A STUDY ON THE TWO COMPONENT PERIODIC SHALLOW WATER SYSTEMS

by

CAIXIA CHEN

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To the Lord Jesus Christ and my family.

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ABSTRACT

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Caixia Chen, Ph.D.

The University of Texas at Arlington, 2012

Supervising Professor: Yue Liu

In this dissertation we study the generalized periodic two-component Camassa-Holm system and the generalized periodic two-component Dullin-Gottwald-Holm system, which can be derived from the Euler equation with nonzero constant vorticity in shallow water waves moving over a linear shear flow. The precise blow-up scenarios of strong solutions and several results of blow-up solutions with certain initial profiles are described in detail. The exact blow-up rates are also determined. Finally, the sufficient conditions for global solutions are established.

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CHAPTER 1

BACKGROUND

1.1 Early Developments

In this section we note the historical development of nonlinear shallow water wave theory following Ablowitz and Clarkson [1]. Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his “Report on Waves”. In Russell’s own words: “I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation”.

Scott Russell was convinced that he had observed an important phenomenon, and he built an experimental tank in his garden to continue his studies of what he

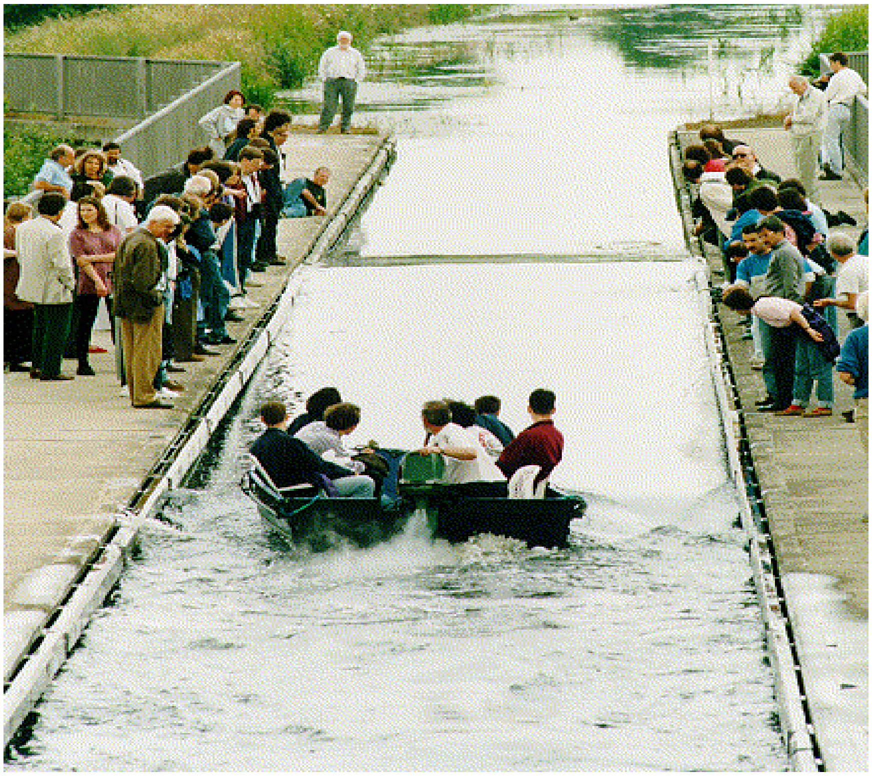

Figure 1.1. Soliton on the Scott Russell Aqueduct on the Union Canal.

dubbed the 'Wave of Translation' (see Figure 1.1). Unfortunately the implications which so excited him (he described the day he made his original observations as the happiest of his life) were ill-understood and largely ignored by his contemporaries, and Scott Russell was remembered instead for his considerable successes in ship hull design, and for conducting the first experimental study of the 'Doppler shift' of sound frequency as a train passes.

Especially, two results of Russell's are of importance to motivate the development of the nonlinear partial differential equations for modeling fluids, etc. i.e. one is that he observed solitary waves and hence deduced their existence. The other one is that he found the speed of propagation c of the solitary wave in a channel of depth

h to be $c = \sqrt{g(h+\alpha)}$, where α is the amplitude of the wave ad g the force due to gravity.

The 'Wave of Translation' itself was regarded as a curiosity until the 1960s when scientists began to use modern digital computers to study non-linear wave propagation. Then an explosion of activity occurred when it was discovered that many phenomena in physics, electronics and biology can be described by the mathematical and physical theory of the 'soliton', as Scott Russell's wave is now known. This work has continued and currently includes modeling high temperature superconductors and energy transport in DNA, as well as in the development of new mathematical techniques and concepts underpinning further developments.

1.2 Recent Developments

1.2.1 The Camassa-Holm Equation

In 1993, Camassa and Holm [7] proposed the following new equation (CH) for shallow water waves:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.1}$$

The Camassa-Holm equation is a well-known integrable equation describing the unidirectional propagation of shallow water waves over a flat bottom [7, 21, 32, 34], as well as water waves moving over an underlying shear flow [35]. The CH equation (1.1) also arises in the study of a certain non-Newtonian fluids [6] and also models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [24]. The CH equation (1.1) was first obtained by Fokas and Fuchssteiner [27, 28] as a bi-Hamiltonian generalized of KdV. The novelty of Camassa and Holm's work was the physical derivation of (1.1) and the discovery that the solitary wave solutions to this equation are solitons.

The CH equation (1.1) has caught a lot of attention in recent years due to two remarkable features. The first is the presence of solutions in the form of peaked solitary waves or "peakons" [2, 7, 39]: $u(t,x) = ce^{-|x-ct|}, c \neq 0$, which are smooth except at the crest, where they are continuous, but have a jump discontinuity in the first derivative. The peakons replicate a feature that is characteristic for the waves of great height waves of the largest amplitude that are exact solutions of the governing equations for water waves [11, 19, 51]. These peakons are shown to be stable [22, 23, 39]. It is worth mentioning that recently it was point out by Lakshmanan [38] that the Camassa-Holm equation could be relevant to the modeling of tsunami waves (see also the discussion in Constantin and Johnson [15] and Segur [46]).

Another remarkable property of the CH equation is the presence of breaking waves (see Figure 1.2. i.e., the solution remains bounded while its slope becomes unbounded in finite time [7, 12, 13, 14, 17, 43, 52]). In [3] and [4], the authors show that the solutions can be uniquely continued after breaking as either global conservative or global dissipative weak solution. It is noted that the KdV equation does not have wave-breaking phenomena [37, 48]. Wave breaking is one of the most intriguing long-standing problems of water wave theory [52]. As mentioned by Whitham [52], it is intriguing to know which mathematical models for shallow water waves exhibit both phenomena of soliton interaction and wave breaking. It is found that the CH equation could be the first such equation and has the potential to become the new master equation for shallow water wave theory, modeling the soliton interaction of peaked traveling waves, wave breaking, admitting solutions as permanent waves, and being integrable Hammiltonian systems.

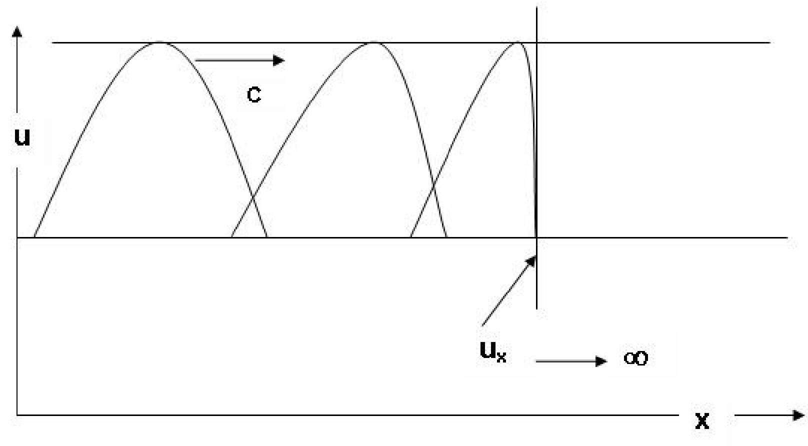

Figure 1.2. An example of wave breaking with velocity $c > 0$.

1.2.2 The Two-component Camassa-Holm System

The Camassa-Holm equation also admits many integrable multi-component generalizations. The most popular one is

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0, \quad m = u - u_{xx}, \\ \rho_t + (u\rho)_x = 0. \end{cases} \tag{1.2}$$

Notice that the CH equation can be obtained via the obvious reduction $\rho \equiv 0$ and $A = 0$. System (1.2) was derived in 1996 [45] (also see [47]). Recently, Constantin-Ivanov [18] and Ivanov [33] established a rigorous justification of the derivation of system (1.2). Mathematical properties of the system have been also studied further in many works, for example [8, 26, 29, 30, 41]. Chen, Liu, and Zhang [8] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher, Lechtenfeld, and Yin [26] investigated local well-posedness for the two-component Camassa-Holm system with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$ by applying Kato's theory [36] and provided some precise blow-up scenarios for strong solutions to the system. The local well-posedness is improved by Gui and Liu [30] to the Besov spaces (especially in the Sobolev space $H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ with $s>3/2$), and they showed that the finite time blowup is determined by either the slope of the first component u or the slope of the second component ρ. The blow-up criterion is made more precise in [41] where the authors showed that the wave breaking in finite time only depends on the slope u. In other words, the wave breaking in u must occur before that in ρ. This blow-up criterion is further improved in [29] to the lowest Sobolev space $H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ with $s>3/2$.

1.2.3 The Dullin-Gottwald-Holm Equation

In 2001, Dullin, Gottwald and Holm [25] studied the following 1+1 quadratically nonlinear equation

$$m_t + c_0 u_x + u m_x + 2m u_x + \gamma u_{xxx} = 0, \quad x\in\mathbb{R}, \quad t>0, \tag{1.3}$$

where $m = u - \alpha^2 u_{xx}$ is a momentum variable. This equation was derived using asymptotic expansions directly in the Hamiltonian for Euler's equation in the shallow water regime, and it is completely integrable with a bi-Hamiltonian as well as a Lax pair in [25].

Using the notation $m = u - \alpha^2 u_{xx}$, Eq.(1.3) can be written as

$$u_t - \alpha^2 u_{txx} + 2\omega_x + 3u_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + u u_{xxx}), \quad x\in\mathbb{R}, \quad t>0, \tag{1.4}$$

where ω and α are two positive constants. Formally, when $\alpha^2=0$, Eq.(1.4) becomes the Korteweg-de Vries (KdV) equation

$$u_t + 2\omega u_x + 3u u_x + \gamma u_{xxx} \quad x\in\mathbb{R}, \quad t>0.$$

While when $\gamma=0$, Eq.(1.3) turns into the Camassa-Holm equation [7, 21, 27]

$$u_t + 2\omega_x - \alpha^2 u_{txx} + 3u_x = \alpha^2(2u_x u_{xx} + u u_{xxx}), \quad x\in\mathbb{R}, \quad t>0. \tag{1.5}$$

Recently, many papers were devoted to the study of the Dullin-Gottwald-Holm (DGH) equation. Gui [49] studied the well-posedness of the Cauchy problem and the scattering problem for the DGH equation. Moreover, the issue of passing to the limit as the dispersive parameter tends to zero for the solution of DGH equation was investigated, and the scattering data of the scattering problem for the equation were explicitly expressed in [49]. And in [54], Yin investigated the local well-posedness, global existence and some blow-up phenomena for the DGH eqaution. Octavian G. Mustafa [44] investigated the low regularity conditions need for the Cauchy problem of the DGH equation via the semigroup approach of quasilinear hyperbolic equations of evolution and the viscosity method. Li and Olver [40] studied the well-posedness, blow-up and the low regular solutions for an integrable nonlinearly dispersive model wave equation. In [20], Adrian Constantin and Jonathan Lenells presented a simple algorithm for the inverse scattering approach to the Camassa-Holm equation. Y. Liu [42] investigated the problems of the existence of global solutions and the formation of singularites for the DGH equation. And the second author et al. [50] studied the limit behavior of the solution to a class of nonlinear dispersive wave equations, which can be seen as some extension of DGH equation. More recently, Christov and Hakkaev [10] computed the Poisson brackets for the scattering data of the DGH equation, and then, the action-angle variables were expressed in terms of the scattering data.

CHAPTER 2

GENERALIZED PERIODIC TWO-COMPONENT CH SYSTEM

2.1 Introduction

In this section, we are concerned with the Cauchy problem of the generalized periodic two-component Camassa-Holm system

$$\begin{cases} m_t - Au_x + \sigma(2mu_x + um_x) + 3(1-\sigma)uu_x + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0,\ x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $m = u - u_{xx}$, or equivalently, in terms of u and ρ,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0,\ x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The generalized two-component Camassa-Holm system was recently derived in [9], following Ivanov's modeling approach [33]. Here $u(t,x)$ describes the horizontal velocity of the fluid, $\rho(t,x)$ is in connection with the horizontal deviation of the surface from equilibrium, and $A \geq 0$ characterizes a linear underlying shear flow, all measured in dimensionless units [9]. We see the appearance of a new free parameter σ. When $\sigma = 1$, it recovers the standard two-component Camassa-Holm system [45].

Notation. Throughout this chapter, we identity all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. The norm of the Lebesgue space $L^p(\mathbb{S}), 1 \le p \le \infty$, is denoted by $\|\cdot\|_{L^p}$ and the Sobolev space $H^s(\mathbb{S}), s \in \mathbb{R}$, by $\|\cdot\|_{H^s}$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity.

2.2 Preliminaries

In this section, we briefly give the needed results to pursue our goal. We first present the local well-posedness for the Cauchy problem of system (2.2) in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$.

Denote the Fourier transform of a function f in the torus $\mathbb{S}$ by $\hat{f}(k)$ with the frequency $k \in \mathbb{Z}$. Then we have $(\widehat{(1-\partial_x^2)^{-1}}f)(k) = (1+k^2)^{-1}\widehat{f}(k) = \hat{G}\cdot\hat{f} = (\widehat{G * f})(k)$, where $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, x \in \mathbb{R}$, $[x]$ stands for the integer part of $x \in \mathbb{R}$, and $\hat{G}(k) = (1+k^2)^{-1}$. Hence $(1-\partial_x^2)^{-1}f = G * f = \int_{\mathbb{S}} G(x-y)f(y)\,dy$ for all $f \in L^2(\mathbb{S})$ and $G * m = u$. Our system (2.2) can be written in the following "transport" type

$$\begin{cases} u_t + \sigma u u_x = -\partial_x G * (-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2), & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \ge 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \ge 0,\ x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Applying the transport equation theory combined with the method of the Besov spaces, one may follow the similar argument as in [30] to obtain the following local well-posedness result for the system (2.2).

Theorem 2.2.1. *If* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *then there exists a maximal time* $T = T(\|(u_0, \rho_0)\|_{H^s \times H^{s-1}}) > 0$ *and a unique solution* (u, ρ) *of (2.2) in* $C([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \bigcap C^1([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$ *with* $(u, \rho)|_{t=0} = (u_0, \rho_0)$. *Moreover, the solution depends continuously on the initial data, and* T *is independent of* s.

Now, we consider the following two associated Lagrangian scales of the generalized two component system (2.2)

$$\begin{cases} \frac{\partial q_1}{\partial t} = u(t, q_1), & 0 < t < T, \\ q_1(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \frac{\partial q_2}{\partial t} = \sigma u(t, q_2), & 0 < t < T, \\ q_2(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.5}$$

where $u \in C^1([0, T), H^{s-1})$ is the first component of the solution (u, ρ) to (2.2). Notice that when $\sigma = 1$, the two characteristics $q_1(t, x)$ and $q_2(t, x)$ are the same.

Lemma 2.2.1. *[9, 18, 26] Let* (u, ρ) *be the solution of system (2.2) with initial data* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *and* T *the maximal time of existence. Then Eq.(2.4) has a unique solution* $q_1 \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$, *and Eq.(2.5) has a unique solution* $q_2 \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. *These two solutions satisfy* $q_i(t, x+1) = q_i(t, x)+1, i = 1, 2$. *Moreover, the map* $q_1(t, \cdot)$ *and* $q_2(t, \cdot)$ *are increasing diffeomorphisms of* $\mathbb{R}$ *with*

$$q_{1,x}(t, x) = exp\left(\int_0^t u_x(\tau, q_1(\tau, x))\right) d\tau > 0, \quad (t, x) \in [0, T) \times \mathbb{R},$$

and

$$q_{2,x}(t, x) = exp\left(\int_0^t \sigma u_x(\tau, q_2(\tau, x))\right) d\tau > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

The above Lemmas indicate that $q_1(t,\cdot):\mathbb{R}\to\mathbb{R}$ and $q_2(t,\cdot):\mathbb{R}\to\mathbb{R}$ are diffeomorphisms of the line for each $t\in[0,T)$. Hence, the L^∞ norm of any function $v(t,\cdot)\in L^\infty(\mathbb{S}), T\in[0,t)$ is preserved under the family of diffeomorphisms $q_1(t,\cdot)$ and $q_2(t,\cdot)$ with $t\in[0,T)$, that is

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{S})}=\|v(t,q_1(t,\cdot))\|_{L^\infty(\mathbb{S})}=\|v(t,q_2(t,\cdot))\|_{L^\infty(\mathbb{S})},\quad t\in[0,T). \tag{2.6}$$

Similarly, we have

$$\inf_{x\in\mathbb{S}}v(t,x)=\inf_{x\in\mathbb{S}}v(t,q_1(t,x))=\inf_{x\in\mathbb{S}}v(t,q_2(t,x)),\quad t\in[0,T), \tag{2.7}$$

$$\sup_{x\in\mathbb{S}}v(t,x)=\sup_{x\in\mathbb{S}}v(t,q_1(t,x))=\sup_{x\in\mathbb{S}}v(t,q_2(t,x)),\quad t\in[0,T). \tag{2.8}$$

Lemma 2.2.2. *[26] Let (u,ρ) be the solution of system (2.2) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s>3/2$, and T the maximal time of existence. Then we have*

$$\rho(t,q(t,x))q_x(t,x)=\rho_0(x),\quad (t,x)\in[0,T)\times\mathbb{R}. \tag{2.9}$$

Moreover if there exists $x_0\in\mathbb{S}$ such that $\rho_0(x_0)=0$, then $\rho(t,q(t,x_0))=0$ for all $t\in[0,T)$.

We may use the following proposition derived in [29] to study the regularity property of solution to (2.2).

Proposition 2.2.1. *Let $0<s<1$. Suppose that $f_0\in H^s, g\in L^1([0,T];H^s)$, and $v,v_x\in L^1([0,T];L^\infty)$ and that $f\in L^\infty([0,T];H^s)\bigcap C([0,T];S')$ solves the one-dimensional linear transport equation*

$$\begin{cases} f_t+vf_x=g,\\ f(0,x)=f_0(x). \end{cases} \tag{2.10}$$

Then $f\in C([0,T];H^s)$. More precisely, there exists a constant C depending only on s such that the following estimate holds:

$$\|f(t)\|_{H^s}\leq\|f_0\|_{H^s}+C\Big(\int_0^t\|g(\tau)\|_{H^s}d\tau+\int_0^t\|f(\tau)\|_{H^s}V'(\tau)d\tau\Big). \tag{2.11}$$

Hence,

$$\|f(t)\|_{H^s} \leq e^{CV(t)}\Big(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau\Big), \tag{2.12}$$

where $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}) d\tau$.

The above proposition was proved using the Littlewood-Palay analysis for the transport equation and the Moser-type estimates. Using this result and performing the same argument as in [29], we can obtain the following blow-up criterion.

Theorem 2.2.2. *Let* (u, ρ) *be the solution of system (2.2) with initial data* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *and* T *the maximal time of existence. Then*

$$T < \infty \Rightarrow \int_0^T \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{2.13}$$

We then give several useful conservation laws of strong solutions to (2.2)

Lemma 2.2.3. *Let* (u, ρ) *be the solution of system (2.2) with initial data* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *and* T *the maximal time of existence. Then for all* $t \in [0, T)$, *we have*

$$\int_{\mathbb{S}} u(t,x)dx = \int_{\mathbb{S}} u_0(x)dx,$$
$$\int_{\mathbb{S}} \rho(t,x)dx = \int_{\mathbb{S}} \rho_0(x)dx.$$

Proof. Integrating the first equation of (2.3) by parts, in view of the periodicity of u and G, we get

$$\frac{d}{dt}\int_{\mathbb{S}} udx = -\int_{\mathbb{S}} \sigma u u_x dx - \int_{\mathbb{S}} \partial_x G * (-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2)dx = 0.$$

On the other hand, integrating the second equation of (2.3) by parts, in view of the periodicity of u and ρ, we get

$$\frac{d}{dt}\int_{\mathbb{S}} \rho dx = -\int_{\mathbb{S}} (u\rho)_x dx = 0.$$

This completes the proof of the lemma. □

Lemma 2.2.4. *Let (u,ρ) be the solution of system (2.2) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then for all $t \in [0,T)$, we have*

$$\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x) + \rho^2(t,x))dx = \int_{\mathbb{S}} (u_0^2(t,x) + u_{0x}^2(t,x) + \rho_0^2(t,x))dx.$$

Proof. Multiplying the first equation of (2.2) by $2u$ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x))dx = \frac{d}{dt}\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)dx.$$

Multiplying the second equation of (2.2) by 2ρ and integrating by parts, we get

$$\frac{d}{dt}\int_{\mathbb{S}} \rho^2(t,x) = -\frac{d}{dt}\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)dx.$$

Adding the above two equalities, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x) + \rho^2(t,x))dx = 0.$$

This completes the proof of the lemma. □

Lemma 2.2.5. *[16] Let $T > 0$ and $v \in C^1([0,T); H^2(\mathbb{R}))$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}} [v_x(t,x)] = v_x(t,\xi(t)).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t,\xi(t)) \quad a.e. \quad on \quad (0,T).$$

Lemma 2.2.6. *[53] For every $f \in H^1(\mathbb{S})$, we have*

$$\max_{x\in[0,1]} f^2(x) \leq \frac{e+1}{2(e-1)}\|f\|^2_{H^1(\mathbb{S})},$$

where the constant $\frac{e+1}{2(e-1)}$ is sharp.

By the conservation law stated in Lemma 2.2.4 and Lemma 2.2.6, we have the following corollary.

Corollary 2.2.1. *Let (u,ρ) be the solution of system (2.3) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then for all $t \in [0,T)$, we have*

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|u(t,\cdot)\|^2_{H^1(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}.$$

Lemma 2.2.7. *[31] If $f \in H^3(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)dx = \frac{a_0}{2}$, then for every $\epsilon > 0$, we have*

$$\max_{x\in\mathbb{S}} f^2(x) \leq \frac{\epsilon+2}{24}\int_{\mathbb{S}} f_x^2 dx + \frac{\epsilon+2}{4\epsilon}a_0^2.$$

2.3 Wave-breaking Phenomenon

In this section, we investigate the wave-breaking phenomena of strong solution to system (2.3). First, we give the wave-breaking criterion for $\sigma \neq 0$.

Theorem 2.3.1 (Waving-breaking criterion)**.** *Let $\sigma \neq 0$ and (u,ρ) be the solution of (2.2) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence, then the solution blows up in finite time if and only if*

$$\liminf_{t\to T^-}\{\inf_{x\in\mathbb{S}} \sigma u_x(t,x)\} = -\infty. \tag{2.14}$$

To prove this wave-breaking criterion, we use the following lemma to show that indeed σu_x is uniformly bounded from above.

Lemma 2.3.1. *Let $\sigma \neq 0$ and (u,ρ) be the solution of (2.2) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then*

(1) If $\sigma > 0$, then

$$\sup_{x\in\mathbb{S}} u_x(t,x) \leq \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty} + C_1^2}{\sigma}}. \tag{2.15}$$

(2) If $\sigma < 0$, then

$$\inf_{x\in\mathbb{S}} u_x(t,x) \geq -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}}. \tag{2.16}$$

The constants above are defined as follows.

$$\begin{aligned}
C_0 =& \|(u_0,\rho_0)\|^2_{H^1\times L^2},\\
C_1 =& C_0\sqrt{\frac{(-1+\sinh 1)A^2}{4\sinh^2(1/2)} + \Big[\frac{|3-\sigma|(e+1)}{2(e-1)} + \frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{2\sinh(1/2)} + \frac{1}{2}\Big]},\\
C_2 =& C_0\sqrt{\frac{(-1+\sinh 1)A^2}{4\sinh^2(1/2)} + \frac{\cosh(1/2)(4-\sigma)}{2\sinh(1/2)}}, \quad for \quad \sigma<0.
\end{aligned} \tag{2.17}$$

Proof. The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Also, we assume that $u_0 \not\equiv 0$. Otherwise, the results become trivial. Note that if $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$ and $G * m = u$. Hence, we can rewrite the first equation in (2.3) as

$$u_t + \sigma u u_x = -\partial_x G * \left(-Au + \frac{3-\sigma}{2} + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2\right). \tag{2.18}$$

Differentiating the above with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$, we obtain

$$u_{tx} + \sigma u u_{xx} + \frac{\sigma}{2}u_x^2 = A\partial_x^2 G * u + \frac{1}{2}\rho^2 + \frac{3-\sigma}{2}u^2 - G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2). \tag{2.19}$$

(1) When $\sigma > 0$, using Lemma 2.2.5 and the fact that

$$\sup_{x\in\mathbb{S}}[v_x(t,x)] = -\inf_{x\in\mathbb{S}}[-v_x(t,x)],$$

we can consider $\bar{m}(t)$ and $\eta(t)$ as follows:

$$\eta(t) \in \mathbb{S} \quad and \quad \bar{m}(t) := u_x(t,\eta(t)) = \sup_{x\in\mathbb{S}}(u_x(t,x)), \quad t \in [0,T). \tag{2.20}$$

Hence,

$$u_{xx}(t,\eta(t)) = 0, \quad a.e. \quad t \in [0,T). \tag{2.21}$$

Take the trajectory $q_1(t,x)$ defined in (2.4). Then we know that $q_1(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for every $t \in [0,T)$. Therefore, there exists $x_1(t) \in \mathbb{R}$ such that

$$q_1(t,x_1(t)) = \eta(t), \quad t \in [0,T). \tag{2.22}$$

Now, let

$$\bar{\xi} = \rho(t,q_1(t,x_1)), \quad t \in [0,T). \tag{2.23}$$

Therefore, along the trajectory $q_1(t,x_1)$, equation (2.19) and the second equation of (2.2) become

$$\begin{aligned} \bar{m}'(t) &= -\frac{\sigma}{2}\bar{m}^2 + \frac{1}{2}\bar{\xi}^2 + f(t,q_1(t,x_1)), \\ \bar{\xi}'(t) &= -\bar{\xi}\bar{m}, \end{aligned} \tag{2.24}$$

for $t \in [0,T)$, where $'$ denotes the derivative with respect to t and $f(t,q_1(t,x_1))$ is given by

$$f = A\partial_x^2 G * u + \frac{3-\sigma}{2}u^2 - G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2). \tag{2.25}$$

We first derive the upper and lower bounds for f for later use in getting the wave-breaking result. Using that $\partial_x^2 G * u = \partial G * \partial_x u$, we have

$$\begin{aligned} f(t,x) &= \frac{3-\sigma}{2}u^2 + A\partial_x G * \partial_x u - G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2) - \frac{1}{2}G * \rho^2 \\ &\le \frac{3-\sigma}{2}u^2 + A|G_x * u_x| + |G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2)|, \end{aligned}$$

for any $x \in \mathbb{S}$ and $t \in [0,T)$. Applying Young's inequality and $G = \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, leads to

$$\begin{aligned} A|G_x * u_x| &\le A\|G_x\|_{L^2}\|u_x\|_{L^2} = A\frac{\sqrt{\frac{1}{2}(-1+\sinh 1)}}{2\sinh(\frac{1}{2})}\|u_x\|_{L^2} \\ &\le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|_{L^2}^2, \end{aligned} \tag{2.26}$$

$$
\begin{aligned}
|G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)| &\le \|G\|_{L^\infty}\|\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\|_{L^1} \\
&= \frac{\cosh(1/2)}{2\sinh(1/2)}\|\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\|_{L^1} \\
&\le \frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2,
\end{aligned} \tag{2.27}
$$

and

$$
\|u(t,\cdot)\|_{L^\infty(\mathbb{S})}^2 \le \frac{e+1}{2(e-1)}\|u(t,\cdot)\|_{H^1(\mathbb{S})}^2 \le \frac{e+1}{2(e-1)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2. \tag{2.28}
$$

Therefore, we obtain the upper bound of f for any $x\in\mathbb{S}$ and $t\in[0,T)$,

$$
\begin{aligned}
f(t,x) &\le \frac{|3-\sigma|}{2}\|u\|_{L^\infty}^2+\frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2 \\
&\quad+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2 \\
&\le \frac{|3-\sigma|(e+1)}{4(e-1)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2+\frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2 \\
&\quad+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2 \\
&\le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\Big[\frac{|3-\sigma|(e+1)}{4(e-1)} \\
&\quad+\frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{4\sinh(1/2)}+\frac{1}{4}\Big]\|(u_0,\rho_0)\|_{H^1\times L^2}^2 \\
&= \frac{1}{2}C_1^2.
\end{aligned} \tag{2.29}
$$

Now, we turn to the lower bound of f. Similar as before, we get

$$
\begin{aligned}
-f &\le \frac{|3-\sigma|}{2}\|u\|_{L^\infty}^2+A|G_x*u_x|+|G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)|+\frac{1}{2}G*\rho^2 \\
&\le \frac{|3-\sigma|(e+1)}{4(e-1)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2+\frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2 \\
&\quad+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2+\frac{\cosh(1/2)}{4\sinh(1/2)}\|\rho\|_{L^2}^2 \\
&\le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\Big[\frac{|3-\sigma|(e+1)}{4(e-1)} \\
&\quad+\frac{\cosh(1/2)(|3-\sigma|+|\sigma|+1)}{4\sinh(1/2)}+\frac{1}{4}\Big]\|(u_0,\rho_0)\|_{H^1\times L^2}^2.
\end{aligned} \tag{2.30}
$$

When $\sigma < 0$, we have a finer estimate

$$
\begin{aligned}
-f &\leq A|G_x * u_x| + |G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2)| + \frac{1}{2}G * \rho^2 \\
&\leq \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|_{L^2}^2 + \frac{\cosh(1/2)(3-\sigma)}{4\sinh(1/2)}\|u\|_{L^2}^2 \\
&\quad - \frac{\cosh(1/2)\sigma}{4\sinh(1/2)}\|u_x\|_{L^2}^2 + \frac{\cosh(1/2)}{4\sinh(1/2)}\|\rho\|_{L^2}^2 \\
&\leq \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{\cosh(1/2)(4-\sigma)}{4\sinh(1/2)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2 \\
&= \frac{1}{2}C_2^2.
\end{aligned} \tag{2.31}
$$

Combining (2.29) and (2.30), we obtain

$$
\begin{aligned}
|f| \leq \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} &+ \Big[\frac{|3-\sigma|(e+1)}{4(e-1)} \\
&+ \frac{\cosh(1/2)(|3-\sigma|+|\sigma|+1)}{4\sinh(1/2)} + \frac{1}{4}\Big]\|(u_0,\rho_0)\|_{H^1\times L^2}^2.
\end{aligned} \tag{2.32}
$$

Since now $s \geq 3$, we have $u \in C_0^1(\mathbb{S})$. Therefore,

$$
\inf_{x\in\mathbb{S}} u_x(t,x) \leq 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) \geq 0, \quad t \in [0,T). \tag{2.33}
$$

Hence, $\bar{m}(t) > 0$ for $t \in [0,T)$. From the second equation of (2.24), we obtain that

$$
\bar{\xi}(t) = \bar{\xi}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}. \tag{2.34}
$$

Hence,

$$
|\rho(t,q_1(t,x_1))| = |\bar{\xi}(t)| \leq |\bar{\xi}(0)| \leq \|\rho_0\|_{L^\infty}.
$$

Now define

$$
P_1(t) = \bar{m}(t) - \|u_{0,x}\|_{L^\infty} - \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}.
$$

Note that $P_1(t)$ is a C^1- differentiable function in $[0,T)$ and satisfies

$$
P_1(0) \leq \bar{m}(0) - \|u_0,x\|_{L^\infty} \leq 0.
$$

We will show that

$$
P_1(t) \leq 0, \qquad t \in [0,T). \tag{2.35}
$$

If not, then suppose there is a $t_0 \in [0, T)$ such that $P_1(t_0) > 0$. Define

$$t_1 = \max\{t < t_0 : P_1(t) = 0\}.$$

Then $P_1(t_1) = 0$ and $P_1' \geq 0$, or equivalently,

$$\begin{aligned}\bar{m}(t_1) =& \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty} + C_1^2}{\sigma}}, \\ \bar{m}'(t_1) =& \geq 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\bar{m}'(t_1) &= -\frac{\sigma}{2}\bar{m}^2(t_1) + \frac{1}{2}\bar{\xi}^2(t_1) + f(t_1, q(t_1, x)) \\ &\leq -\frac{\sigma}{2}\left[\|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty} + C_1^2}{\sigma}}\right]^2 + \frac{1}{2}\|\rho_0\|^2_{L^\infty} + \frac{1}{2}C_1^2 \\ &< 0,\end{aligned}$$

which is a contradiction. Therefore, $P_1(t) \leq 0$, for $t \in [0, T)$, and we obtain (2.15).

(2) To derive a lower bound for u_x in the case of $\sigma < 0$, we consider the functions $m(t)$ and $\xi(t) \in \mathbb{S}$ as in Lemma 2.2.5

$$m(t) := u_x(t, \xi(t)) = \inf_{x \in \mathbb{S}}(u_x(t, x)), \quad t \in [0, T). \tag{2.36}$$

Hence,

$$u_{xx}(t, \xi(t)) = 0, \quad a.e. \quad t \in [0, T). \tag{2.37}$$

Similar as before, we take the characteristic $q_1(t, x)$ defined in (2.4) and choose $x_2(t) \in \mathbb{R}$ such that

$$q_1(t, x_2(t)) = \xi(t) \quad t \in [0, T). \tag{2.38}$$

Let

$$\zeta = \rho(t, q_1(t, x_2)), \quad t \in [0, T). \tag{2.39}$$

Hence, along the trajectory $q_1(t, x_2)$, equation (2.19) and the second equation of (2.2) become

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2}m^2 + \frac{1}{2}\zeta^2 + f(t, q_1(t, x_2)), \\ \zeta'(t) &= -\zeta m. \end{aligned} \tag{2.40}$$

We now define

$$P_2(t) = m(t) - \|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}}.$$

Then $P_2(t)$ is also C^1- differentiable in $[0, T)$ and satisfies

$$P_2(0) \geq m(0) + \|u_{0,x}\|_{L^\infty} \geq 0.$$

We now claim that

$$P_2(t) \geq 0, \qquad t \in [0, T). \tag{2.41}$$

If not, then suppose there is a $\bar{t}_0 \in [0, T)$ such that $P_2(\bar{t}_0) < 0$. Define

$$t_2 = \max\{t < \bar{t}_0 : P_2(t) = 0\}.$$

Then $P_2(t_2) = 0$ and $P_2'(t_2) \leq 0$, or equivalently,

$$m(t_2) = -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}} \quad \textit{and} \quad m'(t_2) \leq 0.$$

On the other hand, we have

$$\begin{aligned} m'(t_2) &= -\frac{\sigma}{2}m^2(t_2) + \frac{1}{2}\zeta^2(t_2) + f(t_2, q(t_2, x)) \\ &\geq -\frac{\sigma}{2}(\|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}})^2 - \frac{1}{2}C_2^2 \\ &> 0. \end{aligned}$$

Again, this is a contradiction. Therefore, $P_2(t) \geq 0$, for $t \in [0, T)$. This in turn implies that (2.16) holds. This completes the proof of Lemma 2.3.1. □

It is found that if σu_x is bounded from below, we may obtain the following estimates for $\|\rho\|_{L^\infty(\mathbb{S})}$.

Proposition 2.3.1. *Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.2) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. If there is an $M \geq 0$, such that*

$$\inf_{(t,x)\in[0,T)\times\mathbb{S}} \sigma u_x \geq -M, \tag{2.42}$$

then

1. *If $\sigma > 0$, then*

$$\|\rho(t,\cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{Mt/\sigma}. \tag{2.43}$$

2. *If $\sigma < 0$, then*

$$\|\rho(t,\cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{Nt}. \tag{2.44}$$

Where $N = \|u_{0,x}\|_{L^\infty} + (C_2/\sqrt{-\sigma})$ and C_2 is given in (2.17).

Proof. (1) For $\sigma > 0$, we define for any give $x \in \mathbb{S}$

$$U(t) = u_x(t, q_1(t,x)), \quad \gamma(t) = \rho(t, q_1(t,x)), \tag{2.45}$$

with $q_1(t, x_1(t)) = x$, for some $x_1(t) \in \mathbb{R}, t \in [0,T)$. Then the ρ equation of system (2.2) becomes

$$\gamma' = -\gamma U. \tag{2.46}$$

Thus,

$$\gamma(t) = \gamma(0) e^{-\int_0^t U(\tau)d\tau}. \tag{2.47}$$

From the assumption (2.42) and $\sigma > 0$, we see

$$U(t) \geq -\frac{M}{\sigma}, \quad t \in [0,T).$$

Hence,

$$|\rho(t, q_1(t,x_1))| = |\gamma(t)| \leq |\gamma(0)| e^{-\int_0^t U(\tau)d\tau} \leq \|\rho_0\|_{L^\infty} e^{Mt/\sigma},$$

which together with (2.6), leads to (2.43).

(2) For $\sigma < 0$, we perform a similar argument as before. Using (2.45), (2.47) and the lower bound (2.16), we have

$$|\rho(t, q_1(t, x_1))| = |\gamma(t)| \leq |\gamma(0)| e^{-\int_0^t U(\tau) d\tau} \leq \|\rho_0\|_{L^\infty} e^{Nt}.$$

Combining the above estimate with (2.6), which implies that (2.44) holds. □

Proof of Theorem 2.3.1. Assume that $T < \infty$ and (2.14) is not valid. Then there is some positive number $M > 0$ such that

$$\sigma u_x(t, x) \geq -M, \quad \forall (t, x) \in [0, T) \times \mathbb{S}.$$

It follows from Lemma 2.3.1 that $|u_x(t, x)| \leq C$, where
$C = C(A, M, \sigma, |(u_0, \rho_0)\|^2_{H^s \times H^{s-1}})$. Therefore, Theorem 2.2.2 in turn implies that the maximal existence time $T = \infty$, which contradicts the assumption that $T < \infty$. Conversely, the Sobolev embedding theorem $H^s(\mathbb{S}) \hookrightarrow L^\infty(\mathbb{S})$ with $s > 1/2$ implies that if (2.14) holds, the corresponding solution blows up in finite time. This completes the proof of Theorem 2.3.1. □

Now, we give the following series of theorems that provide some cases that wave breaks in finite time.

Theorem 2.3.2. *Let $\sigma \neq 0$ and (u, ρ) be the solution of (2.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence.*

1. *When $\sigma > 0$, assume that there is some $x_0 \in \mathbb{S}$ such that*

$$\rho_0(x_0) = 0, \quad u_{0,x}(x_0) = \inf_{x \in \mathbb{S}} u_{0,x}(x),$$

and

$$u_{0,x}(x_0) < -\frac{C_1}{\sqrt{\sigma}}, \tag{2.48}$$

where C_1 is defined in (2.17). Then the corresponding solution to system (2.2) blows up in the following sense: there exists a T_1 with

$$0 < T_1 \le -\frac{2}{\sigma u_{0,x}(x_0) + \sqrt{-\sigma^{3/2} C_1 u_{0,x}(x_0)}}, \tag{2.49}$$

respectively, such that

$$\liminf_{t \to T_1^-} \{\inf_{x \in \mathbb{S}} \sigma u_x(t,x)\} = -\infty.$$

2. *When $\sigma < 0$, assume that there are some $x_0 \in \mathbb{S}$ such that*

$$u_{0,x}(x_0) > -\frac{C_2}{\sqrt{\sigma}}, \tag{2.50}$$

where C_2 is defined in (2.17). Then the corresponding solution to the system (2.2) blows up in finite time in the following sense: there exists a T_2 with

$$0 < T_2 \le -\frac{2}{\sigma u_{0,x}(x_0) - \sqrt{(-\sigma)^{3/2} C_2 u_{0,x}(x_0)}}, \tag{2.51}$$

such that

$$\liminf_{t \to T_2^-} \{\sup_{x \in \mathbb{S}} \sigma u_x(t,x)\} = \infty.$$

Proof. (1) When $\sigma > 0$, similar to the proof of Lemma 2.3.1, it suffices to consider $s \ge 3$. So in the following of this section $s = 3$ is taken for simplicity of notation. We consider along the trajectory $q_1(t, x_2)$ defined in (2.4) and (2.38). In this way, we can write the transport equation of ρ in (2.2) along the trajectory of $q_1(t, x_2)$ as

$$\frac{d\rho(t,\xi(t))}{dt} = -\rho(t,\xi(t)) u_x(t,\xi(t)). \tag{2.52}$$

Form the assumption of the theorem, we see

$$m(0) = u_x(0,\xi(0)) = \inf_{x \in \mathbb{S}} u_{0,x}(x) = u_{0,x}(x_0).$$

Hence, we can choose $\xi(0) = x_0$ and then $\rho_0(\xi(0) = \rho_0(x_0) = 0$. Thus, from (2.52) we see that

$$\rho(t, \xi(t)) = 0, \qquad t \in [0, T). \tag{2.53}$$

Differentiating equation (2.18) with respect to x, evaluating the result at $x = \xi(t)$ and using (2.37) and (2.53), we deduce from Lemma 2.2.5 that

$$m'(t) = -\frac{\sigma}{2} m^2(t) + f(t, q_1(t, x_2)). \tag{2.54}$$

Using the upper bound of f in (2.29), we see that

$$m'(t) \leq -\frac{\sigma}{2} m^2(t) + \frac{1}{2} C_1^2, \quad t \in [0, T).$$

By assumption (2.48), $m(0) = u_{0,x}(x_0) < -C_1/\sqrt{\sigma}$, we see that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2} \frac{C_1^2}{u_{0,x}^2(x_0)\sigma} \in (0, \frac{1}{2}).$$

Using that $m(t) < m(0) = u_{0,x}(x_0) < 0$, we obtain

$$m'(t) \leq -\frac{\sigma}{2} m^2(t) + \frac{1}{2} C_1^2 \leq -\delta\sigma m^2(t), \quad t \in [0, T).$$

Integrating on both sides, we obtain

$$m(t) \leq \frac{u_{0,x}(x_0)}{1 + \delta\sigma u_{0,x}(x_0) t} \to -\infty \quad as \quad t \to -\frac{1}{\delta\sigma u_{0,x}(x_0)}.$$

Hence,

$$T \leq -\frac{1}{\delta\sigma u_{0,x}(x_0)},$$

which proves (2.49).

(2) Similarly as in (1), we consider the function $\bar{m}(t)$and $\eta(t)$ as defined in (2.20). Then we have

$$\bar{m}'(t) = -\frac{\sigma}{2} \bar{m}^2(t) + \frac{1}{2} \rho^2(t, \eta(t)) + f(t, q_1(t, x_1)) \geq -\frac{\sigma}{2} \bar{m}^2(t) + f(t, q_1(t, x_1)). \tag{2.55}$$

Using the lower bound of f as in (2.31), we have

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}C_2^2, \quad t \in [0,T).$$

By assumption (2.50), $\bar{m}(0) = u_{0,x}(x_0) > C_2/\sqrt{-\sigma}$, we see that $\bar{m}'(0) > 0$ and $\bar{m}(t)$ is strictly increasing over $[0,T)$. Set

$$\theta = \frac{1}{2} - \frac{1}{2}\frac{C_2^2}{\sigma u_{0,x}^2(x_0)}.$$

Using that $\bar{m}(t) > \bar{m}(0) = u_{0,x}(x_0) > 0$, we obtain

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}C_2^2 \geq -\theta\sigma\bar{m}^2(t), \quad t \in [0,T).$$

Therefore,

$$\bar{m}(t) \geq \frac{u_{0,x}(x_0)}{1+\theta\sigma u_{0,x}(x_0)t} \to \infty \quad as \quad t \to -\frac{1}{\theta\sigma u_{0,x}(x_0)}.$$

Hence,

$$T \leq -\frac{1}{\theta\sigma u_{0,x}(x_0)},$$

which proves (2.51). □

The following theorem provides another condition for blowup of u_x.

Theorem 2.3.3. *Let $\sigma > 0$ and (u,ρ) be the solution of (2.2) with the initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Assume that $\int_{\mathbb{S}} u_0 dx = \frac{a_0}{2}$. If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$, $u_{0,x}(x_0) = \inf_{x\in\mathbb{S}} u_{0,x}(x)$, and for any $\epsilon > 0$*

$$u_{0,x}(x_0) < -\frac{C_3}{\sqrt{\sigma}}. \tag{2.56}$$

Then the corresponding solution to system (2.2) blows up in the following sense: there exists a T_1 with

$$0 < T \leq -\frac{2}{\sigma u_{0,x}(x_0) + \sqrt{-\sigma^{3/2}C_3 u_{0,x}(x_0)}}, \tag{2.57}$$

respectively, such that

$$\liminf_{t\to T_1^-}\{\inf_{x\in\mathbb{S}}\sigma u_x(t,x)\}=-\infty.$$

The constant above is defined as follows

$$\begin{aligned}C_3=&\Big(\frac{(-1+\sinh 1)A^2}{4\sinh^2(1/2)}+\frac{|3-\sigma|(\epsilon+2)a_0^2}{4\epsilon}\\&+\Big[\frac{|3-\sigma|(\epsilon+2)}{24}+\frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{2\sinh(1/2)}+\frac{1}{2}\Big]\|(u_0,\rho_0)\|^2_{H^1\times\ L^2}\Big)^{\frac{1}{2}}.\end{aligned}$$

Proof. By Lemma 2.2.3, we have $\int_{\mathbb{S}}u(t,x)dx=\int_{\mathbb{S}}u_0(x)dx=\frac{a_0}{2}$. Using Lemma 2.2.7 and the above conservation law, we have

$$\|u\|_{L^\infty(\mathbb{S})}\leq\sqrt{\frac{\epsilon+2}{24}\|(u_0,\rho_0)\|^2_{H^1(\mathbb{S})\times L^2(\mathbb{S})}+\frac{\epsilon+2}{4\epsilon}a_0^2}. \tag{2.58}$$

Similarly as the proof of Theorem 2.3.2(1), we can also get

$$m'(t)=-\frac{\sigma}{2}m^2(t)+f(t,q_1(t,x_2)). \tag{2.59}$$

Using (2.58), we obtain a new upper bound of f

$$\begin{aligned}f=&\frac{3-\sigma}{2}u^2+A\partial_xG*\partial_xu-G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)-\frac{1}{2}G*\rho^2\\
\leq&\frac{3-\sigma}{2}u^2+A|G_x*u_x|+|G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)|\\
\leq&\frac{|3-\sigma|}{2}\|u\|^2_{L^\infty}+\frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|^2_{L^2}\\
&+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|^2_{L^2}+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|^2_{L^2}\\
\leq&\frac{|3-\sigma|(\epsilon+2)}{48}\|(u_0,\rho_0)\|^2_{H^1\times L^2}+\frac{|3-\sigma|(\epsilon+2)a_0^2}{8\epsilon}+\frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}\\
&+\frac{1}{4}\|u_x\|^2_{L^2}+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|^2_{L^2}+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|^2_{L^2}\\
\leq&\frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)}+\frac{|3-\sigma|(\epsilon+2)a_0^2}{8\epsilon}\\
&+\Big[\frac{|3-\sigma|(\epsilon+2)}{48}+\frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{4\sinh(1/2)}+\frac{1}{4}\Big]\|(u_0,\rho_0)\|^2_{H^1\times\ L^2}\\
=&\frac{1}{2}C_3^2.\end{aligned} \tag{2.60}$$

By assumption (2.56), $m(0) = u_{0,x}(x_0) < -C_3/\sqrt{\sigma}$, we see that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2}\frac{C_3^2}{u_{0,x}(x_0)^2\sigma} \in \left(0, \frac{1}{2}\right).$$

Using that $m(t) < m(0) = u_{0,x}(x_0) < 0$, we obtain

$$m'(t) \leq -\frac{\sigma}{2}m^2(t) + \frac{1}{2}C_3^2 \leq -\delta\sigma m^2(t), \quad t \in [0, T).$$

Integrating on both sides, we obtain

$$m(t) \leq \frac{u_{0,x}(x_0)}{1 + \delta\sigma u_{0,x}(x_0)t} \to -\infty \quad as \quad t \to -\frac{1}{\delta\sigma u_{0,x}(x_0)}.$$

Hence,

$$T \leq -\frac{1}{\delta\sigma u_{0,x}(x_0)},$$

which proves (2.57). This complies the proof of Theorem 2.3.3. □

Next, we give a blow-up result if u_0 is odd and ρ_0 is even.

Theorem 2.3.4. *Let $0 < \sigma \leq 3$ and (u, ρ) be the solution of (2.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Assume that u_0 is odd, ρ_0 is even, $u_{0,x} < 0$, and $\rho_0(0) = 0$. Then the corresponding solution to the system (2.2) blows up in finite time. More precisely, there exists a T_0 with $0 < T_0 \leq -(2/\sigma u_{0,x}(0))$ such that*

$$\liminf_{t \to T_0^-}\{\inf_{x \in \mathbb{S}} \sigma u_x(t, x)\} = -\infty.$$

Proof. Similar to the proof of Lemma 2.3.1, it suffices to consider $s \geq 3$. Since u_0 is odd and ρ_0 is even, the corresponding solution $(u(t, x), \rho(t, x))$ satisfies that $u(t, x)$ is odd and $\rho(t, x)$ is even with respect to x for given $0 < t < T$. Hence, $u(t, 0) = 0$ and $\rho_x(t, 0) = 0$. Thanks to the transport equation of ρ in (2.2), we have

$$\begin{cases} \rho_t(t, 0) + \rho(t, 0)u_x(t, 0) = 0, \\ \rho(0, 0) = 0. \end{cases}$$

Thus, $\rho(t,0)=0$. Evaluating (2.19) at $(t,0)$ and denoting $M(t)=u_x(t,0)$, we obtain

$$M'(t)+\frac{\sigma}{2}M^2(t)=A(\partial_x^2 G*u)(t,0)-G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2)(t,0). \qquad (2.61)$$

Notice that $u(t,x)$ is odd and $G(x)$ is even, so

$$A(\partial_x^2 G*u)(t,0)=0.$$

Using $0<\sigma\leq 3$,

$$M'(t)+\frac{\sigma}{2}M^2(t)\leq 0.$$

Hence,

$$M(t)\leq M(0)=u_{0,x}(0)<0, \quad for \quad t\in[0,T),$$

and

$$-\frac{1}{M(t)}+\frac{1}{M(0)}\leq -\frac{\sigma}{2}t,$$

and then

$$u_x(t,0)=M(t)\leq \frac{2M(0)}{2+\sigma M(0)t}\to -\infty, \quad t\to -\frac{2}{\sigma M(0)}, \qquad (2.62)$$

which indicates that the maximal existence time $T\leq -2(2/\sigma u_{0,x}(0))$ and hence it completes the proof of the theorem. □

2.4 Blow-up Rate

We now address the question of the blow-up rate of the slope to a breaking wave for system (2.2) .

Theorem 2.4.1. *Let $\sigma\neq 0$. If $T<\infty$ is the blow-up time of the solution (u,ρ) to (2.2) with the initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s>3/2$, satisfying the assumption of Theorem* 2.3.2*, then*

$$\lim_{t\to T^-}\left[\left(\inf_{x\in\mathbb{S}} u_x(t,x)\right)(T-t)\right]=-\frac{2}{\sigma}, \quad for \quad \sigma>0. \qquad (2.63)$$

$$\lim_{t\to T^-}\left[\left(\sup_{x\in\mathbb{S}} u_x(t,x)\right)(T-t)\right]=-\frac{2}{\sigma}, \quad for \quad \sigma<0. \qquad (2.64)$$

Proof. We may again assume $s = 3$ to prove the theorem. Now, let's consider the first case. Let $\sigma > 0$. From (2.54) we have

$$m'(t) = -\frac{\sigma}{2}m^2(t) + f(t, q_1(t, x_2)).$$

Using (2.32) and denote

$$K = \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \Big[\frac{|3-\sigma|(e+1)}{4(e-1)} + \frac{\cosh(\frac{1}{2})(|3-\sigma|+|\sigma|+1)}{4\sinh(\frac{1}{2})} + \frac{1}{4}\Big]\|(u_0,\rho_0)\|^2_{H^1\times L^2}. \tag{2.65}$$

We know

$$-\frac{\sigma}{2}m^2(t) - K \le m'(t) \le -\frac{\sigma}{2}m^2(t) + K. \tag{2.66}$$

Choose $0 < \varepsilon < \sigma/2$. Since $m(t) \to -\infty$ as $t \to T^-$, we can find $t_0 \in (0, T)$ such that

$$m(t_0) < -\sqrt{2\sigma K + \frac{K}{\varepsilon}}.$$

Since $m(t)$ is absolutely continuous on $[0, T)$. It is then inferred from the above differential inequality that $m(t)$ is strictly decreasing on $[t_0, T)$ and hence

$$m(t) < -\sqrt{2\sigma K + \frac{K}{\varepsilon}} < -\sqrt{\frac{K}{\varepsilon}} \qquad t \in [t_0, T).$$

Then (2.66) implies that

$$\frac{\sigma}{2} - \varepsilon < \frac{d}{dt}\Big(\frac{1}{m(t)}\Big) < \frac{\sigma}{2} + \varepsilon, \quad a.e. \quad t \in [t_0, T).$$

Integrating the above relation on (t, T) with $t \in [t_0, T)$ and noticing that $m(t) \to -\infty$ as $t \to T^-$, we obtain

$$(\frac{\sigma}{2} - \varepsilon)(T - t) < -\frac{1}{m(t)} < (\frac{\sigma}{2} + \varepsilon)(T - t).$$

Since $\varepsilon \in (0, \sigma/2)$ is arbitrary, in view of the definition of $m(t)$, the above inequality implies (2.63).

Next, we consider the second case. Let $\sigma < 0$. From (2.55) we have

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - K,$$

where K is defined in (2.65). Since $\bar{m}(t) \to \infty$ as $t \to T^-$, we can choose a $t_0 \in (0, T)$ such that

$$\bar{m}(t) > \sqrt{-2\sigma K}.$$

Therefore, we have that $\bar{m}(t)$ is strictly increasing on $[t_0, T)$ and

$$\bar{m}(t) > \bar{m}(t_0) > \sqrt{-2\sigma K} > 0.$$

Using the transport equation for ρ, we have that

$$\rho'(t, \eta(t)) = -\bar{m}(t)\rho(t, \eta(t)).$$

Hence,

$$\rho(t, \eta(t)) = \rho(t_0, \eta(t_0))e^{-\int_{t_0}^{t} \bar{m}(\tau)d\tau}, \quad t \in [t_0, T).$$

Then

$$\rho^2(t, \eta(t)) \leq \rho^2(t_0, \eta(t_0)), \quad t \in [t_0, T).$$

Therefore, using (2.55) again, we have

$$-\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}\rho^2(t_0, \eta(t_0)) - K \leq \bar{m}'(t) \leq -\frac{\sigma}{2}\bar{m}^2(t) + \frac{1}{2}\rho^2(t_0, \eta(t_0)) + K. \quad (2.67)$$

Now let

$$\bar{K} = \frac{1}{2}\rho^2(t_0, \eta(t_0)) + K,$$

and choose $0 < \varepsilon < -\sigma/2$. We can pick a $t_0 \in [t_0, T)$ such that

$$\bar{m}(t_1) > \sqrt{-2\sigma\bar{K} + \frac{\bar{K}}{\varepsilon}}.$$

Then

$$\bar{m}(t) > \bar{m}(t_1) > \sqrt{-2\sigma\bar{K} + \frac{\bar{K}}{\varepsilon}} > \sqrt{\frac{\bar{K}}{\varepsilon}}.$$

Hence, (2.67) implies that

$$\frac{\sigma}{2} - \varepsilon < \frac{d}{dt}\Big(\frac{1}{\bar{m}(t)}\Big) < \frac{\sigma}{2} + \varepsilon, \quad a.e. \quad t \in [t_1, T).$$

Integrating the above relation on (t, T) with $t \in [t_1, T)$ and noticing that $\bar{m}(t) \to \infty$ as $t \to T^-$, we obtain

$$(\frac{\sigma}{2} - \varepsilon)(T - t) < -\frac{1}{\bar{m}(t)} < (\frac{\sigma}{2} + \varepsilon)(T - t).$$

Since $\varepsilon \in (0, -\sigma/2)$ is arbitrary, in view of the definition of $\bar{m}(t)$, the above inequality implies (2.64). □

2.5 Global Existence

In this section, we provide a sufficient condition for the global solution of system (2.2) in the case when $0 < \sigma < 2$ and $\sigma = 0$.

Theorem 2.5.1. *Let $0 < \sigma < 2$ and (u, ρ) be the solution of (2.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. If*

$$\inf_{x \in \mathbb{S}} \rho_0(x) > 0, \tag{2.68}$$

then $T = +\infty$ and the solution (u, ρ) is global.

We need the following lemma to prove the above theorem.

Lemma 2.5.1. *Let $0 < \sigma < 2$ and (u, ρ) be the solution of (2.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Assume that $\inf_{x \in \mathbb{S}} \rho_0(x) > 0$.*

1. *If $0 < \sigma \leq 1$, then*

$$\Big|\inf_{x \in \mathbb{S}} u_x(t, x)\Big| \leq \frac{1}{\inf_{x \in \mathbb{S}} \rho_0(x)} C_5 e^{C_4 t}, \tag{2.69}$$

$$\Big|\sup_{x \in \mathbb{S}} u_x(t, x)\Big| \leq \frac{1}{\inf_{x \in \mathbb{S}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_5^{\frac{1}{2-\sigma}} e^{\frac{C_4 t}{2-\sigma}}, \tag{2.70}$$

2. If $1 \le \sigma < 2$, then

$$\left|\inf_{x\in\mathbb{S}} u_x(t,x)\right| \le \frac{1}{\inf_{x\in\mathbb{S}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_5^{\frac{1}{2-\sigma}} e^{\frac{C_4 t}{2-\sigma}}, \tag{2.71}$$

$$\left|\sup_{x\in\mathbb{S}} u_x(t,x)\right| \le \frac{1}{\inf_{x\in\mathbb{S}} \rho_0(x)} C_5 e^{C_4 t}. \tag{2.72}$$

The constants C_4 and C_5 are defined by

$$\begin{aligned} C_4 = 1 + \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \Big[\frac{|3-\sigma|(e+1)}{4(e-1)} + \frac{\cosh(\frac{1}{2})(|3-\sigma|+|\sigma|+1)}{4\sinh(\frac{1}{2})} \\ + \frac{1}{4}\Big] \|(u_0,\rho_0)\|^2_{H^1\times L^2}, \end{aligned} \tag{2.73}$$

$$C_5 = 1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_{0,x}\|^2_{L^\infty}. \tag{2.74}$$

Proof. Similar as before, a density argument indicates that it suffices to prove the desired results for $s \ge 3$. Thus, we have

$$\inf_{x\in\mathbb{S}} u_x(t,x) < 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) > 0, \quad t \in [0,T).$$

(1) First we will derive an estimate for $|\inf_{x\in\mathbb{S}} u_x(t,x)|$. Define $m(t)$ and $\xi(t)$ as in (2.36), and consider along the characteristics $q_1(t,x_1(t))$ as in (2.4) and (2.22). Thus, from (2.33),

$$m(t) \le 0 \quad for \quad t \in [0,T). \tag{2.75}$$

Let $\zeta(t) = \rho(t,\xi(t))$ and evaluate (2.19) and the second equation of the system (3.2) at $(t,\xi(t))$. We have

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2}m^2(t) + \frac{1}{2}\zeta^2(t) + f(t,q_1(t,x_1)), \\ \zeta'(t) &= -\zeta m, \end{aligned} \tag{2.76}$$

for $t \in [0,T)$ where f is defined in (2.25). The second equation above implies that $\zeta(t)$ and $\zeta(0)$ are of the same sign.

Now, we want to construct a Lyapunov function for our system, as in [15]. Since here we have a free parameter σ, we could not find a uniform Lyapunov function. Instead, we will split the case $0 < \sigma \leq 1$ and the case $1 < \sigma < 2$. From the assumption of the theorem, we know that $\zeta(0) = \rho(0, \xi(0)) > 0$.

When $0 < \sigma \leq 1$, we define the following Lyapunov function

$$w_1(t) = \zeta(0)\zeta(t) - \frac{\zeta(0)}{\zeta(t)}\Big(1 + m^2(t)\Big), \tag{2.77}$$

which is always positive for $t \in [0, T)$. Differentiating $w_1(t)$ and using (2.76), we obtain

$$\begin{aligned} w_1'(t) &= \zeta(0)\zeta' - \frac{\zeta(0)}{\zeta^2}(1+m^2)\zeta' + \frac{2}{\zeta}\zeta(0)mm' \\ &= \frac{2\zeta(0)m}{\zeta}\Big[\frac{1-\sigma}{2}m^2 + \frac{1}{2} + f(t, q_1(t, x_1))\Big] \\ &\leq \frac{\zeta(0)}{\zeta}(1+m^2)\Big[|f(t, q_1(t, x_1))| + \frac{1}{2}\Big] \leq C_4 w_1(t), \end{aligned} \tag{2.78}$$

where we have used (2.75) and the bound (2.32) for f. Hence,

$$\begin{aligned} w_1(t) &\leq w_1(0)e^{C_4 t} = [\zeta^2(0) + 1 + m^2(0)]e^{C_4 t} \\ &\leq (1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty})e^{C_4 t} = C_5 e^{C_4 t}. \end{aligned} \tag{2.79}$$

Recalling that $\zeta(t)$ and $\zeta(0)$ are of the same sign, we have

$$\zeta(0)\zeta(t) \leq w_1(t) \quad and \quad |\zeta(0)||m(t)| \leq w_1(t).$$

Then from (2.79), we have

$$\Big|\inf_{x\in\mathbb{S}} u_x(t,x)\Big| = |m(t)| \leq \frac{w_1(t)}{\zeta(0)} \leq \frac{1}{\inf_{x\in\mathbb{S}} \rho_0(x)} C_5 e^{C_4 t}, \quad for \quad t \in [0, T),$$

which proves (2.69).

If $1 \leq \sigma < 2$, we may define the Lyapunov function to be

$$w_2(t) = \zeta^\sigma(0)\frac{\zeta^2(t) + 1 + m^2(t)}{\zeta^\sigma(t)}. \tag{2.80}$$

Differentiating $w_2(t)$ and using (2.76), we obtain

$$\begin{aligned} w_2'(t) &= \frac{2\zeta^\sigma(0)m}{\zeta^\sigma}\Big[\frac{\sigma-1}{2}\zeta^2+\frac{\sigma}{2}+f(t,q_1(t,x_1)\Big] \\ &\le \frac{\zeta^\sigma(0)}{\zeta^\sigma}(1+m^2)[|f(t,q_1(t,x_1)|+\frac{\sigma}{2}] \le C_4 w_2(t). \end{aligned} \tag{2.81}$$

Thus,

$$\begin{aligned} w_2(t) &\le w_2(0)e^{C_4t} = [\zeta^2(0)+1+m^2(0)]e^{C_4t} \\ &\le (1+\|u_{0,x}\|^2_{L^\infty}+\|\rho_0\|^2_{L^\infty})e^{C_4t} = C_5e^{C_4t}. \end{aligned} \tag{2.82}$$

Applying Young's inequality $ab \le a^p/p + b^q/q$ to (2.80) with

$$p=\frac{2}{\sigma} \quad and \quad q=\frac{2}{2-\sigma},$$

we have

$$\begin{aligned} \frac{w_2(t)}{\zeta^\sigma(0)} &= [\zeta^{\frac{\sigma(2-\sigma)}{2}}]^{2/\sigma} + \Big[\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\Big]^{2/(2-\sigma)} \\ &\ge \frac{\sigma}{2}[\zeta^{\frac{\sigma(2-\sigma)}{2}}]^{2/\sigma} + \frac{2-\sigma}{2}\Big[\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\Big]^{2/(2-\sigma)} \\ &\ge (1+m^2)^{\frac{2-\sigma}{2}} \ge |m(t)|^{2-\sigma}. \end{aligned}$$

Therefore,

$$\Big|\inf_{x\in\mathbb{S}} u_x(t,x)\Big| \le \Big[\frac{w_2(t)}{\zeta^\sigma(0)}\Big]^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in\mathbb{S}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)}C_5^{\frac{1}{2-\sigma}}e^{\frac{C_4t}{2-\sigma}}, \quad t\in[0,T),$$

which proves (2.71).

(2) Next we try to control $|\sup_{x\in\mathbb{S}} u_x(t,x)|$. Similarly as before, we consider $\bar{m}(t), \eta(t), q_1(t,x_2(t))$ as in (2.20) and (2.38). Then (2.76) becomes

$$\begin{aligned} &\bar{m}'(t) = -\frac{\sigma}{2}\bar{m}^2(t)+\frac{1}{2}\bar{\zeta}^2(t)+f(t,q_1(t,x_2)), \\ &\bar{\zeta}'(t) = -\bar{\zeta}\bar{m}, \end{aligned} \tag{2.83}$$

for $t\in[0,T)$, where $\bar{\zeta}(t)=\rho(t,\eta(t))$. From (2.33), we have

$$\bar{m}(t)\ge 0, \qquad t\in[0,T). \tag{2.84}$$

When $0 < \sigma \leq 1$, the corresponding Lyapunov function is

$$\bar{w}_1(t) = \bar{\zeta}^\sigma(0)\frac{\bar{\zeta}^2(t) + 1 + \bar{m}^2(t)}{\bar{\zeta}^\sigma(t)}. \tag{2.85}$$

Then from (2.81) and (2.84), we see that

$$\bar{w}_1'(t) \leq C_4\bar{w}_1(t), \quad then \quad \bar{w}_1(t) \leq C_5 e^{C_4 t}.$$

Hence, by the similar argument as before, we obtain

$$\frac{\bar{w}_1(t)}{\bar{\zeta}^\sigma(0)} \geq |m(t)|^{2-\sigma}.$$

Therefore,

$$\Big|\sup_{x\in\mathbb{S}} u_x(t,x)\Big| \leq \Big[\frac{\bar{w}_1(t)}{\bar{\zeta}^\sigma(0)}\Big]^{\frac{1}{2-\sigma}} \leq \frac{1}{\inf_{x\in\mathbb{S}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_5^{\frac{1}{2-\sigma}} e^{\frac{C_4 t}{2-\sigma}}, \quad t \in [0,T),$$

which proves (2.70).

When $1 \leq \sigma < 2$, consider the Lyapunov function

$$\bar{w}_2(t) = \bar{\zeta}(0)\bar{\zeta}(t) = \frac{\bar{\zeta}(0)}{\bar{\zeta}(t)}\Big(1 + \bar{m}^2(t)\Big). \tag{2.86}$$

From (2.78) and (2.84), it follows that $\bar{w}_2'(t) \leq C_4\bar{w}_1(t)$. This in turn implies that $\bar{w}_1(t) \leq C_5 e^{C_4 t}$.

Thus, we have

$$\Big|\sup_{x\in\mathbb{S}} u_x(t,x)\Big| = |\bar{m}(t)| \leq \frac{\bar{w}_1(t)}{\bar{\zeta}(0)} \leq \frac{1}{\inf_{x\in\mathbb{S}}\rho_0(x)} C_5 e^{C_4 t}, \quad t \in [0,T),$$

which proves (2.72). □

Proof of Theorem 2.5.1. Assume on the contrary that $T < \infty$ and the solution blows up in finite time. It then follows Theorem 2.2.2 that

$$\int_0^T \|u_x(t,x)\|_{L^\infty} dt = \infty. \tag{2.87}$$

However, from the assumption of the theorem and Lemma 2.5.1, we have

$$|u_x(t,x)| < \infty,$$

for all $(t,x) \in [0,T) \times \mathbb{S}$, a contradiction to (2.87). Thus, $T = +\infty$, and the solution (u,ρ) is global. □

We are now in a position to consider the case $\sigma = 0$.

Theorem 2.5.2. *Let $\sigma = 0$. If $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, then there exists a unique solution (u,ρ) of system (2.2) with the initial data (u_0, ρ_0). Moreover, the solution depends continuously on the initial data. Then $T = +\infty$ and the solution (u,ρ) is global.*

When $\sigma = 0$, we can rewrite system (2.3) as

$$\begin{cases} u_t = -\partial_x G * (-Au + \frac{3}{2}u^2 + \frac{1}{2}\rho^2), \\ \rho_t + u\rho_x = -u_x \rho, \\ u(0,x) = u_0(x), \\ \rho(0,x) = \rho_0(x), \\ u(t,x+1) = u(t,x), \\ \rho(t,x+1) = \rho(t,x). \end{cases} \tag{2.88}$$

To prove Theorem 2.5.2 of global well-posedness of solutions, we need the following estimates for u_x.

Lemma 2.5.2. *Let $\sigma = 0$ and (u,ρ) be the solution of (2.88) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then*

$$\sup_{x \in \mathbb{S}} u_x(t,x) \leq \sup_{x \in \mathbb{S}} u_{0,x}(x) + \frac{1}{2}(\sup_{x \in \mathbb{S}} \rho_0^2(x) + C_6^2)t, \tag{2.89}$$

$$\inf_{x \in \mathbb{S}} u_x(t,x) \geq \inf_{x \in \mathbb{S}} u_{0,x}(x) + \frac{1}{2}(\inf_{x \in \mathbb{S}} \rho_0^2(x) - C_7^2)t, \tag{2.90}$$

where the constants C_6 and C_7 are defined as follows

$$C_6 = \sqrt{\frac{(-1+\sinh 1)A^2}{4\sinh^2(1/2)} + \frac{3(e+1)+1}{4(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}}, \tag{2.91}$$

$$C_7 = \sqrt{\frac{(-1+\sinh 1)A^2}{4\sinh^2(1/2)} + \frac{3\cosh(1/2)+1}{4\sinh(1/2)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}}. \tag{2.92}$$

Proof. The local well-posedness theorem and a density argument implies that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Also we may assume that

$$u_0 \not\equiv 0, \tag{2.93}$$

otherwise the results become trivial. Since now $s = 3$, we have $u \in C_0^1(\mathbb{S})$. Therefore

$$\inf_{x\in\mathbb{S}} u_x(t,x) \leq 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) \geq 0, \quad t \in [0,T). \tag{2.94}$$

Differentiating the first equation of (2.88) with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$ we obtain

$$u_{tx} = \frac{1}{2}\rho^2 + \frac{3}{2}u^2 + A\partial_x^2 G * u - G * (\frac{3}{2}u^2 + \frac{1}{2}\rho^2). \tag{2.95}$$

Using Lemma 2.2.5 and the fact that

$$\sup_{x\in\mathbb{S}}[u_x(t,x)] = -\inf_{x\in\mathbb{S}}[v_x(t,x)],$$

we can consider $\bar{m}(t)$ and $\bar{\xi}(t)$ as follows

$$\bar{m}(t) := u_x(t,\bar{\xi}(t)) = \sup_{x\in\mathbb{S}}(u_x(t,x)), \quad t \in [0,T). \tag{2.96}$$

Hence

$$u_{xx}(t,\bar{\xi}(t)) = 0, \quad a.e. \quad t \in [0,T). \tag{2.97}$$

Take the trajectory $q(t,x)$ defined in (2.4). Then we know that $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for every $t \in [0,T)$. Therefore, there exists $x_1(t) \in \mathbb{S}$ such that

$$q_(t, x_1(t)) = \xi(t), \quad t \in [0,T). \tag{2.98}$$

Now let

$$\bar{\zeta}(t) = \rho(t, q(t, x_1)), \quad t \in [0, T). \tag{2.99}$$

Therefore along this trajectory $q(t, x_1)$, equation (2.95) and the second equation of (2.88) become

$$\begin{aligned} \bar{m}'(t) &= \frac{1}{2}\bar{\zeta}^2 + f(t, q(t, x_1)), \\ \bar{\zeta}'(t) &= -\bar{\zeta}\bar{m}, \end{aligned} \tag{2.100}$$

for $t \in [0, T)$, where $'$ denotes the derivative with respect to t and $f(t, q(t, x_1))$ is given by

$$f = A\partial_x^2 G * u + \frac{3}{2}u^2 - G * (\frac{3}{2}u^2 + \frac{1}{2}\rho^2). \tag{2.101}$$

We first derive the upper and lower bounds for f for later use in getting the wave-breaking result. Using that $\partial_x^2 G * u = \partial_x G * \partial_x u$, we have

$$f = \frac{3}{2}u^2 + A\partial_x G * \partial_x u - G * (\frac{3}{2}u^2) - \frac{1}{2}G * \rho^2 \le \frac{3}{2}u^2 + A|G_x * u_x|.$$

Using (2.26) and (2.28), we obtain the upper bound of f

$$\begin{aligned} f &\le \frac{3}{2}\|u(t,\cdot)\|^2_{L^\infty(\mathbb{S})} + \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} \\ &\le \frac{3(e+1)}{4(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2} + \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} \\ &\le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{3(e+1)+1}{4(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2} = \frac{1}{2}C_6^2. \end{aligned} \tag{2.102}$$

Now we turn to the lower bound of f. Similarly as before, we get

$$\begin{aligned} -f &\le A|G_x * u_x| + \frac{3}{2}|G * u^2| + \frac{1}{2}G * \rho^2 \\ &\le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} + \frac{3\cosh(1/2)}{4\sinh(1/2)}\|u\|^2_{L^2} + \frac{\cosh(1/2)}{4\sinh(1/2)}\|\rho\|^2_{L^2} \\ &\le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \Big(\frac{3\cosh(1/2)+1}{4\sinh(1/2)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}\Big) = \frac{1}{2}C_7^2. \end{aligned} \tag{2.103}$$

Combining (2.102) and (2.103), we obtain

$$|f| \le \frac{(-1+\sinh 1)A^2}{8\sinh^2(1/2)} + \frac{3\cosh(1/2)+1}{4\sinh(1/2)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}. \tag{2.104}$$

From (2.94) we know $\bar{m}(t) \geq 0$ for $t \in [0, T)$. From the second equation of (2.100) we obtain that

$$\bar{\zeta}(t) = \bar{\zeta}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}. \tag{2.105}$$

Hence

$$|\rho(t, q(t, x_1)) = |\bar{\zeta}(t)| \leq |\bar{\zeta}(0)|.$$

Therefore, we have

$$\bar{m}'(t) = \frac{1}{2}\bar{\zeta}^2(t) + f \leq \frac{1}{2}\bar{\zeta}^2(0) + \frac{1}{2}C_6^2 \leq \frac{1}{2}\Big(\sup_{x\in\mathbb{S}} \rho_0^2(x) + C_6^2\Big).$$

Integrating the above over $[0, t]$, we prove (2.91).

To obtain a lower bound for $\inf_{x\in\mathbb{S}} u_x(t, x)$, we use the similar idea. Consider the function $m(t)$ and $\xi(t)$ as in Lemma 2.2.5

$$m(t) := u_x(t, \xi(t)) = \inf_{x\in\mathbb{S}}(u_x(t, x)), \quad t \in [0, T). \tag{2.106}$$

Hence

$$u_{xx}(t, \xi(t)) = 0, \quad a.e. \quad t \in [0, T). \tag{2.107}$$

Again take the trajectory $q(t, x)$ defined in (2.4) and choose $x_2(t) \in \mathbb{S}$ such that

$$q(t, x_2(t)) = \xi(t) \quad t \in [0, T). \tag{2.108}$$

Now let

$$\bar{\zeta}(t) = \rho(t, q(t, x_2)), \quad t \in [0, T). \tag{2.109}$$

Hence along this trajectory $q(t, x_2)$, equation (2.95) and the second equation of (2.88) become

$$\begin{aligned} m'(t) &= \frac{1}{2}\zeta^2 + f(t, q(t, x_2)), \\ \zeta'(t) &= -\zeta m. \end{aligned} \tag{2.110}$$

Since $m(t) \geq 0$, we have from the second equation of the above that

$$|\rho(t, q(t, x_2)) = |\zeta(t)| \geq |\zeta(0)|.$$

Then

$$m'(t) = \frac{1}{2}\zeta^2(0) - \frac{1}{2}C_7^2 \geq \frac{1}{2}\Big(\inf_{x \in \mathbb{S}} \rho_0^2(x) + C_7^2\Big).$$

Integrating the above over $[0, t]$, we obtain (5.90). This completes the proof of the Lemma 2.5.2. □

Proof of Theorem 2.5.2. Similarly as the proof of Lemma 2.5.1, assume on the contrary that $T < \infty$ and the solution blows up in finite time. It then follows Theorem 2.2.2 that

$$\int_0^T \|u_x(t, x)\|_{L^\infty} dt = \infty. \tag{2.111}$$

However, from the assumption of the theorem and Lemma 2.5.2, we have

$$|u_x(t, x)| < \infty,$$

for all $(t, x) \in [0, T) \times \mathbb{S}$, a contradiction to (2.111). Thus, $T = +\infty$, and the solution (u, ρ) is global. □

CHAPTER 3

GENERALIZED PERIODIC TWO-COMPONENT DGH SYSTEM

3.1 Introduction

In this section, we are concerned with the Cauchy problem of the generalized periodic two-component Dullin-Gottwald-Holm (DGH) system

$$\begin{cases} m_t - Au_x + \sigma(2mu_x + um_x) + 3(1-\sigma)uu_x + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0,\ x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $m = u - u_{xx}$, and σ is a real parameter. It is a model from the shallow water theory with nonzero constant vorticity, where $u(t,x)$ is the horizontal velocity and $\rho(t,x)$ is related to the free surface elevation from equilibrium. The scalar $A > 0$ characterizes a linear underlying shear flow and hence the system in (3.1) models wave-current interactions. The real dimensionless constant σ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching. System (3.1) can be written in terms of u and ρ,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t \geq 0,\ x \in \mathbb{R}. \end{cases} \tag{3.2}$$

System (3.2) has the following two Hamiltonians:

$$\begin{aligned} H_1 =& \frac{1}{2}\int_{\mathbb{S}} (u^2 + u_x^2 + \rho^2)dx, \\ H_2 =& \frac{1}{2}\int_{\mathbb{S}} (u^3 + \sigma u u_x^2 - Au^2 - \gamma_x^2 + 2u\rho + u\rho^2)dx. \end{aligned}$$

Notation. Throughout this chapter, we identity all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. The norm of the Lebesegue space $L^p(\mathbb{S}), 1 \leq p \leq \infty$, is denoted by $\|\cdot\|_{L^p}$ and the Sobolev space $H^s(\mathbb{S}), s \in \mathbb{R}$, by $\|\cdot\|_{H^s}$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity.

3.2 Derivation of the Model

In this section, we will follow Ivanov's approach in [33] to derive system (3.1). Consider the motion of an inviscid incompressible fluid with a constant density ρ governed by Euler's equations:

$$\begin{cases} \vec{v}_t + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \vec{g}, \\ \nabla\cdot v = 0, \end{cases} \tag{3.3}$$

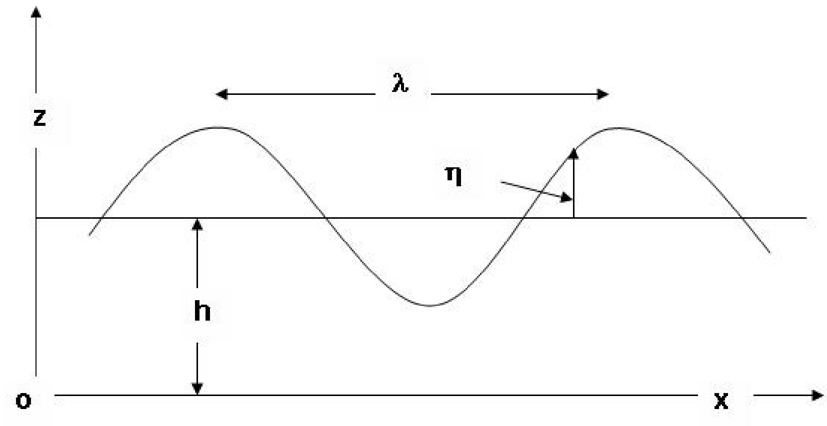

Figure 3.1. A fundamental shallow water wave model.

where $\vec{v}(x, y, z, t)$ is the velocity of the fluid at the point (x, y, z) at the time t, $P(x, y, z, t)$ is the pressure, and $\vec{g}(0, 0, -g)$ is the gravity acceleration.

Using the shallow water approximation and non-dimensionalization, the above equations can be written as

$$\begin{cases} u_t + \varepsilon(uu_x + wu_z) = -p_x, \\ \delta^2(w_t + \varepsilon(uw_x + ww_z)) = -p_z, \\ u_x + w_z = 0, \\ w = \eta_t + \varepsilon u\eta_x, \quad p = \eta \quad on \quad z = 1 + \varepsilon\eta, \\ w = 0 \quad on \quad z = 0, \end{cases}$$

where now $\vec{v} = (u, 0, w)$, $p(x, z, t)$ is the pressure variable measuring the deviation from the hydrostatic pressure distribution, $\eta(t, x)$ is the deviation from the mean level $z = h$ of the water surface, and $\varepsilon = \frac{a}{h}$ and $\delta = \frac{h}{\lambda}$ are the two dimensionless parameters with a being the typical amplitude of the water and λ being the typical wavelength of the wave (see Figure 3.1.).

Let us consider waves in the presence of a shear flow. In such case the horizontal velocity of the flow will be $u + U(z)$, where $u = U(z)$, $0 \leq z \leq h$, $w \equiv 0$, $p \equiv 0$, $\eta \equiv 0$

is an exact solution of the governing equation (3.3) and this solution represents an arbitrary underlying shear flow. Taking the simplest case : $\tilde{U}(z) = Az$ where $A > 0$ is a constant.

In the case of constant vorticity $\omega = A$, one obtains at the order of $O(\varepsilon, \delta^2)$ the following equations for u_0 and η, where u_0 is the leading order approximation for u

$$\left(u_0 - \delta^2 \frac{1}{2} u_{0xx}\right)_t + \varepsilon u_0 u_{0x} + \eta_x - \delta^2 \frac{A}{3} u_{0xxx} = 0, \tag{3.4}$$

and

$$\eta_t + A\eta_x + \left[(1+\varepsilon\eta)u_0 + \varepsilon \frac{A}{2}\eta^2\right]_x - \delta^2 \frac{1}{6} u_{0xxx} = 0. \tag{3.5}$$

Let both the parameters ε and δ go to 0, one obtains from (3.4)-(3.5) the system of linear equations

$$\begin{aligned} u_{0t} + \eta_x =& 0, \\ \eta_t + A\eta_x + u_{0x} =& 0, \end{aligned}$$

hence,

$$\eta_{tt} + A\eta_{tx} - \eta_{xx} = 0. \tag{3.6}$$

This equation has a traveling wave solution $\eta = \eta(x - ct)$ with a velocity c satisfying

$$c^2 - Ac - 1 = 0.$$

This gives the same solution for c that follows the Burns condition [5].

Introducing a new variable

$$\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\mu u_{0xx},$$

for some constants α, β and μ satisfying

$$\begin{aligned} \frac{\mu}{\alpha} =& \frac{1}{6(c-A)}, \\ \alpha =& 1 + \frac{Ac}{2} + \frac{\beta}{\alpha}, \end{aligned}$$

equation (3.4) and (3.5) become

$$\begin{cases} m_t + Am_x - Au_{0x} + \delta^2(\frac{A}{6} - \frac{1}{6(c-A)})u_{0xxx} \\ \quad +\varepsilon(1 - \frac{\alpha^2+2\beta}{\alpha}c^2)u_0u_{0x} + \frac{\rho\rho_x}{\varepsilon\alpha} = 0, \\ \rho_t + A\rho_x + \alpha\varepsilon(\rho u_0)_x = 0. \end{cases} \tag{3.7}$$

where $m = u_0 - \frac{1}{2}\delta^2 u_{0xx}$. At the order of $O(1)$, we may break u_0u_{0x} as

$$u_0u_{0x} = s(2mu_{0x} + u_0m_x) + (1-3s)u_0u_{0x} + O(\delta^2),$$

for any $s \in \mathbb{S}$. Thus, the first equation of the system (3.7) can be written at the order $O(\varepsilon, \delta^2)$ as

$$m_t + Am_x - Au_{0x} - \frac{1}{6c}\delta^2 u_{0xxx} + \varepsilon(1 - \frac{\alpha^2+2\beta}{\alpha}c^2)s(2mu_{0x} + u_0m_x) \\ +\varepsilon(1 - \frac{\alpha^2+2\beta}{\alpha}c^2)(1-3s)u_0u_{0x} + \frac{\rho\rho_x}{\varepsilon\alpha} = 0.$$

Using the scaling : $u_0 \to \frac{1}{\alpha\varepsilon}$, $x \to \delta x$, $t \to \delta t$, then system (3.7) becomes

$$\begin{cases} m_t + Am_x - Au_{0x} - \frac{1}{6c}u_{0xxx} + \frac{1}{\alpha}(1 - \frac{\alpha^2+2\beta}{\alpha}c^2)s(2mu_{0x} + u_0m_x) \\ +\frac{1}{\alpha}(1 - \frac{\alpha^2+2\beta}{\alpha}c^2)(1-3s)u_0u_{0x} + \rho\rho_x = 0, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{cases}$$

Now if we choose

$$\frac{1}{3\alpha}(1 - \frac{\alpha^2+2\beta}{\alpha}c^2) = 1$$

and denote $\sigma = 3s$ and $\gamma = -\frac{1}{6c}$, then we arrive at

$$\begin{cases} m_t + Am_x - Au_{0x} + \sigma(2mu_{0x} + u_0m_x) + 3(1-\sigma)u_0u_{0x} + \gamma u_{0xxx} + \rho\rho_x = 0, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{cases} \tag{3.8}$$

Thus, the constants α, β, μ and c satisfy

$$
\begin{aligned}
&\alpha = \frac{1}{3(1 = c^2)} + \frac{c^2}{3},\\
&\beta = \alpha^2 - \alpha(1 + \frac{Ac}{2}),\\
&\mu = \frac{\alpha}{6(c - A)},\\
&c^2 - Ac - 1 = 0.
\end{aligned}
$$

With a further Galilean transformation $x \to x - At$, $t \to t$, as used in (3.8), we can drop the terms Am_x and $A\rho_x$ in (3.8) and hence get the generalized periodic two-component DGH system (3.1).

3.3 Preliminaries

In this section, we briefly give the needed results to pursue our goal. We first present the local well-posedness for the Cauchy problem of system (3.2) in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$.

Let $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, x \in \mathbb{R}$. Then $(1 - \partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$ and $G * m = u$. Our system (3.2) can be written in the following "transport" type

$$
\begin{cases}
u_t + (\sigma u - \gamma)u_x = -\partial_x G * [\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2], & t > 0,\ x \in \mathbb{R},\\
\rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R},\\
u(0, x) = u_0(x), & x \in \mathbb{R},\\
\rho(0, x) = \rho_0(x), & x \in \mathbb{R},\\
u(t, x + 1) = u(t, x), & t \geq 0,\ x \in \mathbb{R},\\
\rho(t, x + 1) = \rho(t, x), & t \geq 0,\ x \in \mathbb{R}.
\end{cases}
\tag{3.9}
$$

Applying the transport equation theory combined with the method of the Besov spaces, one may follow the similar argument as in [29] to obtain the following local well-posedness result for the system (3.2).

Theorem 3.3.1. *If* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *then there exists a maximal time* $T = T(\|(u_0, \rho_0)\|_{H^s \times H^{s-1}}) > 0$ *and a unique solution* (u, ρ) *of (3.2) in* $C([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \bigcap C^1([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$ *with* $(u,\rho)|_{t=0} = (u_0, \rho_0)$. *Moreover, the solution depends continuously on the initial data, and* T *is independent of* s.

Now, we consider the following two associated Lagrangian scales of the generalized two component system (3.2)

$$\begin{cases} \frac{\partial q_1}{\partial t} = u(t, q_1), & 0 < t < T, \\ q_1(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \frac{\partial q_2}{\partial t} = \sigma u(t, q_2) - \gamma, & 0 < t < T, \\ q_2(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.11}$$

where $u \in C^1([0,T), H^{s-1})$ is the first component of the solution (u, ρ) to (3.2).

Lemma 3.3.1. *[9, 14, 26] Let* (u, ρ) *be the solution of system (3.2) with initial data* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *and* T *the maximal time of existence. Then Eq.(3.10) has a unique solution* $q_1 \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$, *and Eq.(3.11) has a unique solution* $q_2 \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. *These two solutions satisfy* $q_i(t, x+1) = q_i(t,x) + 1, i = 1, 2$. *Moreover, the map* $q_1(t, \cdot)$ *and* $q_2(t, \cdot)$ *are increasing diffeomorphisms of* $\mathbb{R}$ *with*

$$q_{1,x}(t,x) = exp\left(\int_0^t u_x(\tau, q_1(\tau, x))\right) d\tau > 0, \quad (t,x) \in [0,T) \times \mathbb{R},$$

and

$$q_{2,x}(t,x) = exp\left(\int_0^t \sigma u_x(\tau, q_2(\tau, x))\right) d\tau > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

The above Lemmas indicate that $q_1(t, \cdot) : \mathbb{R} \to \mathbb{R}$ and $q_2(t, \cdot) : \mathbb{R} \to \mathbb{R}$ are diffeomorphisms of the line for each $t \in [0,T)$. Hence, the L^∞ norm of any function

$v(t,\cdot) \in L^\infty(\mathbb{S}), T \in [0,t)$ is preserved under the family of diffeomorphisms $q_1(t,\cdot)$ and $q_2(t,\cdot)$ with $t \in [0,T)$, that is

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{S})} = \|v(t,q_1(t,\cdot))\|_{L^\infty(\mathbb{S})} = \|v(t,q_2(t,\cdot))\|_{L^\infty(\mathbb{S})}, \quad t \in [0,T). \tag{3.12}$$

Similarly, we have

$$\inf_{x\in\mathbb{S}} v(t,x) = \inf_{x\in\mathbb{S}} v(t,q_1(t,x)) = \inf_{x\in\mathbb{S}} v(t,q_2(t,x)), \quad t \in [0,T). \tag{3.13}$$

$$\sup_{x\in\mathbb{S}} v(t,x) = \sup_{x\in\mathbb{S}} v(t,q_1(t,x)) = \sup_{x\in\mathbb{S}} v(t,q_2(t,x)), \quad t \in [0,T). \tag{3.14}$$

Lemma 3.3.2. *[26] Let (u,ρ) be the solution of system (3.2) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then we have*

$$\rho(t,q(t,x))q_x(t,x) = \rho_0(x), \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{3.15}$$

Moreover if there exists $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$, then $\rho(t,q(t,x_0)) = 0$ for all $t \in [0,T)$.

We may use the following proposition derived in [29] to study the regularity property of solution to system (3.2).

Proposition 3.3.1. *Let $0 < s < 1$. Suppose that $f_0 \in H^s, g \in L^1([0,T];H^s)$, and $v, v_x \in L^1([0,T];L^\infty)$ and that $f \in L^\infty([0,T];H^s)\bigcap C([0,T];S')$ solves the one-dimensional linear transport equation*

$$\begin{cases} f_t + vf_x = g, \\ f(0,x) = f_0(x). \end{cases} \tag{3.16}$$

Then $f \in C([0,T];H^s)$. More precisely, there exists a constant C depending only on s such that the following estimate holds:

$$\|f(t)\|_{H^s} \leq \|f_0\|_{H^s} + C\Big(\int_0^t \|g(\tau)\|_{H^s} d\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau\Big). \tag{3.17}$$

Hence,

$$\|f(t)\|_{H^s} \le e^{CV(t)}\Big(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau\Big), \tag{3.18}$$

where $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}) d\tau$.

The above proposition was proved using the Littlewood-Palay analysis for the transport equation and the Moser-type estimates. Using this result and performing the same argument as in [29], we can obtain the following blow-up criterion.

Theorem 3.3.2. *Let* (u, ρ) *be the solution of system (3.2) with initial data* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *and* T *the maximal time of existence. Then*

$$T < \infty \Rightarrow \int_0^T \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{3.19}$$

We then give several useful conservation laws of strong solutions to system (3.2).

Lemma 3.3.3. *Let* (u, ρ) *be the solution of system (3.2) with initial data* $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, *and* T *the maximal time of existence. Then for all* $t \in [0, T)$, *we have*

$$\int_{\mathbb{S}} u(t,x)dx = \int_{\mathbb{S}} u_0(x)dx,$$
$$\int_{\mathbb{S}} \rho(t,x)dx = \int_{\mathbb{S}} \rho_0(x)dx.$$

Proof. Integrating the first equation of (3.9) by parts, in view of the periodicity of u and G, we get

$$\frac{d}{dt}\int_{\mathbb{S}} u dx = -\int_{\mathbb{S}} (\sigma u - \gamma)u_x dx - \int_{\mathbb{S}} \partial_x G * [\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 - (\gamma - A)u + \frac{1}{2}\rho^2] dx = 0.$$

On the other hand, integrating the second equation of (3.12) by parts, in view of the periodicity of u and ρ, we get

$$\frac{d}{dt}\int_{\mathbb{S}} \rho dx = -\int_{\mathbb{S}} (u\rho)_x dx = 0.$$

This completes the proof of the lemma. ∎

Lemma 3.3.4. *Let (u,ρ) be the solution of system (3.2) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then for all $t \in [0,T)$, we have*

$$\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x) + \rho^2(t,x))dx = \int_{\mathbb{S}} (u_0^2(t,x) + u_{0x}^2(t,x) + \rho_0^2(t,x))dx.$$

Proof. Multiplying the first equation of (3.2) by $2u$ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x))dx = \frac{d}{dt}\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)dx.$$

Multiplying the second equation of (3.2) by 2ρ and integrating by parts, we get

$$\frac{d}{dt}\int_{\mathbb{S}} \rho^2(t,x) = -\frac{d}{dt}\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)dx.$$

Adding the above two equalities, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} (u^2(t,x) + u_x^2(t,x) + \rho^2(t,x))dx = 0.$$

This completes the proof of the lemma. □

Lemma 3.3.5. *[16] Let $T > 0$ and $v \in C^1([0,T); H^2(\mathbb{R}))$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}} [v_x(t,x)] = v_x(t,\xi(t)).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t,\xi(t)) \quad a.e. \quad on \quad (0,T).$$

Lemma 3.3.6. *[53] For every $f \in H^1(\mathbb{S})$, we have*

$$\max_{x\in[0,1]} f^2(x) \le \frac{e+1}{2(e-1)}\|f\|^2_{H^1(\mathbb{S})},$$

where the constant $\frac{e+1}{2(e-1)}$ is sharp.

By the conservation law stated in Lemma 3.3.4 and Lemma 3.3.6, we have the following corollary.

Corollary 3.3.1. *Let (u, ρ) be the solution of system (3.9) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then for all $t \in [0, T)$, we have*

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|u(t,\cdot)\|^2_{H^1(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}.$$

Lemma 3.3.7. *[31] If $f \in H^3(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)dx = \frac{a_0}{2}$, then for every $\varepsilon > 0$, we have*

$$\max_{x\in\mathbb{S}} f^2(x) \leq \frac{\varepsilon+2}{24}\int_{\mathbb{S}} f_x^2 dx + \frac{\varepsilon+2}{4\epsilon}a_0^2.$$

3.4 Wave-breaking Phenomenon

In this section, we investigate the wave-breaking phenomena of strong solution to system (3.2). First, we give the wave-breaking criterion for $\sigma \neq 0$.

Theorem 3.4.1 (Waving-breaking criterion)**.** *Let $\sigma \neq 0$ and (u, ρ) be the solution of (3.2) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence, then the solution blows up in finite time if and only if*

$$\liminf_{t\to T^-}\{\inf_{x\in\mathbb{S}} \sigma u_x(t,x)\} = -\infty. \tag{3.20}$$

To prove this wave-breaking criterion, we use the following lemma to show that indeed σu_x is uniformly bounded from above.

Lemma 3.4.1. *Let $\sigma \neq 0$ and (u, ρ) be the solution of (3.2) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then*

(1) If $\sigma > 0$, then

$$\sup_{x\in\mathbb{S}} u_x(t,x) \leq \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty} + C_9^2}{\sigma}}. \tag{3.21}$$

(2) If $\sigma < 0$, then

$$\inf_{x\in\mathbb{S}} u_x(t,x) \geq -\|u_{0,x}\|_{L^\infty} - \frac{C_{10}}{\sqrt{-\sigma}}. \tag{3.22}$$

The constants above are defined as follows.

$$\begin{aligned}
C_8 =& \|(u_0,\rho_0)\|^2_{H^1\times L^2},\\
C_9 =& C_8\sqrt{\frac{(-1+\sinh 1)|\gamma-A|^2}{4\sinh^2(1/2)} + \Big[\frac{|3-\sigma|(e+1)}{2(e-1)} + \frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{2\sinh(1/2)} + \frac{1}{2}\Big]},\\
C_{10} =& C_8\sqrt{\frac{(-1+\sinh 1)|\gamma-A|^2}{4\sinh^2(1/2)} + \frac{\cosh(1/2)(4-\sigma)}{2\sinh(1/2)}}, \quad for \quad \sigma<0.
\end{aligned} \tag{3.23}$$

Proof. The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Also, we assume that $u_0 \not\equiv 0$. Otherwise, the results become trivial. Note that if $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$ and $G * m = u$. Hence, we can rewrite the first equation in (3.9) as

$$u_t + (\sigma u - \gamma)u_x = -\partial_x G * \left(\frac{3-\sigma}{2} + \frac{\sigma}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\right). \tag{3.24}$$

Differentiating the above with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$, we obtain

$$u_{tx} + (\sigma u-\gamma)u_{xx} + \frac{\sigma}{2}u_x^2 = \frac{1}{2}\rho^2 + \frac{3-\sigma}{2}u^2 - (\gamma-A)\partial_x^2 G*u - G*(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2). \tag{3.25}$$

(1) When $\sigma > 0$, using Lemma 3.3.5 and the fact that

$$\sup_{x\in\mathbb{S}}[v_x(t,x)] = -\inf_{x\in\mathbb{S}}[-v_x(t,x)],$$

we can consider $\bar{m}(t)$ and $\eta(t)$ as follows:

$$\eta(t) \in \mathbb{S} \quad and \quad \bar{m}(t) := u_x(t,\eta(t)) = \sup_{x\in\mathbb{S}}(u_x(t,x)), \quad t \in [0,T). \tag{3.26}$$

Hence,

$$u_{xx}(t,\eta(t))=0, \quad a.e. \quad t\in[0,T). \tag{3.27}$$

Take the trajectory $q_1(t,x)$ defined in (3.10). Then we know that $q_1(t,\cdot):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism for every $t\in[0,T)$. Therefore, there exists $x_1(t)\in\mathbb{R}$ such that

$$q_1(t,x_1(t))=\eta(t), \quad t\in[0,T). \tag{3.28}$$

Now, let

$$\bar{\xi}=\rho(t,q_1(t,x_1)), \quad t\in[0,T). \tag{3.29}$$

Therefore, along the trajectory $q_1(t,x_1)$, equation (3.25) and the second equation of (3.2) become

$$\begin{aligned} \bar{m}'(t)=&-\frac{\sigma}{2}\bar{m}^2+\frac{1}{2}\bar{\xi}^2+f(t,q_1(t,x_1)),\\ \bar{\xi}'(t)=&-\bar{\xi}\bar{m}, \end{aligned} \tag{3.30}$$

for $t\in[0,T)$, where $'$ denotes the derivative with respect to t and $f(t,q_1(t,x_1))$ is given by

$$f=\frac{3-\sigma}{2}u^2-(\gamma-A)\partial_x^2G*u-G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2). \tag{3.31}$$

We first derive the upper and lower bounds for f for later use in getting the wave-breaking result. Using that $\partial_x^2G*u=\partial G*\partial_x u$, we have

$$\begin{aligned} f(t,x)=&\frac{3-\sigma}{2}u^2-(\gamma-A)\partial_xG*\partial_xu-G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)-\frac{1}{2}G*\rho^2\\ \leq&\frac{3-\sigma}{2}u^2+|\gamma-A||G_x*u_x|+|G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)|, \end{aligned}$$

for any $x\in\mathbb{S}$ and $t\in[0,T)$. Applying Young's inequality and $G=\frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, leads to

$$\begin{aligned} |\gamma-A||G_x*u_x|\leq|\gamma-A|\|G_x\|_{L^2}\|u_x\|_{L^2}=&|\gamma-A|\frac{\sqrt{\frac{1}{2}(-1+\sinh 1)}}{2\sinh(\frac{1}{2})}\|u_x\|_{L^2}\\ \leq&\frac{(-1+\sinh 1)|\gamma-A|^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2, \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
|G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)| &\le \|G\|_{L^\infty}\|\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\|_{L^1}\\
&= \frac{\cosh(1/2)}{2\sinh(1/2)}\|\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\|_{L^1}\\
&\le \frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2,
\end{aligned}
\tag{3.33}
$$

and

$$
\|u(t,\cdot)\|_{L^\infty(\mathbb{S})}^2 \le \frac{e+1}{2(e-1)}\|u(t,\cdot)\|_{H^1(\mathbb{S})}^2 \le \frac{e+1}{2(e-1)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2. \tag{3.34}
$$

Therefore, we obtain the upper bound of f for any $x\in\mathbb{S}$ and $t\in[0,T)$,

$$
\begin{aligned}
f(t,x) &\le \frac{|3-\sigma|}{2}\|u\|_{L^\infty}^2+\frac{(-1+\sinh 1)|\gamma-A|^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2\\
&\quad+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2\\
&\le \frac{|3-\sigma|(e+1)}{4(e-1)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2+\frac{(-1+\sinh 1)|\gamma-A|^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2\\
&\quad+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2\\
&\le \frac{(-1+\sinh 1)|\gamma-A|^2}{8\sinh^2(1/2)}\\
&\quad+\Big[\frac{|3-\sigma|(e+1)}{4(e-1)}+\frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{4\sinh(1/2)}+\frac{1}{4}\Big]\|(u_0,\rho_0)\|_{H^1\times L^2}^2\\
&= \frac{1}{2}C_9^2.
\end{aligned}
\tag{3.35}
$$

Now, we turn to the lower bound of f. Similar as before, we get

$$
\begin{aligned}
-f &\le \frac{|3-\sigma|}{2}\|u\|_{L^\infty}^2+|\gamma-A||G_x*u_x|+|G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2)|+\frac{1}{2}G*\rho^2\\
&\le \frac{|3-\sigma|(e+1)}{4(e-1)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2+\frac{(-1+\sinh 1)|\gamma-A|^2}{8\sinh^2(1/2)}+\frac{1}{4}\|u_x\|_{L^2}^2\\
&\quad+\frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|_{L^2}^2+\frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|_{L^2}^2+\frac{\cosh(1/2)}{4\sinh(1/2)}\|\rho\|_{L^2}^2\\
&\le \frac{(-1+\sinh 1)|\gamma-A|^2}{8\sinh^2(1/2)}\\
&\quad+\Big[\frac{|3-\sigma|(e+1)}{4(e-1)}+\frac{\cosh(1/2)(|3-\sigma|+|\sigma|+1)}{4\sinh(1/2)}+\frac{1}{4}\Big]\|(u_0,\rho_0)\|_{H^1\times L^2}^2.
\end{aligned}
\tag{3.36}
$$

When $\sigma < 0$, we have a finer estimate

$$
\begin{aligned}
-f &\leq |\gamma - A||G_x * u_x| + |G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2)| + \frac{1}{2}G * \rho^2 \\
&\leq \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|_{L^2}^2 + \frac{\cosh(1/2)(3-\sigma)}{4\sinh(1/2)}\|u\|_{L^2}^2 \\
&\quad - \frac{\cosh(1/2)\sigma}{4\sinh(1/2)}\|u_x\|_{L^2}^2 + \frac{\cosh(1/2)}{4\sinh(1/2)}\|\rho\|_{L^2}^2 \\
&\leq \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{\cosh(1/2)(4-\sigma)}{4\sinh(1/2)}\|(u_0,\rho_0)\|_{H^1\times L^2}^2 \\
&= \frac{1}{2}C_{10}^2.
\end{aligned}
\tag{3.37}
$$

Combining (3.35) and (3.36), we obtain

$$
\begin{aligned}
|f| \leq & \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} \\
&+ \Big[\frac{|3-\sigma|(e+1)}{4(e-1)} + \frac{\cosh(1/2)(|3-\sigma| + |\sigma| + 1)}{4\sinh(1/2)} + \frac{1}{4}\Big]\|(u_0,\rho_0)\|_{H^1\times L^2}^2.
\end{aligned}
\tag{3.38}
$$

Since now $s \geq 3$, we have $u \in C_0^1(\mathbb{S})$. Therefore,

$$
\inf_{x\in\mathbb{S}} u_x(t,x) \leq 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) \geq 0, \quad t \in [0,T). \tag{3.39}
$$

Hence, $\bar{m}(t) > 0$ for $t \in [0,T)$. From the second equation of (3.30), we obtain that

$$
\bar{\xi}(t) = \bar{\xi}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}. \tag{3.40}
$$

Hence,

$$
|\rho(t, q_1(t,x_1))| = |\bar{\xi}(t)| \leq |\bar{\xi}(0)| \leq \|\rho_0\|_{L^\infty}.
$$

Now define

$$
P_1(t) = \bar{m}(t) - \|u_{0,x}\|_{L^\infty} - \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_9^2}{\sigma}}.
$$

Note that $P_1(t)$ is a C^1- differentiable function in $[0,T)$ and satisfies

$$
P_1(0) \leq \bar{m}(0) - \|u_0, x\|_{L^\infty} \leq 0.
$$

We will show that

$$
P_1(t) \leq 0, \quad for \quad t \in [0,T). \tag{3.41}
$$

If not, then suppose there is a $t_0 \in [0,T)$ such that $P_1(t_0) > 0$. Define

$$t_1 = \max\{t < t_0 : P_1(t) = 0\}.$$

Then $P_1(t_1) = 0$ and $P_1' \geq 0$, or equivalently,

$$\begin{aligned}
\bar{m}(t_1) =& \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_9^2}{\sigma}}, \\
\bar{m}'(t_1) =& \geq 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\bar{m}'(t_1) &= -\frac{\sigma}{2}\bar{m}^2(t_1) + \frac{1}{2}\bar{\xi}^2(t_1) + f(t_1, q(t_1, x)) \\
&\leq -\frac{\sigma}{2}\left[\|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_9^2}{\sigma}}\right]^2 + \frac{1}{2}\|\rho_0\|_{L^\infty}^2 + \frac{1}{2}C_9^2 \\
&< 0,
\end{aligned}$$

which is a contradiction. Therefore, $P_1(t) \leq 0$, for $t \in [0,T)$, and we obtain (3.21).

(2) To derive a lower bound for u_x in the case of $\sigma < 0$, we consider the functions $m(t)$ and $\xi(t) \in \mathbb{S}$ as in Lemma 3.3.5

$$m(t) := u_x(t, \xi(t)) = \inf_{x \in \mathbb{S}}(u_x(t,x)), \quad t \in [0,T). \tag{3.42}$$

Hence,

$$u_{xx}(t, \xi(t)) = 0, \quad a.e. \quad t \in [0,T). \tag{3.43}$$

Similar as before, we take the characteristic $q_1(t,x)$ defined in (3.10) and choose $x_2(t) \in \mathbb{R}$ such that

$$q_1(t, x_2(t)) = \xi(t) \quad t \in [0,T). \tag{3.44}$$

Let

$$\zeta = \rho(t, q_1(t, x_2)), \quad t \in [0,T). \tag{3.45}$$

Hence, along the trajectory $q_1(t, x_2)$, equation (3.25) and the second equation of (3.2) become

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2}m^2 + \frac{1}{2}\zeta^2 + f(t, q_1(t, x_2)), \\ \zeta'(t) &= -\zeta m. \end{aligned} \tag{3.46}$$

We now define

$$P_2(t) = m(t) - \|u_{0,x}\|_{L^\infty} + \frac{C_{10}}{\sqrt{-\sigma}}.$$

Then $P_2(t)$ is also C^1- differentiable in $[0, T)$ and satisfies

$$P_2(0) \geq m(0) + \|u_{0,x}\|_{L^\infty} \geq 0.$$

We now claim that

$$P_2(t) \geq 0, \quad for \quad t \in [0, T). \tag{3.47}$$

If not, then suppose there is a $\bar{t}_0 \in [0, T)$ such that $P_2(\bar{t}_0) < 0$. Define

$$t_2 = \max\{t < \bar{t}_0 : P_2(t) = 0\}.$$

Then $P_2(t_2) = 0$ and $P_2'(t_2) \leq 0$, or equivalently,

$$m(t_2) = -\|u_{0,x}\|_{L^\infty} - \frac{C_{10}}{\sqrt{-\sigma}} \quad and \quad m'(t_2) \leq 0.$$

On the other hand, we have

$$m'(t_2) = -\frac{\sigma}{2}m^2(t_2) + \frac{1}{2}\zeta^2(t_2) + f(t_2, q(t_2, x)) \geq -\frac{\sigma}{2}(\|u_{0,x}\|_{L^\infty} + \frac{C_{10}}{\sqrt{-\sigma}})^2 - \frac{1}{2}C_{10}^2 > 0.$$

Again, this is a contradiction. Therefore, $P_2(t) \geq 0$, for $t \in [0, T)$. This in turn implies that (3.22) holds. This completes the proof of Lemma 3.4.1. □

It is found that if σu_x is bounded from below, we may obtain the following estimates for $\|\rho\|_{L^\infty(\mathbb{S})}$.

Proposition 3.4.1. *Let $\sigma \neq 0$ and (u, ρ) be the solution of (3.2) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. If there is an $M \geq 0$, such that*

$$\inf_{(t,x)\in[0,T)\times\mathbb{S}} \sigma u_x \geq -M, \tag{3.48}$$

then

1. *If $\sigma > 0$, then*

$$\|\rho(t, \cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{Mt/\sigma}. \tag{3.49}$$

2. *If $\sigma < 0$, then*

$$\|\rho(t, \cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{Nt}. \tag{3.50}$$

Where $N = \|u_{0,x}\|_{L^\infty} + (C_{10}/\sqrt{-\sigma})$ and C_{10} is given in (3.23).

Proof. (1) For $\sigma > 0$, we define for any give $x \in \mathbb{S}$

$$U(t) = u_x(t, q_1(t, x)), \quad \alpha(t) = \rho(t, q_1(t, x)), \tag{3.51}$$

with $q_1(t, x_1(t)) = x$, for some $x_1(t) \in \mathbb{R}, t \in [0, T)$. Then the ρ equation of system (3.2) becomes

$$\alpha' = -\alpha U. \tag{3.52}$$

Thus,

$$\alpha(t) = \alpha(0)e^{-\int_0^t U(\tau)d\tau}. \tag{3.53}$$

From the assumption (3.48) and $\sigma > 0$, we see

$$U(t) \geq -\frac{M}{\sigma}, \quad t \in [0, T).$$

Hence,

$$|\rho(t, q_1(t, x_1))| = |\alpha(t)| \leq |\alpha(0)|e^{-\int_0^t U(\tau)d\tau} \leq \|\rho_0\|_{L^\infty} e^{Mt/\sigma},$$

which together with (3.12), leads to (3.49).

(2) For $\sigma < 0$, we perform a similar argument as before. Using (3.51), (3.53) and the lower bound (3.22), we have

$$|\rho(t, q_1(t, x_1))| = |\alpha(t)| \le |\alpha(0)| e^{-\int_0^t U(\tau) d\tau} \le \|\rho_0\|_{L^\infty} e^{Nt}.$$

Combining the above estimate with (3.12), which implies that (3.50) holds. □

Proof of Theorem 3.4.1. Assume that $T < \infty$ and (3.20) is not valid. Then there is some positive number $M > 0$ such that

$$\sigma u_x(t, x) \ge -M, \quad \forall (t, x) \in [0, T) \times \mathbb{S}.$$

It now follows from Lemma 3.4.1 that $|u_x(t, x)| \le C$, where $C = C(A, M, \sigma, |(u_0, \rho_0)\|^2_{H^s \times H^{s-1}})$. Therefore, Theorem 3.3.2 in turn implies that the maximal existence time $T = \infty$, which contradicts the assumption that $T < \infty$. Conversely, the Sobolev embedding theorem $H^s(\mathbb{S}) \hookrightarrow L^\infty(\mathbb{S})$ with $s > 1/2$ implies that if (3.20) holds, the corresponding solution blows up in finite time. This completes the proof of Theorem 3.4.1. □

Now, we give the following series of theorems that provide some cases that wave breaks in finite time.

Theorem 3.4.2. *Let $\sigma \neq 0$ and (u, ρ) be the solution of (3.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence.*

1. *When $\sigma > 0$, assume that there is some $x_0 \in \mathbb{S}$ such that*

$$\rho_0(x_0) = 0, \quad u_{0,x}(x_0) = \inf_{x \in \mathbb{S}} u_{0,x}(x),$$

and

$$u_{0,x}(x_0) < -\frac{C_9}{\sqrt{\sigma}}, \tag{3.54}$$

where C_9 is defined in (3.23). Then the corresponding solution to system (3.2) blows up in the following sense: there exists a T_1 with

$$0 < T_1 \leq -\frac{2}{\sigma u_{0,x}(x_0) + \sqrt{-\sigma^{3/2} C_9 u_{0,x}(x_0)}}, \tag{3.55}$$

respectively, such that

$$\liminf_{t \to T_1^-} \{\inf_{x \in \mathbb{S}} \sigma u_x(t, x)\} = -\infty.$$

2. *When $\sigma < 0$, assume that there are some $x_0 \in \mathbb{S}$ such that*

$$u_{0,x}(x_0) > -\frac{C_{10}}{\sqrt{\sigma}}, \tag{3.56}$$

where C_{10} is defined in (3.23). Then the corresponding solution to the system (3.2) blows up in finite time in the following sense: there exists a T_2 with

$$0 < T_2 \leq -\frac{2}{\sigma u_{0,x}(x_0) - \sqrt{(-\sigma)^{3/2} C_{10} u_{0,x}(x_0)}}, \tag{3.57}$$

such that

$$\liminf_{t \to T_2^-} \{\sup_{x \in \mathbb{S}} \sigma u_x(t, x)\} = \infty.$$

Proof. (1) When $\sigma > 0$, similar to the proof of Lemma 3.4.1, it suffices to consider $s \geq 3$. So in the following of this section $s = 3$ is taken for simplicity of notation. We consider along the trajectory $q_1(t, x_2)$ defined in (3.10) and (3.44). In this way, we can write the transport equation of ρ in (3.2) along the trajectory of $q_1(t, x_2)$ as

$$\frac{d\rho(t, \xi(t))}{dt} = -\rho(t, \xi(t)) u_x(t, \xi(t)). \tag{3.58}$$

Form the assumption of the theorem, we see

$$m(0) = u_x(0, \xi(0)) = \inf_{x \in \mathbb{S}} u_{0,x}(x) = u_{0,x}(x_0).$$

Hence, we can choose $\xi(0) = x_0$ and then $\rho_0(\xi(0) = \rho_0(x_0) = 0$. Thus, from (3.61) we see that

$$\rho(t, \xi(t)) = 0, \qquad t \in [0, T). \tag{3.59}$$

Differentiating equation (3.24) with respect to x, evaluating the result at $x = \xi(t)$ and using (3.43) and (3.59), we deduce from Lemma 3.3.5 that

$$m'(t) = -\frac{\sigma}{2} m^2(t) + f(t, q_1(t, x_2)). \tag{3.60}$$

Using the upper bound of f in (3.35), we see that

$$m'(t) \le -\frac{\sigma}{2} m^2(t) + \frac{1}{2} C_9^2, \quad t \in [0, T).$$

By assumption (3.57), $m(0) = u_{0,x}(x_0) < -C_9/\sqrt{\sigma}$, we see that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2} \frac{C_9^2}{u_{0,x}^2(x_0)\sigma} \in (0, \frac{1}{2}).$$

Using that $m(t) < m(0) = u_{0,x}(x_0) < 0$, we obtain

$$m'(t) \le -\frac{\sigma}{2} m^2(t) + \frac{1}{2} C_9^2 \le -\delta\sigma m^2(t), \quad t \in [0, T).$$

Integrating on both sides, we obtain

$$m(t) \le \frac{u_{0,x}(x_0)}{1 + \delta\sigma u_{0,x}(x_0)t} \to -\infty \quad as \quad t \to -\frac{1}{\delta\sigma u_{0,x}(x_0)}.$$

Hence,

$$T \le -\frac{1}{\delta\sigma u_{0,x}(x_0)},$$

which proves (3.55).

(2) Similarly as in (1), we consider the function $\bar{m}(t)$and $\eta(t)$ as defined in (3.26). Then we have

$$\bar{m}'(t) = -\frac{\sigma}{2}\bar{m}^2(t) + \frac{1}{2}\rho^2(t, \eta(t)) + f(t, q_1(t, x_1)) \ge -\frac{\sigma}{2}\bar{m}^2(t) + f(t, q_1(t, x_1)). \tag{3.61}$$

Using the lower bound of f as in (3.40), we have

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}C_{10}^2, \quad t \in [0, T).$$

By assumption (3.56), $\bar{m}(0) = u_{0,x}(x_0) > C_{10}/\sqrt{-\sigma}$, we see that $\bar{m}'(0) > 0$ and $\bar{m}(t)$ is strictly increasing over $[0, T)$. Set

$$\theta = \frac{1}{2} - \frac{1}{2}\frac{C_{10}^2}{\sigma u_{0,x}^2(x_0)}.$$

Using that $\bar{m}(t) > \bar{m}(0) = u_{0,x}(x_0) > 0$, we obtain

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}C_{10}^2 \geq -\theta\sigma\bar{m}^2(t), \quad t \in [0, T).$$

Therefore,

$$\bar{m}(t) \geq \frac{u_{0,x}(x_0)}{1 + \theta\sigma u_{0,x}(x_0)t} \to \infty \quad as \quad t \to -\frac{1}{\theta\sigma u_{0,x}(x_0)}.$$

Hence,

$$T \leq -\frac{1}{\theta\sigma u_{0,x}(x_0)},$$

which proves (3.57). □

The following theorem provides another condition for blowup of u_x.

Theorem 3.4.3. *Let $\sigma > 0$ and (u, ρ) be the solution of (3.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Assume that $\int_{\mathbb{S}} u_0 dx = \frac{a_0}{2}$. If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$, $u_{0,x}(x_0) = \inf_{x\in\mathbb{S}} u_{0,x}(x)$, and for any $\varepsilon > 0$*

$$u_{0,x}(x_0) < -\frac{C_{11}}{\sqrt{\sigma}}. \tag{3.62}$$

Then the corresponding solution to system (3.2) blows up in the following sense: there exists a T_1 with

$$0 < T \leq -\frac{2}{\sigma u_{0,x}(x_0) + \sqrt{-\sigma^{3/2}C_{11}u_{0,x}(x_0)}}, \tag{3.63}$$

respectively, such that

$$\liminf_{t\to T^-}\{\inf_{x\in\mathbb{S}}\sigma u_x(t,x)\} = -\infty.$$

The constant above is defined as follows

$$C_{11} = \Big(\frac{(-1+\sinh 1)A^2}{4\sinh^2(1/2)} + \frac{|3-\sigma|(\varepsilon+2)a_0^2}{4\varepsilon} + \Big[\frac{|3-\sigma|(\varepsilon+2)}{24} + \frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{2\sinh(1/2)} + \frac{1}{2}\Big]\|(u_0,\rho_0)\|^2_{H^1\times\ L^2}\Big)^{\frac{1}{2}}.$$

Proof. By Lemma 3.3.3, we have $\int_{\mathbb{S}} u(t,x)dx = \int_{\mathbb{S}} u_0(x)dx = \frac{a_0}{2}$. Using Lemma 3.3.7 and the above conservation law, we have

$$\|u\|_{L^\infty(\mathbb{S})} \le \sqrt{\frac{\varepsilon+2}{24}\|(u_0,\rho_0)\|^2_{H^1(\mathbb{S})\times L^2(\mathbb{S})} + \frac{\varepsilon+2}{4\epsilon}a_0^2}. \tag{3.64}$$

Similarly as the proof of Theorem 3.4.2(1), we can also get

$$m'(t) = -\frac{\sigma}{2}m^2(t) + f(t, q_1(t,x_2)). \tag{3.65}$$

Using (3.64), we obtain a new upper bound of f

$$\begin{aligned}
f &= \frac{3-\sigma}{2}u^2 - (\gamma - A)\partial_x G * \partial_x u - G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2) - \frac{1}{2}G * \rho^2 \\
&\le \frac{3-\sigma}{2}u^2 + |\gamma - A||G_x * u_x| + |G * (\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2)| \\
&\le \frac{|3-\sigma|}{2}\|u\|^2_{L^\infty} + \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} \\
&\quad + \frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|^2_{L^2} + \frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|^2_{L^2} \\
&\le \frac{|3-\sigma|(\varepsilon+2)}{48}\|(u_0,\rho_0)\|^2_{H^1\times L^2} + \frac{|3-\sigma|(\varepsilon+2)a_0^2}{8\varepsilon} + \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} \\
&\quad + \frac{1}{4}\|u_x\|^2_{L^2} + \frac{\cosh(1/2)|3-\sigma|}{4\sinh(1/2)}\|u\|^2_{L^2} + \frac{\cosh(1/2)|\sigma|}{4\sinh(1/2)}\|u_x\|^2_{L^2} \\
&\le \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{|3-\sigma|(\varepsilon+2)a_0^2}{8\varepsilon} \\
&\quad + \Big[\frac{|3-\sigma|(\varepsilon+2)}{48} + \frac{\cosh(1/2)(|3-\sigma|+|\sigma|)}{4\sinh(1/2)} + \frac{1}{4}\Big]\|(u_0,\rho_0)\|^2_{H^1\times\ L^2} \\
&= \frac{1}{2}C_{11}^2.
\end{aligned} \tag{3.66}$$

By assumption (3.62), $m(0) = u_{0,x}(x_0) < -C_{11}/\sqrt{\sigma}$, we see that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2}\frac{C_{11}^2}{u_{0,x}(x_0)^2\sigma} \in \left(0, \frac{1}{2}\right).$$

Using that $m(t) < m(0) = u_{0,x}(x_0) < 0$, we obtain

$$m'(t) \le -\frac{\sigma}{2}m^2(t) + \frac{1}{2}C_{11}^2 \le -\delta\sigma m^2(t), \quad t \in [0, T).$$

Integrating on both sides, we obtain

$$m(t) \le \frac{u_{0,x}(x_0)}{1 + \delta\sigma u_{0,x}(x_0)t} \to -\infty \quad as \quad t \to -\frac{1}{\delta\sigma u_{0,x}(x_0)}.$$

Hence,

$$T \le -\frac{1}{\delta\sigma u_{0,x}(x_0)},$$

which proves (3.63). This complies the proof of Theorem 3.3.3. □

Next, we give a blow-up result if u_0 is odd and ρ_0 is even.

Theorem 3.4.4. *Let $0 < \sigma \le 3$ and (u, ρ) be the solution of (3.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Assume that u_0 is odd, ρ_0 is even, $u_{0,x} < 0$, and $\rho_0(0) = 0$. Then the corresponding solution to the system (3.2) blows up in finite time. More precisely, there exists a T_0 with $0 < T_0 \le -(2/\sigma u_{0,x}(0))$ such that*

$$\liminf_{t \to T_0^-} \{\inf_{x \in \mathbb{S}} \sigma u_x(t, x)\} = -\infty.$$

Proof. Similar to the proof of Lemma 3.4.1, it suffices to consider $s \ge 3$. Since u_0 is odd and ρ_0 is even, the corresponding solution $(u(t, x), \rho(t, x))$ satisfies that $u(t, x)$ is odd and $\rho(t, x)$ is even with respect to x for given $0 < t < T$. Hence, $u(t, 0) = 0$ and $\rho_x(t, 0) = 0$. Thanks to the transport equation of ρ in (3.2), we have

$$\begin{cases} \rho_t(t, 0) + \rho(t, 0)u_x(t, 0) = 0, \\ \rho(0, 0) = 0. \end{cases}$$

Thus, $\rho(t,0)=0$. Evaluating (3.28) at $(t,0)$ and denoting $M(t)=u_x(t,0)$, we obtain

$$M'(t)+\frac{\sigma}{2}M^2(t)=-(\gamma-A)(\partial_x^2G*u)(t,0)-G*(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2)(t,0). \quad (3.67)$$

Notice that $u(t,x)$ is odd and $G(x)$ is even, so

$$(\gamma-A)(\partial_x^2G*u)(t,0)=0.$$

Using $0<\sigma\leq 3$,

$$M'(t)+\frac{\sigma}{2}M^2(t)\leq 0.$$

Hence,

$$M(t)\leq M(0)=u_{0,x}(0)<0, \quad for \quad t\in[0,T),$$

and

$$-\frac{1}{M(t)}+\frac{1}{M(0)}\leq-\frac{\sigma}{2}t,$$

and then

$$u_x(t,0)=M(t)\leq\frac{2M(0)}{2+\sigma M(0)t}\to-\infty, \quad t\to-\frac{2}{\sigma M(0)}, \quad (3.68)$$

which indicates that the maximal existence time $T_0\leq-2(2/\sigma u_{0,x}(0))$ and hence it completes the proof of the theorem. □

3.5 Blow-up Rate

We now address the question of the blow-up rate of the slope to a breaking wave for system (3.2) .

Theorem 3.5.1. *Let $\sigma\neq 0$. If $T<\infty$ is the blow-up time of the solution (u,ρ) to (3.2) with the initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s>3/2$, satisfying the assumption of Theorem* 3.4.2, *then*

$$\lim_{t\to T^-}\left[\left(\inf_{x\in\mathbb{S}}u_x(t,x)\right)(T-t)\right]=-\frac{2}{\sigma}, \quad for \quad \sigma>0. \quad (3.69)$$

$$\lim_{t\to T^-}\left[\left(\sup_{x\in\mathbb{S}}u_x(t,x)\right)(T-t)\right]=-\frac{2}{\sigma}, \quad for \quad \sigma<0. \quad (3.70)$$

Proof. We may again assume $s = 3$ to prove the theorem. Now, let's consider the first case. Let $\sigma > 0$. From (3.60) we have

$$m'(t) = -\frac{\sigma}{2}m^2(t) + f(t, q_1(t, x_2)).$$

Using (3.38) and denote

$$\begin{aligned} K = &\frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} \\ &+ \Big[\frac{|3-\sigma|(e+1)}{4(e-1)} + \frac{\cosh(\frac{1}{2})(|3-\sigma| + |\sigma| + 1)}{4\sinh(\frac{1}{2})} + \frac{1}{4}\Big]\|(u_0, \rho_0)\|^2_{H^1\times L^2}. \end{aligned} \tag{3.71}$$

We know

$$-\frac{\sigma}{2}m^2(t) - K \le m'(t) \le -\frac{\sigma}{2}m^2(t) + K. \tag{3.72}$$

Choose $0 < \varepsilon < \sigma/2$. Since $m(t) \to -\infty$ as $t \to T^-$, we can find $t_0 \in (0, T)$ such that

$$m(t_0) < -\sqrt{2\sigma K + \frac{K}{\varepsilon}}.$$

Since $m(t)$ is absolutely continuous on $[0, T)$. It is then inferred from the above differential inequality that $m(t)$ is strictly decreasing on $[t_0, T)$ and hence

$$m(t) < -\sqrt{2\sigma K + \frac{K}{\varepsilon}} < -\sqrt{\frac{K}{\varepsilon}} \qquad t \in [t_0, T).$$

Then (3.72) implies that

$$\frac{\sigma}{2} - \varepsilon < \frac{d}{dt}\Big(\frac{1}{m(t)}\Big) < \frac{\sigma}{2} + \varepsilon, \quad a.e. \quad t \in [t_0, T).$$

Integrating the above relation on (t, T) with $t \in [t_0, T)$ and noticing that $m(t) \to -\infty$ as $t \to T^-$, we obtain

$$(\frac{\sigma}{2} - \varepsilon)(T - t) < -\frac{1}{m(t)} < (\frac{\sigma}{2} + \varepsilon)(T - t).$$

Since $\varepsilon \in (0, \sigma/2)$ is arbitrary, in view of the definition of $m(t)$, the above inequality implies (3.69).

Next, we consider the second case. Let $\sigma < 0$. From (3.61) we have

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - K,$$

where K is defined in (3.71). Since $\bar{m}(t) \to \infty$ as $t \to T^-$, we can choose a $t_0 \in (0, T)$ such that

$$\bar{m}(t) > \sqrt{-2\sigma K}.$$

Therefore, we have that $\bar{m}(t)$ is strictly increasing on $[t_0, T)$ and

$$\bar{m}(t) > \bar{m}(t_0) > \sqrt{-2\sigma K} > 0.$$

Using the transport equation for ρ, we have that

$$\rho'(t, \eta(t)) = -\bar{m}(t)\rho(t, \eta(t)).$$

Hence,

$$\rho(t, \eta(t)) = \rho(t_0, \eta(t_0))e^{-\int_{t_0}^t \bar{m}(\tau)d\tau}, \quad t \in [t_0, T).$$

Then

$$\rho^2(t, \eta(t)) \leq \rho^2(t_0, \eta(t_0)), \quad t \in [t_0, T).$$

Therefore, using (3.64) again, we have

$$-\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}\rho^2(t_0, \eta(t_0)) - K \leq \bar{m}'(t) \leq -\frac{\sigma}{2}\bar{m}^2(t) + \frac{1}{2}\rho^2(t_0, \eta(t_0)) + K. \quad (3.73)$$

Now let

$$\bar{K} = \frac{1}{2}\rho^2(t_0, \eta(t_0)) + K,$$

and choose $0 < \varepsilon < -\sigma/2$. We can pick a $t_0 \in [t_0, T)$ such that

$$\bar{m}(t_1) > \sqrt{-2\sigma\bar{K} + \frac{\bar{K}}{\varepsilon}}.$$

Then

$$\bar{m}(t) > \bar{m}(t_1) > \sqrt{-2\sigma\bar{K} + \frac{\bar{K}}{\varepsilon}} > \sqrt{\frac{\bar{K}}{\varepsilon}}.$$

Hence, (3.76) implies that

$$\frac{\sigma}{2} - \varepsilon < \frac{d}{dt}\Big(\frac{1}{\bar{m}(t)}\Big) < \frac{\sigma}{2} + \varepsilon, \quad a.e. \quad t \in [t_1, T).$$

Integrating the above relation on (t, T) with $t \in [t_1, T)$ and noticing that $\bar{m}(t) \to \infty$ as $t \to T^-$, we obtain

$$(\frac{\sigma}{2} - \varepsilon)(T - t) < -\frac{1}{\bar{m}(t)} < (\frac{\sigma}{2} + \varepsilon)(T - t).$$

Since $\varepsilon \in (0, -\sigma/2)$ is arbitrary, in view of the definition of $\bar{m}(t)$, the above inequality implies (3.70. □

3.6 Global Existence

In this section, we provide a sufficient condition for the global solution of system (3.2) in the case when $0 < \sigma < 2$ and $\sigma = 0$.

Theorem 3.6.1. *Let $0 < \sigma < 2$ and (u, ρ) be the solution of (3.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. If*

$$\inf_{x\in\mathbb{S}} \rho_0(x) > 0, \tag{3.74}$$

then $T = +\infty$ and the solution (u, ρ) is global.

We need the following lemma to prove the above theorem.

Lemma 3.6.1. *Let $0 < \sigma < 2$ and (u, ρ) be the solution of (3.2) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Assume that $\inf\limits_{x\in\mathbb{S}} \rho_0(x) > 0$.*

1. *If $0 < \sigma \leq 1$, then*

$$\Big|\inf_{x\in\mathbb{S}} u_x(t, x)\Big| \leq \frac{1}{\inf_{x\in\mathbb{S}} \rho_0(x)} C_{13} e^{C_{12} t}, \tag{3.75}$$

$$\Big|\sup_{x\in\mathbb{S}} u_x(t, x)\Big| \leq \frac{1}{\inf_{x\in\mathbb{S}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_{13}^{\frac{1}{2-\sigma}} e^{\frac{C_{12} t}{2-\sigma}}, \tag{3.76}$$

2. If $1 \le \sigma < 2$, *then*

$$\left|\inf_{x\in\mathbb{S}} u_x(t,x)\right| \le \frac{1}{\inf_{x\in\mathbb{S}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_{13}^{\frac{1}{2-\sigma}} e^{\frac{C_{12}t}{2-\sigma}}, \tag{3.77}$$

$$\left|\sup_{x\in\mathbb{S}} u_x(t,x)\right| \le \frac{1}{\inf_{x\in\mathbb{S}} \rho_0(x)} C_{13} e^{C_{12}t}. \tag{3.78}$$

The constants C_{12} *and* C_{13} *are defined as follows*

$$C_{12} = 1 + \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} \tag{3.79}$$

$$+ \Big[\frac{|3-\sigma|(e+1)}{4(e-1)} + \frac{\cosh(\frac{1}{2})(|3-\sigma| + |\sigma| + 1)}{4\sinh(\frac{1}{2})} + \frac{1}{4}\Big] \|(u_0,\rho_0)\|^2_{H^1\times L^2} \tag{3.80}$$

$$C_{13} = 1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_{0,x}\|^2_{L^\infty}. \tag{3.81}$$

Proof. Similar as before, a density argument indicates that it suffices to prove the desired results for $s \ge 3$. Thus, we have

$$\inf_{x\in\mathbb{S}} u_x(t,x) < 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) > 0, \quad t \in [0,T).$$

(1) First we will derive an estimate for $|\inf_{x\in\mathbb{S}} u_x(t,x)|$. Define $m(t)$ and $\xi(t)$ as in (3.42), and consider along the characteristics $q_1(t, x_1(t))$ as in (3.10) and (3.28). Thus, from (3.39),

$$m(t) \le 0 \quad for \quad t \in [0,T). \tag{3.82}$$

Let $\zeta(t) = \rho(t, \xi(t))$ and evaluate (3.25) and the second equation of the system (3.2) at $(t, \xi(t))$. We have

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2} m^2(t) + \frac{1}{2}\zeta^2(t) + f(t, q_1(t,x_1)), \\ \zeta'(t) &= -\zeta m, \end{aligned} \tag{3.83}$$

for $t \in [0,T)$ where f is defined in (3.31). The second equation above implies that $\zeta(t)$ and $\zeta(0)$ are of the same sign.

Now, we want to construct a Lyapunov function for our system, as in [15]. Since here we have a free parameter σ, we could not find a uniform Lyapunov function. Instead, we will split the case $0 < \sigma \leq 1$ and the case $1 < \sigma < 2$. From the assumption of the theorem, we know that $\zeta(0) = \rho(0, \xi(0)) > 0$.

When $0 < \sigma \leq 1$, we define the following Lyapunov function

$$w_1(t) = \zeta(0)\zeta(t) - \frac{\zeta(0)}{\zeta(t)}\Big(1 + m^2(t)\Big), \tag{3.84}$$

which is always positive for $t \in [0, T)$. Differentiating $w_1(t)$ and using (3.82), we obtain

$$\begin{aligned} w_1'(t) &= \zeta(0)\zeta' - \frac{\zeta(0)}{\zeta^2}(1 + m^2)\zeta' + \frac{2}{\zeta}\zeta(0)mm' \\ &= \frac{2\zeta(0)m}{\zeta}\Big[\frac{1-\sigma}{2}m^2 + \frac{1}{2} + f(t, q_1(t, x_1))\Big] \\ &\leq \frac{\zeta(0)}{\zeta}(1 + m^2)\Big[|f(t, q_1(t, x_1))| + \frac{1}{2}\Big] \leq C_{12}w_1(t), \end{aligned} \tag{3.85}$$

where we have used (3.81) and the bound (3.38) for f. Hence,

$$\begin{aligned} w_1(t) &\leq w_1(0)e^{C_{12}t} = [\zeta^2(0) + 1 + m^2(0)]e^{C_{12}t} \\ &\leq (1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty})e^{C_{12}t} = C_{13}e^{C_{12}t}. \end{aligned} \tag{3.86}$$

Recalling that $\zeta(t)$ and $\zeta(0)$ are of the same sign, we have

$$\zeta(0)\zeta(t) \leq w_1(t) \quad \textit{and} \quad |\zeta(0)||m(t)| \leq w_1(t).$$

Then from (3.85), we have

$$\Big|\inf_{x\in\mathbb{S}} u_x(t, x)\Big| = |m(t)| \leq \frac{w_1(t)}{\zeta(0)} \leq \frac{1}{\inf_{x\in\mathbb{S}}\rho_0(x)}C_{13}e^{C_{12}t}, \quad \textit{for} \quad t \in [0, T),$$

which proves (3.75).

If $1 \leq \sigma < 2$, we may define the Lyapunov function to be

$$w_2(t) = \zeta^\sigma(0)\frac{\zeta^2(t) + 1 + m^2(t)}{\zeta^\sigma(t)}. \tag{3.87}$$

Differentiating $w_2(t)$ and using (3.82), we obtain

$$\begin{aligned} w_2'(t) &= \frac{2\zeta^\sigma(0)m}{\zeta^\sigma}\Big[\frac{\sigma-1}{2}\zeta^2+\frac{\sigma}{2}+f(t,q_1(t,x_1)\Big] \\ &\le \frac{\zeta^\sigma(0)}{\zeta^\sigma}(1+m^2)[|f(t,q_1(t,x_1)|+\frac{\sigma}{2}] \le C_{12}w_2(t). \end{aligned} \tag{3.88}$$

Thus,

$$\begin{aligned} w_2(t) &\le w_2(0)e^{C_{12}t} = [\zeta^2(0)+1+m^2(0)]e^{C_{12}t} \\ &\le (1+\|u_{0,x}\|^2_{L^\infty}+\|\rho_0\|^2_{L^\infty})e^{C_{12}t} = C_{13}e^{C_{12}t}. \end{aligned} \tag{3.89}$$

Applying Young's inequality $ab \le a^p/p + b^q/q$ to (3.86) with

$$p=\frac{2}{\sigma} \quad \textit{and} \quad q=\frac{2}{2-\sigma},$$

we have

$$\begin{aligned} \frac{w_2(t)}{\zeta^\sigma(0)} &= [\zeta^{\frac{\sigma(2-\sigma)}{2}}]^{2/\sigma} + \Big[\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\Big]^{2/(2-\sigma)} \\ &\ge \frac{\sigma}{2}[\zeta^{\frac{\sigma(2-\sigma)}{2}}]^{2/\sigma} + \frac{2-\sigma}{2}\Big[\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\Big]^{2/(2-\sigma)} \\ &\ge (1+m^2)^{\frac{2-\sigma}{2}} \ge |m(t)|^{2-\sigma}. \end{aligned}$$

Therefore,

$$\Big|\inf_{x\in\mathbb{S}} u_x(t,x)\Big| \le \Big[\frac{w_2(t)}{\zeta^\sigma(0)}\Big]^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in\mathbb{S}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_{13}^{\frac{1}{2-\sigma}} e^{\frac{C_{12}t}{2-\sigma}}, \quad t\in[0,T),$$

which proves (3.77).

(2) Next we try to control $|\sup_{x\in\mathbb{S}} u_x(t,x)|$. Similarly as before, we consider $\bar{m}(t), \eta(t), q_1(t,x_2(t))$ as in (3.26) and (3.44). Then (3.82) becomes

$$\begin{aligned} \bar{m}'(t) &= -\frac{\sigma}{2}\bar{m}^2(t)+\frac{1}{2}\bar{\zeta}^2(t)+f(t,q_1(t,x_2)), \\ \bar{\zeta}'(t) &= -\bar{\zeta}\bar{m}, \end{aligned} \tag{3.90}$$

for $t\in[0,T)$, where $\bar{\zeta}(t)=\rho(t,\eta(t))$. From (3.39), we have

$$\bar{m}(t) \ge 0, \qquad t\in[0,T). \tag{3.91}$$

When $0<\sigma\le 1$, the corresponding Lyapunov function is

$$\bar{w}_1(t) = \bar{\zeta}^{\sigma}(0)\frac{\bar{\zeta}^2(t)+1+\bar{m}^2(t)}{\bar{\zeta}^{\sigma}(t)}. \tag{3.92}$$

Then from (3.90) and (3.93), we see that

$$\bar{w}_1'(t) \le C_{12}\bar{w}_1(t), \quad then \quad \bar{w}_1(t) \le C_{13}e^{C_{12}t}.$$

Hence, by the similar argument as before, we obtain

$$\frac{\bar{w}_1(t)}{\bar{\zeta}^{\sigma}(0)} \ge |m(t)|^{2-\sigma}.$$

Therefore,

$$\left|\sup_{x\in\mathbb{S}} u_x(t,x)\right| \le \left[\frac{\bar{w}_1(t)}{\bar{\zeta}^{\sigma}(0)}\right]^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in\mathbb{S}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_{13}^{\frac{1}{2-\sigma}} e^{\frac{C_{12}t}{2-\sigma}}, \quad t\in[0,T),$$

which proves (3.76).

When $1\le\sigma<2$, consider the Lyapunov function

$$\bar{w}_2(t) = \bar{\zeta}(0)\bar{\zeta}(t) = \frac{\bar{\zeta}(0)}{\bar{\zeta}(t)}\Big(1+\bar{m}^2(t)\Big). \tag{3.93}$$

From (3.84) and (3.90), it follows that $\bar{w}_2'(t)\le C_{12}\bar{w}_1(t)$. This in turn implies that $\bar{w}_1(t)\le C_{13}e^{C_{12}t}$.

Thus, we have

$$\left|\sup_{x\in\mathbb{S}} u_x(t,x)\right| = |\bar{m}(t)| \le \frac{\bar{w}_1(t)}{\bar{\zeta}(0)} \le \frac{1}{\inf_{x\in\mathbb{S}}\rho_0(x)} C_{13}e^{C_{12}t}, \quad t\in[0,T),$$

which proves (3.78). □

Proof of Theorem 3.6.1. Assume on the contrary that $T<\infty$ and the solution blows up in finite time. It then follows Theorem 3.3.2 that

$$\int_0^T \|u_x(t,x)\|_{L^\infty}dt = \infty. \tag{3.94}$$

However, from the assumption of the theorem and Lemma 3.6.1, we have

$$|u_x(t,x)| < \infty,$$

for all $(t,x) \in [0,T) \times \mathbb{S}$, a contradiction to (3.93). Thus, $T = +\infty$, and the solution (u,ρ) is global. □

We are now in a position to consider the case $\sigma = 0$.

Theorem 3.6.2. *Let $\sigma = 0$. If $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, then there exists a unique solution (u,ρ) of system (3.2) with the initial data (u_0,ρ_0). Moreover, the solution depends continuously on the initial data. Then $T = +\infty$ and the solution (u,ρ) is global.*

When $\sigma = 0$, we can rewrite system (3.2) as

$$\begin{cases} u_t - \gamma u_x = -\partial_x G * (\frac{3}{2}u^2 + (\gamma - A)u + \frac{1}{2}\rho^2), \\ \rho_t + u\rho_x = -u_x\rho, \\ u(0,x) = u_0(x), \\ \rho(0,x) = \rho_0(x), \\ u(t,x+1) = u(t,x), \\ \rho(t,x+1) = \rho(t,x). \end{cases} \tag{3.95}$$

To prove Theorem 3.6.2 of global well-posedness of solutions, we need the following estimates for u_x.

Lemma 3.6.2. *Let $\sigma = 0$ and (u,ρ) be the solution of (3.95) with the initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 3/2$, and T the maximal time of existence. Then*

$$\sup_{x\in\mathbb{S}} u_x(t,x) \le \sup_{x\in\mathbb{S}} u_{0,x}(x) + \frac{1}{2}(\sup_{x\in\mathbb{S}} \rho_0^2(x) + C_{14}^2)t, \tag{3.96}$$

$$\inf_{x\in\mathbb{S}} u_x(t,x) \ge \inf_{x\in\mathbb{S}} u_{0,x}(x) + \frac{1}{2}(\inf_{x\in\mathbb{S}} \rho_0^2(x) - C_{15}^2)t, \tag{3.97}$$

where the constants C_{14} *and* C_{15} *are defined as follows*

$$C_{14} = \sqrt{\frac{(-1+\sinh 1)|\gamma - A|^2}{4\sinh^2(1/2)} + \frac{3(e+1)+1}{4(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}}, \tag{3.98}$$

$$C_{15} = \sqrt{\frac{(-1+\sinh 1)|\gamma - A|^2}{4\sinh^2(1/2)} + \frac{3\cosh(1/2)+1}{4\sinh(1/2)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}}. \tag{3.99}$$

Proof. The local well-posedness theorem and a density argument implies that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Also we may assume that

$$u_0 \not\equiv 0, \tag{3.100}$$

otherwise the results become trivial. Since now $s = 3$, we have $u \in C_0^1(\mathbb{S})$. Therefore

$$\inf_{x\in\mathbb{S}} u_x(t,x) \leq 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) \geq 0, \quad t \in [0,T). \tag{3.101}$$

Differentiating the first equation of (3.97) with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$ we obtain

$$u_{tx} - \gamma u_{xx} = \frac{1}{2}\rho^2 + \frac{3}{2}u^2 + A\partial_x^2 G * u - G * (\frac{3}{2}u^2 + \frac{1}{2}\rho^2). \tag{3.102}$$

Using Lemma 3.3.5 and the fact that

$$\sup_{x\in\mathbb{S}}[u_x(t,x)] = -\inf_{x\in\mathbb{S}}[v_x(t,x)],$$

we can consider $\bar{m}(t)$ and $\bar{\xi}(t)$ as follows

$$\bar{m}(t) := u_x(t,\bar{\xi}(t)) = \sup_{x\in\mathbb{S}}(u_x(t,x)), \quad t \in [0,T). \tag{3.103}$$

Hence

$$u_{xx}(t,\bar{\xi}(t)) = 0, \quad a.e. \quad t \in [0,T). \tag{3.104}$$

Take the trajectory $q(t,x)$ defined in (3.13). Then we know that $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for every $t \in [0,T)$. Therefore, there exists $x_1(t) \in \mathbb{S}$ such that

$$q_(t, x_1(t)) = \xi(t), \quad t \in [0,T). \tag{3.105}$$

Now let

$$\bar{\zeta}(t) = \rho(t, q(t, x_1)), \quad t \in [0, T). \tag{3.106}$$

Therefore along this trajectory $q(t, x_1)$, equation (3.101) and the second equation of (3.94) become

$$\begin{aligned} \bar{m}'(t) &= \frac{1}{2}\bar{\zeta}^2 + f(t, q(t, x_1)), \\ \bar{\zeta}'(t) &= -\bar{\zeta}\bar{m}, \end{aligned} \tag{3.107}$$

for $t \in [0, T)$, where $'$ denotes the derivative with respect to t and $f(t, q(t, x_1))$ is given by

$$f = A\partial_x^2 G * u + \frac{3}{2}u^2 - G * (\frac{3}{2}u^2 + \frac{1}{2}\rho^2). \tag{3.108}$$

We first derive the upper and lower bounds for f for later use in getting the wave-breaking result. Using that $\partial_x^2 G * u = \partial_x G * \partial_x u$, we have

$$f = \frac{3}{2}u^2 + A\partial_x G * \partial_x u - G * (\frac{3}{2}u^2) - \frac{1}{2}G * \rho^2 \leq \frac{3}{2}u^2 + A|G_x * u_x|.$$

Using (3.32) and (3.34), we obtain the upper bound of f

$$\begin{aligned} f &\leq \frac{3}{2}\|u(t,\cdot)\|^2_{L^\infty(\mathbb{S})} + \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} \\ &\leq \frac{3(e+1)}{4(e-1)}\|(u_0, \rho_0)\|^2_{H^1\times L^2} + \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} \\ &\leq \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{3(e+1)+1}{4(e-1)}\|(u_0, \rho_0)\|^2_{H^1\times L^2} = \frac{1}{2}C_{14}^2. \end{aligned} \tag{3.109}$$

Now we turn to the lower bound of f. Similarly as before, we get

$$\begin{aligned} -f &\leq A|G_x * u_x| + \frac{3}{2}|G * u^2| + \frac{1}{2}G * \rho^2 \\ &\leq \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|^2_{L^2} + \frac{3\cosh(1/2)}{4\sinh(1/2)}\|u\|^2_{L^2} + \frac{\cosh(1/2)}{4\sinh(1/2)}\|\rho\|^2_{L^2} \\ &\leq \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \Big(\frac{3\cosh(1/2)+1}{4\sinh(1/2)}\|(u_0, \rho_0)\|^2_{H^1\times L^2}\Big) = \frac{1}{2}C_{15}^2. \end{aligned} \tag{3.110}$$

Combining (3.108) and (3.109), we obtain

$$|f| \le \frac{(-1+\sinh 1)|\gamma - A|^2}{8\sinh^2(1/2)} + \frac{3\cosh(1/2)+1}{4\sinh(1/2)}\|(u_0,\rho_0)\|^2_{H^1\times L^2}. \tag{3.111}$$

From (3.100) we know $\bar{m}(t) \ge 0$ for $t \in [0,T)$. From the second equation of (3.106) we obtain that

$$\bar{\zeta}(t) = \bar{\zeta}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}. \tag{3.112}$$

Hence

$$|\rho(t, q(t, x_1)) = |\bar{\zeta}(t)| \le |\bar{\zeta}(0)|.$$

Therefore, we have

$$\bar{m}'(t) = \frac{1}{2}\bar{\zeta}^2(t) + f \le \frac{1}{2}\bar{\zeta}^2(0) + \frac{1}{2}C_{14}^2 \le \frac{1}{2}\Big(\sup_{x\in\mathbb{S}}\rho_0^2(x) + C_{14}^2\Big).$$

Integrating the above over $[0,t]$, we prove (3.97).

To obtain a lower bound for $\inf_{x\in\mathbb{S}} u_x(t,x)$, we use the similar idea. Consider the function $m(t)$ and $\xi(t)$ as in Lemma 3.3.5

$$m(t) := u_x(t,\xi(t)) = \inf_{x\in\mathbb{S}}(u_x(t,x)), \quad t\in[0,T). \tag{3.113}$$

Hence

$$u_{xx}(t,\xi(t)) = 0, \quad a.e. \quad t\in[0,T). \tag{3.114}$$

Again take the trajectory $q(t,x)$ defined in (3.10) and choose $x_2(t)\in\mathbb{S}$ such that

$$q(t,x_2(t)) = \xi(t) \quad t\in[0,T). \tag{3.115}$$

Now let

$$\bar{\zeta}(t) = \rho(t, q(t,x_2)), \quad t\in[0,T). \tag{3.116}$$

Hence along this trajectory $q(t,x_2)$, equation (3.101) and the second equation of (3.94) become

$$\begin{aligned} m'(t) &= \frac{1}{2}\zeta^2 + f(t, q(t,x_2)), \\ \zeta'(t) &= -\zeta m. \end{aligned} \tag{3.117}$$

Since $m(t) \geq 0$, we have from the second equation of the above that

$$|\rho(t, q(t, x_2)) = |\zeta(t)| \geq |\zeta(0)|.$$

Then

$$m'(t) = \frac{1}{2}\zeta^2(0) - \frac{1}{2}C_{15}^2 \geq \frac{1}{2}\Big(\inf_{x\in\mathbb{S}} \rho_0^2(x) + C_{15}^2\Big).$$

Integrating the above over $[0, t]$, we obtain (3.96). This completes the proof of the Lemma 3.6.2. □

Proof of Theorem 3.6.2. Similarly as the proof of Lemma 3.6.1, assume on the contrary that $T < \infty$ and the solution blows up in finite time. It then follows Theorem 3.3.2 that

$$\int_0^T \|u_x(t, x)\|_{L^\infty} dt = \infty. \tag{3.118}$$

However, from the assumption of the theorem and Lemma 3.6.2, we have

$$|u_x(t, x)| < \infty,$$

for all $(t, x) \in [0, T) \times \mathbb{S}$, a contradiction to (3.117). Thus, $T = +\infty$, and the solution (u, ρ) is global. □

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

4.1 Conclusions

The goal of the present Chapter 2 is to derive some conditions of blow-up solutions and determine blow-up rate for the system (2.2). The global existence of solutions is also studied. Similarly as in [18, 26], we can use the method of Besov spaces together with the transport equation theory to show that system (2.2) is locally well posed in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s > 3/2$. For system (2.2) , when $\sigma \neq 1$, it has two different characteristics, which are described in (2.4) and (2.5). Since there is no uniform characteristics, in order to obtain similar estimates, one needs more regularity arguments. The way to resolve this issue is to employ the method of characteristics along a properly chosen q_1 which is showed in (2.4) to capture the maximum/minimum of u_x. Moreover, we use the method of characteristics together with the use of the conservation laws to get the improved estimate of u_x, which is always uniformly bounded from above. To study the problem of the global existence of solutions, we use the method of Lyapunov functions introduced in [18, 8]. We find a sufficient condition for global solutions which is determined only by a positive profile of the free surface component ρ of the system, in the case $0 < \sigma < 2$ and $\sigma = 0$. However, the case when $\sigma < 0$ or $\sigma \geq 2$ still remain open at this moment.

The goal of the present Chapter 3 is to derive the generalized periodic two-component DGH system (3.2) by the shallow water theory, then we derive some conditions of blow-up solutions and determine blow-up rate for the generalized periodic system (3.2). Similarly as in [18, 26], we can use the method of Besov spaces together with the transport equation theory to show that system (3.2) is locally well posed in $H^s(\mathbb{S}) \times H^s(\mathbb{S})$ with $s > 3/2$. Moreover, we use the method of characteristics together with the use of the conservation laws to get the improved estimate of u_x, which is always uniformly bounded from above. To study the problem of the global existence of solutions, we use the method of Lyapunov functions introduced in [18]. We find a sufficient condition for global solutions which is determined only by a nonzero profile to the free surface component ρ of the system.

4.2 Future Work

- Specific analysis and numerical simulation of stability of solitary waves of the generalized periodic two-component Camassa-Holm system and the generalized periodic two-component Dullin-Gottwald-Holm System.
- Global weak solutions for the generalized periodic two-component Dullin-Gottwald-Holm System with $0 \leq \sigma < 2$.
- Global existence and global weak solutions for the generalized periodic two-component Camassa-Holm system and the generalized periodic two-component Dullin-Gottwald-Holm System as $\sigma < 0$ or $\sigma \geq 2$.

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BIOGRAPHICAL STATEMENT

Caixia Chen was born in Shandong, China, in 1982. She received her B.S. degree from Ludong University, China, in 2004, her M.S. degree from Jiangsu University, China, in 2007. In 2008, She joined the doctoral program in the Department of Mathematics, University of Texas at Arlington. Apart from Mathematical research she takes a keen interest in Physics and space science. She is a very passionate teacher and encourages young people to study the sciences. Her current research interests are, but not limited to, the mathematical questions of well-posedness, creation of singularities, wave-breaking phenomena, the existence and stability of steady solutions and traveling waves as well as the physical derivation of the models.