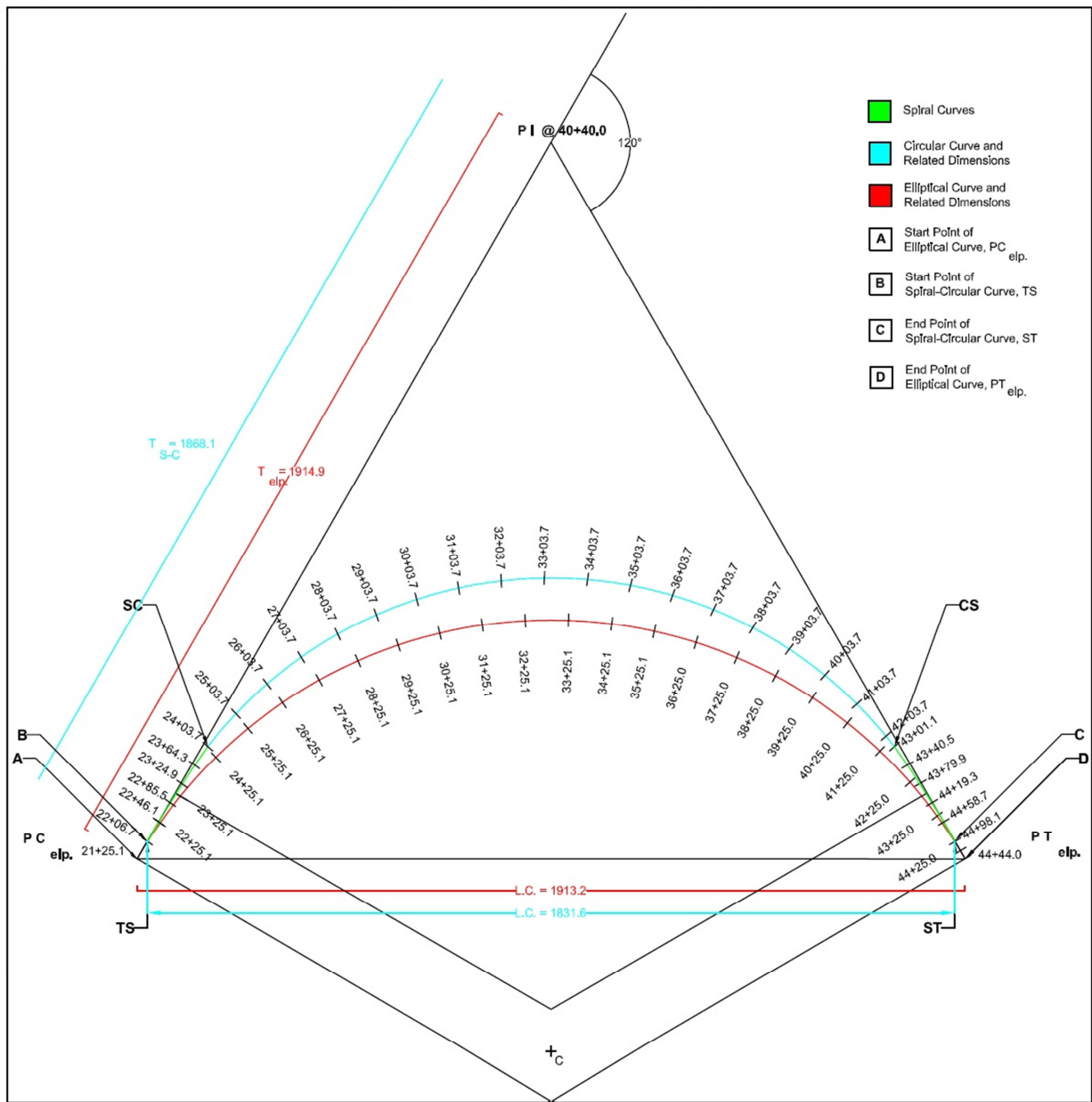


Figure 3.3 Final Profile. Elliptical curve vs. spiral-circular curve.

CHAPTER 4

CONCLUSIONS AND RECOMMENDATIONS

According to the results, elliptical curves can be used as horizontal transition curves to provide a smoother and safer directional transition in comparison with simple circular or spiral-circular curves. A possible advantage of using elliptical curves instead of all other alternatives is that elliptical curves can shorten the length of the roadway as shown in the application example. Since elliptical curves provide a smoother transition, another possible advantage is that the transition from the normal crown to the superelevated cross-section can be achieved more gradually through the entire length of the elliptical arc. Therefore, it can also provide a smoother cross-sectional transition and one that is likely more aesthetically pleasing.

As a result, elliptical curves should be considered as an alternative design for horizontal alignments. For instance, for each specific horizontal alignment problem with a given intersection angle, Δ , and design speed, V_d , alternative calculations for simple circular, spiral-circular-spiral, compound circular, and elliptical can be conducted. Then, the results obtained for each alternative should be compared with respect to the arc length and ROW requirements to optimize the design. Regarding the sight distance, the middle ordinate distance, which is the distance between the middle point of the curve and middle point of long chord, should be calculated for elliptical curves to make sure

there is no horizontal sight restriction for drivers.

In terms of calculations, the key equation to find the elliptical arc length is an elliptic integral. There are some difficulties to find a good estimation of this integral. This integral should be estimated for each feasible ellipse satisfying the intersection angle and the design speed.

In addition, many other calculations need to be done for algorithms (A) and (B), as described in the previous chapter. Also, this integral is a determinant equation for algorithm (C) to calculate station numbers. Therefore, it is recommended to develop a software, which includes all three algorithms and their calculations to find the most suitable elliptical curve for a given Δ and V_d . Also, elliptical calculations as an alternative design to circular, circular compound, or spiral-circular alignments should be incorporated in highway design software packages such as Geopak (Bentley Systems, 2012) and Microstation (Bentley Systems, 2012). Also, there may be benefits in using elliptical arcs for reverse curves. This aspect can be investigated as an extension of this work. As mentioned earlier, since the elliptical curve is found for a specific problem, using elliptical curves in lieu of circular, spiral-circular, and compound curve should be examined for various combination of intersection angles, Δ , and design speeds, V_d , to investigate possible advantages regarding the arc length and ROW.

Regarding environmental issues, using elliptical curves can reduce the mass of air pollutants. Elliptical curves can shorten the length of the roadway as well as provide a smoother transition from the normal crown to full-superelevated cross-section. Both of these properties could reduce vehicle fuel consumption. During a roadway's design

life, an elliptical curve can therefore save road users a significant amount of fuel. As a result, less fuel consumption could also translate to less air pollution. In addition, in the case of asphalt pavements, the shorter length of the roadway will decrease solar radiation absorbed by the asphalt surface. Therefore, elliptical curves can be more environmentally beneficial as they have the potential to substantially reduce air pollution and solar radiation absorbed by the asphalt surface over the design life of the roadway. One possible extension of this work could be a user-cost study of elliptical versus the more conventional horizontal alignments. The user cost could be quantified in terms of fuel consumption and air pollutants over the design life of a project and be used in the evaluation of alternative designs.

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