**ESTIMATING TRANSITION PROBABILITIES
FROM AGGREGATE SAMPLES AUGMENTED
BY HAPHAZARD RECAPTURES**

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Technical Report #302

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Abstract

Transition probabilities provide a convenient summary of changes in a categorical trait over time in a population. The difficulties of estimating such probabilities based on only aggregate data from repeated sampling are well known. We give here a method for augmenting aggregate data with haphazard recapture data, which can dramatically improve the estimation precision of transition probabilities. The method requires a rather high sampling fraction to provide sufficient numbers of recaptures. It is based on a generalized nonlinear least squares strategy which yields transition probability estimates preserving their natural parameter space, and which are asymptotically efficient. The asymptotic theory is given under finite population sampling assumptions which are typical in practice.

1. Introduction

The problem of measuring change over time, based on repeated samples, in a population parameter descriptive of a trait Z is an old one. It is widely recognized (see eg. Cochran (1977), Lee et al. (1977)) that a better estimate of the change in the parameter is obtained if one can measure Z on the same units at each stage of sampling (so that change at the unit level is measured), than if one can only measure Z in aggregate samples of units at each stage. However, logistical constraints often preclude tracking individual units through time, so that aggregate sampling for assessment of change is the best one can do.

For the case where Z is a qualitative trait, we discuss here a strategy for estimating transition probabilities from aggregate data, augmented by haphazard recaptures (i.e. units which haphazardly occur in the samples at two consecutive stages of the sampling). Such augmentation

is motivated by the generally poor estimates which result from using only the aggregate data. However, for such recaptures to be available in sufficient numbers to be useful, the sampling fraction must be reasonably high, so this strategy will not always be viable. We have applied it here (see Section 3) to a situation with sampling fraction .2 with good results.

The paper is organized as follows. The method is detailed in section 2. In 2.1 we use a finite population sampling (*fps*) framework for defining transition probabilities relative to a time-homogeneous Markov model. In 2.2 we develop an estimation scheme by incorporating the *fps* estimates of these parameters into a nonlinear regression model in order to efficiently estimate the transition probabilities while preserving their natural parameter space. We then obtain the limit distribution of the transition probability estimates as the population size $N \rightarrow \infty$. Unfortunately the theory of the asymptotic distributions of nonlinear least squares estimates in our non-standard setup was not available to the authors' knowledge, and had to be derived. This is done in 2.5. An example application is given in Section 3. Brief sketches of the proofs of the main theorems are given in the appendix. Complete proofs and other details are available in the companion technical report Hawkins et al. (1995) (henceforth called HHE).

2. Methods

2.1. The Target Parameters.

We adopt a *fps* strategy. Let $l=1, \dots, N$ index the members of the finite population of units under study. Let $t=1, \dots, r$ index the sampling waves. Let Z be a qualitative trait with $m \geq 2$ categories defined for each unit. For $1 \leq i \leq m$, let $Z_{lti}=1$ if unit l has $Z=i$ at wave t , and otherwise be zero. Finally define the $m \times 1$ vector $Z_{lt}^0 = (Z_{lt1}, \dots, Z_{ltm})^T$. (Note that Z_{lt}^0 is non-random.)

We introduce a time-homogeneous Markov model as follows. Define the $m \times 1$ vector parameter

$$\underline{q}_{Nt}^0 = \frac{1}{N} \sum_{l=1}^N Z_{lt}^0 ,$$

whose typical element $q_{Nti}^0, 1 \leq i \leq m$ is the fraction of the population with

$Z=i$ at wave t . Consider first a more general time-dependent Markov model consisting of the assertion that the sequence $q_{N1}^0, \dots, q_{Nr}^0$ satisfies

$$q_{N,t+1}^0 = P_{t,t+1}^T q_{Nt}^0, \quad t=1, \dots, r-1 \quad (2.1)$$

for some $m \times m$ transition matrices $P_{t,t+1} = (p_{ij}^{t,t+1}; i, j = 1, \dots, m)$. Model (2.1) is vacuous unless there actually exist such transition matrices $P_{t,t+1}$ defined in terms of relevant population quantities. To see that this is indeed the case, define

$$p_{ij}^{t,t+1} = \frac{1}{N} \sum_{l=1}^N Z_{lti} Z_{l,t+1,j} / \frac{1}{N} \sum_{l=1}^N Z_{lti}. \quad (2.2)$$

Then it may be verified that (2.1) holds with $p_{ij}^{t,t+1}$ defined by (2.2). Thus, $p_{ij}^{t,t+1}$ represents the fraction of the sub-population satisfying $Z=i$ at wave t , who subsequently satisfied $Z=j$ at wave $t+1$.

The time homogeneity of the Markov model (2.1) consists of $p_{ij}^{t,t+1}$ being the same for all t ; i.e. $p_{ij}^{t,t+1} =: p_{ij}, t=1, \dots, r-1$. If this condition holds, then, it may be verified (see HHE) that p_{ij} must have the representation.

$$p_{ij} = \sum_{t=1}^{r-1} \sum_{l=1}^N Z_{lti} Z_{l,t+1,j} / \sum_{t=1}^{r-1} \sum_{l=1}^N Z_{lti}. \quad (2.3)$$

Since $p_{ij} = p_{ij}^{t,t+1}$, $t=1, \dots, r-1$ must satisfy (2.1), time homogeneity implies that (2.1) holds with $P = (p_{ij})$:

$$q_{N,t+1}^0 = P^T q_{Nt}^0, \quad t=1, \dots, r-1. \quad (2.4)$$

The ensuing methods assume model (2.4) - i.e. the time homogeneous Markov model. The representation (2.3) is utilized in Section 2.4 to construct estimates of the p_{ij} .

A few comments about (2.4) are in order here.

(1) Since we are using a finite population sampling approach, the q_t^0 's

are not, strictly speaking, probability distributions, so the p_{ij} 's are not, strictly speaking, transition probabilities. Rather, since the $\underline{q}_t^0$'s are summaries of the distribution of the (non-random) trait Z in the population at the waves t , p_{ij} should be viewed as a summary measure of between-waves movement across the values of Z at the population level. Of course, the p_{ij} 's are particularly effective measures of change, since by their conditional nature they avoid problematic issues of unequal initial distributions $\underline{q}_1^0$ when comparing changes across several study populations. (2) The view of the p_{ij} 's in comment (1) above is in contrast with that in other treatments of the aggregate data problem in the literature where (see e.g. Kabfleish and Lawless (1984)) the sequence $\{Z_{lti}; t=1, \dots, r\}$ for unit l is assumed to be stochastic, specifically a realization of a Markov chain governed by a transition matrix of probabilities p_{ij} common to a conceptual population represented by unit l . The essential difference between views (1) and (2) is modeling at the population level (as in (1)) rather than at the unit level (as in (2)).

To avoid linear redundancies in our estimation maneuvers, we can restrict to "sufficient parts" of the parameters. To this end, define $\underline{q}_t$ as the $(m-1) \times 1$ vector containing the first $m-1$ elements of $\underline{q}_t^0$. Similarly, for $i=1, \dots, m$ let $\underline{p}_i^T = (p_{i1}, \dots, p_{i,m-1})$ (recalling that $p_{im} = 1 - \sum_{j=1}^{m-1} p_{ij}$).

2.2 The Estimation Scheme

Standard approaches to the problem of estimating the p_{ij} 's from aggregate data are based on (2.4) with estimates plugged in for the $\underline{q}_t$'s (see Lee et al (1977), Kabfleish & Lawless (1984)). These methods either fail to constrain the p_{ij} 's to their natural parameter space (Kabfleish & Lawless (1984)) or fall short of offering statistical properties of resulting constrained estimates (Lee et al). In either case the estimates perform poorly - see Lawless & McLeish (1984).

The approach here uses a reparameterization of the p_{ij} 's to preserve their parameter space, yielding a nonlinear model in the new parameters β_{ij} in place of (2.4). The model is then augmented (in the Goldberger - Theil

sense of combining information) with the "recapture" estimates of the p_{ij} 's, for which an asymptotic expectation model in terms of the β_{ij} 's may be formulated. The augmented model parameters β_{ij} are then efficiently estimated by generalized nonlinear least squares (gls), using an efficient weight matrix derived under our *fps* assumptions. Delta method approximations are used to produce the covariance matrix of the estimated p_{ij} 's.

The remainder of this section defines the key features of the nonlinear model described above. Details of the estimates involved are given in section 2.4. In this direction let $\hat{q}_t$, $t=1, \dots, r$ denote the estimate of q_t and let $\hat{p}_i$ denote the estimate of p_i , $i=1, \dots, m$.

The augmented reparameterized nonlinear model has the form

$$\underline{Y} = \underline{F}(\underline{\beta}; \underline{\hat{q}}^*) + \underline{\epsilon}, \text{ where} \quad (2.5)$$

$$\underline{Y} = \begin{bmatrix} \underline{Y}_1 \\ \dots \\ \underline{Y}_2 \end{bmatrix} \begin{matrix} (r-1)(m-1) \times 1 \\ \\ m(m-1) \times 1 \end{matrix}, \quad \underline{\epsilon} = \begin{bmatrix} \underline{\epsilon}_1 \\ \dots \\ \underline{\epsilon}_2 \end{bmatrix}, \quad (2.6)$$

$$\underline{F} = \begin{bmatrix} \underline{F}_1(\underline{\beta}; \underline{\hat{q}}^*) \\ \dots \\ \underline{F}_2(\underline{\beta}) \end{bmatrix} \begin{matrix} (r-1)(m-1) \times 1 \\ \\ m(m-1) \times 1 \end{matrix}, \quad (2.7)$$

$$\underline{\hat{q}}^* = [\underline{\hat{q}}_1^T, \dots, \underline{\hat{q}}_r^T]^T \quad (r(m-1) \times 1),$$

$$\underline{Y}_1 = A \underline{\hat{q}}^* = \begin{bmatrix} \underline{\hat{q}}_2^* \\ \dots \\ \underline{\hat{q}}_r^* \end{bmatrix}, \quad A = \begin{bmatrix} 0_{(r-1)(m-1) \times (m-1)} & I_{(r-1)(m-1)} \end{bmatrix}, \quad (2.8)$$

$$\underline{Y}_2 = [\hat{p}_1^T, \dots, \hat{p}_m^T]^T \quad (m(m-1) \times 1). \quad (2.9)$$

Thus model (2.5) has two parts, the first part representing (via $\underline{F}_1$) the aggregate estimates in $\underline{Y}_1$ in terms of model (2.4), the second part representing (via the mean function $\underline{F}_2$) the transition probability estimates $\underline{Y}_2$ from the haphazard recaptures.

The mean functions $F_1(\underline{\beta}; \hat{\underline{q}}^*)$ and $F_2(\underline{\beta})$ involve the reparameterization of the p_{ij} 's in terms of $\underline{\beta} = (\beta_{11}, \dots, \beta_{1,m-1}, \beta_{21}, \dots, \beta_{2,m-1}, \dots, \beta_{m1}, \dots, \beta_{m,m-1})^T$ ($m(m-1) \times 1$) as follows:

$$p_{ij} = p_{ij}(\underline{\beta}) = e^{\beta_{ij}} / [1 + \sum_{k=1}^{m-1} e^{\beta_{ik}}], \quad 1 \leq j \leq m-1, \quad 1 \leq i \leq m. \quad (2.10)$$

Clearly $0 \leq p_{ij}(\underline{\beta}) \leq 1$, $\sum_{j=1}^{m-1} p_{ij}(\underline{\beta}) \leq 1$ for each i and

$$p_{im}(\underline{\beta}) = 1 / [1 + \sum_{k=1}^{m-1} e^{\beta_{ik}}]. \quad (2.11)$$

Also, $\underline{\beta} \in IR^{m(m-1)}$ is an unconstrained parameter.

To get the elements of $F_1(\underline{\beta}; \hat{\underline{q}}^*)$, write

$$F_1(\underline{\beta}; \hat{\underline{q}}^*) = \begin{bmatrix} F_{11} \\ F_{12} \\ \dots \\ F_{1,r-1} \end{bmatrix}, \quad F_{1t} = \begin{bmatrix} F_{1t1} \\ F_{1t2} \\ \dots \\ F_{1t,m-1} \end{bmatrix} \quad (2.12)$$

and then write out the right side of (2.4) in scalar form to get

$$F_{1ti}(\underline{\beta}; \hat{\underline{q}}^*) = \sum_{j=1}^m \hat{q}_{tj} p_{ji}(\underline{\beta}), \quad 1 \leq t \leq r-1, 1 \leq i \leq m-1. \quad (2.13)$$

This part (from aggregate data) of model (2.5) arises essentially by plugging estimates of the $\underline{q}_t$'s into both sides of (2.4). The justification for casting this part of the model in the form (2.5) is that the difference $\underline{\epsilon}_1 = \underline{Y}_1 - F_1(\underline{\beta}_T; \hat{\underline{q}}^*)$ may be considered as being approximately multivariate normal with zero mean if model (2.4) actually holds, based on *fps* central limit theory (see Lemma 2.3 in Section 2.5.3).

The "recaptures" part of (2.5) ($\underline{Y}_2$ and $F_2(\underline{\beta})$) is justified as follows. The estimate $\hat{\underline{p}}_i$ of $\underline{p}_i$ may be considered to be approximately multivariate normal with mean $\underline{p}_i$ and variance matrix $\text{Var}(\hat{\underline{p}}_i)$ by virtue of *fps* central

limit theory (see Theorem 2.1 of Section 2.5.2). Thus, if we define

$$F_2(\beta) = \begin{bmatrix} F_{21} \\ F_{22} \\ \vdots \\ F_{2m} \end{bmatrix}, \quad F_{2k} = \begin{bmatrix} F_{2k1} \\ \vdots \\ F_{2k,m-1} \end{bmatrix}, \text{ and}$$

$$F_{2kj}(\beta) = p_{kj}(\beta), 1 \leq k \leq m, 1 \leq j \leq m-1 \quad (2.14)$$

with $p_{kj}(\beta)$ given by (2.10), then $F_2(\beta)$ is the asymptotic mean of Y_2 so long as (2.4) holds. The fact that ϵ_1 and $\epsilon_2 = Y_2 - F_2(\beta)$ are approximately jointly normal supports suitably weighted least squares as an estimation strategy. However, compensation for the "random regressor" $\hat{q}^*$ in F_1 will need to be made if an asymptotically optimal generalized least squares estimator $\hat{\beta}_{GLS}$ is to be constructed; see section 2.5.3.

2.3 The Sampling Design

Since this directly impacts the form of the matrix $Var(Y)$, we detail our assumptions here.

Briefly, we assume that at wave t , we select a simple random sample of size n_t , where n_t is determined in advance. Further we assume that the samples at waves t and t' are independent if $t \neq t'$. Let $f_t = n_t/N$ denote the sampling fraction.

In more precise terms, for wave t define random indicator variables $I_{lt} = 1$ if unit l is selected in the sample and $I_{lt} = 0$ otherwise, $l = 1, \dots, N$. Let $\alpha_{lt} = E(I_{lt})$ be the design probability. Then we assume that $\sum_{l=1}^N I_{lt} = n_t$, that I_{lt} and $I_{l't'}$ are independent for $t \neq t'$ and all l, l' and that $\alpha_{lt} = n_t/N$.

These assumptions are reasonable in our application in section 3, but in principle the methods discussed here could be adapted to other sampling designs (e.g. unequal α_{lt} 's, correlated sampling indicators across waves,

etc.). For brevity such extensions are not considered here.

2.4 The Estimates of q_t and p_{ij}

Let

$$\hat{q}_t = \frac{1}{n_t} \sum_{l=1}^N Z_{lt} I_{lt} \quad (2.15)$$

where $\underline{Z}_{lt}^T = (Z_{lt1}, \dots, Z_{lt, m-1})$. Then under our assumptions in section 2.3 about the sampling process, $\hat{q}_t$ is (design) unbiased for q_t , with covariance matrix

$$\text{Var}(\hat{q}_t) = \frac{1}{n_t}(1-f_t) \frac{1}{N-1} \sum_{l=1}^N (\underline{Z}_{lt} - \underline{q}_t)(\underline{Z}_{lt} - \underline{q}_t)^T. \quad (2.16)$$

A consistent estimate of $\text{Var}(\hat{q}_t)$ is

$$\begin{aligned} \hat{\text{Var}}(\hat{q}_t) &= \frac{1}{n_t}(1-f_t) \frac{1}{n_t-1} \sum_{l=1}^N (\underline{Z}_{lt} - \hat{q}_t)(\underline{Z}_{lt} - \hat{q}_t)^T I_{lt} \\ &= \frac{1}{n_t-1}(1-f_t) \{ \text{Diag}(\hat{q}_t) - \hat{q}_t \hat{q}_t^T \}, \end{aligned} \quad (2.17)$$

which requires that the sampling fractions f_t be either known or ignorable.

By virtue of the independence of the I_{lt} 's for distinct t , $\hat{q}_t$ and $\hat{q}_{t'}$ are independent if $t \neq t'$, so $\text{Var}(\hat{q}^*) = \text{Diag}(\text{Var}(\hat{q}_t): t=1, \dots, r)$ and $\text{Var}(\underline{Y}_1) = \underline{A} \text{Var}(\hat{q}^*) \underline{A}^T$.

In view of (2.3), a natural estimate of p_{ij} is $\hat{p}_{ij} = R_{ij}/D_i$, where

$$\begin{aligned} R_{ij} &= \sum_{t=1}^{r-1} (n_t n_{t+1})^{-1} \sum_{l=1}^N Z_{lti} Z_{l,t+1,j} I_{lt} I_{l,t+1} \quad \text{and} \\ D_i &= \sum_{t=1}^{r-1} (n_t n_{t+1})^{-1} \sum_{l=1}^N Z_{lti} I_{lt} I_{l,t+1}. \end{aligned} \quad (2.18)$$

It is easy to check that $N \cdot R_{ij}$ and $N \cdot D_i$ are, respectively, unbiased for the numerator and denominator of p_{ij} in (2.3). (This is the reason for using the divisor $n_t n_{t+1}/N$, which is the expected number of recaptures, rather than the observed number $n_{t,t+1}$ of recaptures, which is a random variable.)

Define $\hat{p}_i = (\hat{p}_{i1}, \dots, \hat{p}_{i, m-1})^T$, which naturally estimates p_i . The $\hat{p}_i, i=1, \dots, m$ will be arranged in the $\underline{Y}_2$ - component of the data vector in

model (2.5) as shown in (2.9). Estimates of $Var(\underline{Y}_2)$ and $cov(\underline{Y}_1, \underline{Y}_2)$ are developed in section 2.5.4.

Since the data will typically be available in some summarized form (and not in terms of the indicators Z_{lti}), we now give the sample estimates $\hat{q}_{ti}$ and $\hat{p}_{ij}$ in terms of summarized data quantities.

The marginal aggregate data (for the $\hat{q}_t$'s) may be arranged into the form of Table 2.1.

Table 2.1. Marginal Aggregate Data Frequency Table

wave	Value of Z		
	$i = 1$	$\dots$	m
$t = 1$	f_{11}	$\dots$	f_{1m}
$\vdots$	$\vdots$		$\vdots$
r	f_{r1}	$\dots$	f_{rm}
			n_r

$$n_t = \sum_{i=1}^m f_{ti}$$

$$f_{ti} = \sum_{l=1}^N Z_{lti} I_{lt} = \text{number of sampled units}$$

with $Z = i$ at wave t

Of course, for $1 \leq t \leq r$, $\hat{q}_t^0 = [\hat{q}_{t1}, \dots, \hat{q}_{tm}]^T$, with $\hat{q}_{ti} = f_{ti}/n_t$.

The recapture data may be arranged into the form of Table 2.2.

Table 2.2 Transition Data from Haphazard Recaptures

t	wave pair	$i = 1$				$i = m$			
		$Z = 1$ at wave t , $Z = j$ at wave $t+1$				$Z = m$ at wave t , $Z = j$ at wave $t+1$			
		$n_{t,t+1}$	$j = 1$	...	m	- - -	$j = 1$	...	m
1	(1,2)	n_{12}	$n_{11}^{1,2}$	...	$n_{1m}^{1,2}$	...	$n_{m1}^{1,2}$	...	$n_{mm}^{1,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$		
t	($t, t+1$)	$n_{t,t+1}$	$n_{11}^{t,t+1}$	...	$n_{1m}^{t,t+1}$	...	$n_{m1}^{t,t+1}$	...	$n_{mm}^{t,t+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$		
$r-1$	($r-1, r$)	$n_{r-1,r}$	$n_{11}^{r-1,r}$	...	$n_{1m}^{r-1,r}$	...	$n_{m1}^{r-1,r}$	...	$n_{mm}^{r-1,r}$
		R_{ij}	R_{11}	...	R_{1m}	...	R_{m1}	...	R_{mm}
		D_i		D_1		...		D_m	
		$\hat{p}_{ij}$	$\hat{p}_{11}$	...	$\hat{p}_{1m}$	...	$\hat{p}_{m1}$	...	$\hat{p}_{mm}$

Here

$$n_{ij}^{t,t+1} = \sum_{l=1}^N Z_{l ti} Z_{l, t+1, j} I_{lt} I_{l, t+1} = \text{number of recaptures for wave pair}$$

($t, t+1$) which had $Z=i$ at wave t and $Z=j$ at wave $t+1$, and

$$n_{t,t+1} = \sum_{i=1}^m \sum_{j=1}^m n_{ij}^{t,t+1} \text{ is the total number of recaptures in the data for}$$

wave pair ($t, t+1$). The R_{ij} 's are the weighted column sums

$$R_{ij} = \sum_{t=1}^{r-1} (n_t n_{t+1})^{-1} n_{ij}^{t,t+1}, \text{ and } \hat{p}_{ij} = R_{ij} / D_i. \text{ The quantity } D_i \text{ may be shown to}$$

$$\text{equal } \sum_{j=1}^m R_{ij}.$$

2.5. The Asymptotically Optimal Nonlinear Least Squares Estimate of β and its Asymptotic Distribution

2.5.1. The Asymptotic Framework

To obtain an asymptotic theory for the finite population sampling model, we must make assumptions about the behavior of certain population quantities as the population size $N \rightarrow \infty$. These assumptions are to some degree artificial, but serve as rough guides to the data requirements needed for reliable results.

We assume the following of the finite population trait vectors $Z_{Nlt}^0 \equiv Z_{Nlt}^0$ as $N \rightarrow \infty$:

$$(A1) \quad \bar{Z}_{Nt}^0 = \frac{1}{N} \sum_{l=1}^N Z_{Nlt}^0 \rightarrow \bar{Z}_t^0, \text{ a finite vector, } t=1, \dots, r.$$

$$(A2) \quad \frac{1}{N} \sum_{l=1}^N (Z_{Nlt}^0 - \bar{Z}_{Nt}^0)(Z_{Nlt}^0 - \bar{Z}_{Nt}^0)^T \rightarrow \Gamma_t^0, \text{ a finite matrix, } t=1, \dots, r.$$

$$(A3) \quad \text{The wave sample sizes } n_t, \quad 1 \leq t \leq r, \text{ satisfy } n_t = n_N \lambda_t, \text{ where } \lambda_t > 0, \lambda_1 + \dots + \lambda_r = 1 \text{ and } n_N \rightarrow \infty, \text{ but } n_N/N \rightarrow 0, \text{ as } N \rightarrow \infty. \quad (\text{Note that this implies } \sum_{t=1}^r n_t = n_N).$$

$$(A4) \quad n_N^{3/2}/N \rightarrow \infty.$$

$$(A5) \quad \text{The marginal wave vectors } \{q_1, \dots, q_{r-1}\} \text{ span } \mathbb{R}^m \text{ (so } r-1 \geq m \text{ must hold)}.$$

$$\text{For later use let } g_{Nt} = n_t n_{t+1}/N \text{ and } g_N = \sum_{t=1}^r g_{Nt}.$$

In regard to these assumptions, (A4) implies (actually more than) that the expected number of repeats between waves t and $t+1$, which is $n_t n_{t+1}/N$, is $O(n_N^2/N) \rightarrow \infty$ as $N \rightarrow \infty$. This is a necessary assumption if the $\hat{p}_i$'s are to be approximately normal. Assumption (A5), which is needed for both the consistency and asymptotic normality of the nonlinear least squares estimator, is discussed (essentially) in Kabfleisch and Lawless (1984). It implies that if the population is near a stationary distribution, then the methods discussed here will not work.

2.5.2 Asymptotic Normality of the Basic Estimates

Since the result is used in the sequel, and is of interest in its own right, we first obtain the asymptotic joint normality of the components of $\underline{Y}$ in model (2.5). Of some note here are the different rates of convergence of the $\underline{Y}_1$ and $\underline{Y}_2$ components.

Put $\underline{W}_N^T = [\hat{\underline{q}}_N^{*T}, \underline{Y}_{2N}^T]$. (Note the addition of the subscript N for these asymptotic results.) It will be convenient to write the i^{th} component ($1 \leq i \leq m$) $\hat{\underline{p}}_i$ of $\underline{Y}_{2N}$ in the form $\hat{\underline{p}}_i = \underline{R}_i / D_i$, where

$$\underline{R}_i = \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N Z_{lti} Z_{l,t+1} I_{lt} I_{l,t+1},$$

$$D_i = \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N Z_{lti} I_{lt} I_{l,t+1}.$$

Let

$$\underline{B}_N = \text{Blockdiag} \left\{ n_1^{\frac{1}{2}} \underline{I}_{m-1}, \dots, n_r^{\frac{1}{2}} \underline{I}_{m-1}, \underbrace{g_N^{\frac{1}{2}} \underline{I}_{m-1}, \dots, g_N^{\frac{1}{2}} \underline{I}_{m-1}}_{m \text{ times}} \right\},$$

$$\underline{R}_i^* = \lim_{N \rightarrow \infty} E\left(\frac{1}{N} \underline{R}_i\right), \quad D_i^* = \lim_{N \rightarrow \infty} E\left(\frac{1}{N} D_i\right), \quad \underline{q}^* = \lim_{N \rightarrow \infty} E(\hat{\underline{q}}_N^*)$$

(all assumed to exist), and set $\underline{p}_i^* = \underline{R}_i^* / D_i^*$.

Let $\underline{W}^* = [\underline{q}^{*T}, \underline{p}_1^{*T}, \dots, \underline{p}_m^{*T}]^T$.

Theorem 2.1. Under model (2.4) and assumptions (A1)-(A4),

$$\underline{B}_N \{\underline{W}_N - \underline{W}^*\} \xrightarrow{L} N(\underline{0}, \underline{\Gamma}_W) \text{ as } N \rightarrow \infty,$$

for some p.d. covariance matrix $\underline{\Gamma}_W$.

The exact form of $\underline{\Gamma}_W$ is of no interest, since we shall never use it. The scaling matrix $\underline{B}_N$ gives the different rates of convergence of the aggregate estimate $\hat{\underline{q}}_N^*$ and the transition estimates $\hat{\underline{p}}_i$.

2.5.3 The Estimate of β and its Asymptotic Distribution

There are several complicating factors. First, the asymptotically optimal weight matrix (for *gls*) depends on the random regressor $\hat{q}_N^*$ in model (2.5). Second, the sampling process assumed here (*fps*) is not the usual one (iid sampling) assumed for nonlinear least squares distribution theory, so the arguments needed to get the asymptotic distribution of the nonlinear least squares estimate are non-standard. Finally, the "data" vector $\underline{Y}$ in (2.5) contains two types of estimates: aggregate and transition estimates, which converge at different rates as $N \rightarrow \infty$.

The following general theorem accomodates these issues, and provides the framework by which we will establish the limit distribution of the *gls* estimator. The theorem generalizes a corresponding result in Kabfleisch and Lawless (1984) for the linear case and common convergence rates.

Theorem 2.2 Consider the model

$$\tilde{Y}_n = f(\tilde{X}_n; \theta_T) + \tilde{\epsilon}_n$$

where $\tilde{Y}_n$ is $s \times 1$, $\tilde{X}_n$ is $s \times a$, θ_T is $v \times 1$, $\tilde{\epsilon}_n$ is $s \times 1$, and $\tilde{X}_n$, $\tilde{Y}_n$ and $\tilde{\epsilon}_n$ are random matrices satisfying (as $n \rightarrow \infty$)

- (i) $\tilde{X}_n \xrightarrow{P} \tilde{H}, \tilde{Y}_n \xrightarrow{P} \tilde{Q}$, where $\tilde{H}, \tilde{Q}$ are constant and satisfy $\tilde{Q} = f(\tilde{H}; \theta_T)$;
- (ii) $\tilde{B}_n \tilde{\epsilon}_n \xrightarrow{L} N_s(\underline{0}, \sum_{\epsilon})$, where $\sum_{\epsilon}(s \times s)$ is positive definite (*p.d.*) and $\tilde{B}_n$ is an $s \times s$ diagonal matrix of scaling sequences (not necessarily all the same) such that there exists a real sequence $h_n > 0$, $h_n \rightarrow 0$ for which $h_n \tilde{B}_n \rightarrow \tilde{B}$;
- (iii) $\text{rank}(\tilde{B} \dot{f}(\tilde{H}; \theta_T)) = v$, where

$\dot{f}(\tilde{H}; \theta)$ is the $s \times v$ θ - Jacobian matrix of f (assumed suitably differentiable).

Further, for arbitrary $s \times s$ *p.d.* Q , define

$$S_Q(\underline{\theta}) = [\tilde{Y}_n - f(\tilde{X}_n; \underline{\theta})]^T \tilde{B}_n^T Q \tilde{B}_n [\tilde{Y}_n - f(\tilde{X}_n; \underline{\theta})]$$

and let $\hat{\underline{\theta}}_n$ satisfy conditions (iv) and (v) below:

(iv) $\frac{d}{d\underline{\theta}} S_Q(\hat{\underline{\theta}}_n) = \underline{0}$, or equivalently,

$$\dot{f}^T(\tilde{X}_n; \hat{\underline{\theta}}_n) \tilde{B}_n^T Q \tilde{B}_n [\tilde{Y}_n - f(\tilde{X}_n; \hat{\underline{\theta}}_n)] = \underline{0};$$

(v) $\hat{\underline{\theta}}_n \xrightarrow{P} \underline{\theta}_T$ as $n \rightarrow \infty$.

Then $\underline{M}(Q) = : \dot{f}^T(\tilde{H}; \underline{\theta}_T) \tilde{B}^T Q \tilde{B} \dot{f}(\tilde{H}; \underline{\theta}_T)$ is p.d. and

$$h_n^{-1}(\hat{\underline{\theta}}_n - \underline{\theta}_T) \xrightarrow{L} N_v(\underline{0}, \Gamma(\underline{\theta}_T; Q)) \text{ as } n \rightarrow \infty,$$

where $\Gamma(\underline{\theta}_T; Q) = \underline{M}^{-1}(Q) \dot{f}^T(\tilde{H}; \underline{\theta}_T) \tilde{B}^T Q \sum_{\epsilon} \epsilon \tilde{B} \dot{f}(\tilde{H}; \underline{\theta}_T) \underline{M}^{-1}(Q)$.

Further, if $Q = \sum_{\epsilon} \epsilon^{-1}$, then $\hat{\underline{\theta}}_n$ is optimal (in the sense that $\Gamma(\underline{\theta}_T; Q) - \Gamma(\underline{\theta}_T; \sum_{\epsilon} \epsilon^{-1})$ is positive semi-definite for all p.d. Q) and $\Gamma(\underline{\theta}_T; \sum_{\epsilon} \epsilon^{-1}) = \underline{M}^{-1}(\sum_{\epsilon} \epsilon^{-1})$. The asymptotic optimality continues to hold if $\underline{Q} = \sum_{\epsilon} \epsilon^{-1}$, a consistent estimate of $\sum_{\epsilon} \epsilon^{-1}$.

To apply this result in the present setting let $s = \dim(\underline{Y}) = (r+m-1)(m-1)$, $\nu = m(m-1)$ and let Q ($s \times s$) denote an arbitrary p.d. matrix. The corresponding gls estimate $\hat{\underline{\beta}}_Q$ of $\underline{\beta}$ in (2.5) satisfies $\hat{\underline{\beta}}_Q = \argmin_{\underline{\beta}} S_N(\underline{\beta})$, where, with

$$\tilde{B}_N = \text{Blockdiag} \left\{ \underbrace{n_N^{\frac{1}{2}} I_{m-1}, \dots, n_N^{\frac{1}{2}} I_{m-1}}_{r-1 \text{ blocks}}, \underbrace{g_N^{\frac{1}{2}} I_{m-1}, \dots, g_N^{\frac{1}{2}} I_{m-1}}_{m \text{ blocks}} \right\},$$

$$S_N(\underline{\beta}) = : n_N^{-1} \{ \underline{Y}_N - F(\underline{\beta}; \hat{\underline{q}}_N^*) \}^T \tilde{B}_N^T Q \tilde{B}_N \{ \underline{Y}_N - F(\underline{\beta}; \hat{\underline{q}}_N^*) \} \quad (2.19)$$

is the error sum of squares function. We shall apply Theorem 2.2 with

$$\tilde{Y}_N = : \underline{Y}_N \xrightarrow{P} \begin{bmatrix} A \underline{q}^* \\ \underline{p}^* \end{bmatrix} = : \tilde{G} \quad (2.20)$$

$$\tilde{X}_N = : \hat{\underline{q}}_N^* \xrightarrow{P} \underline{q}^* = : \tilde{H}, \quad (2.21)$$

$$f(\tilde{X}_N; \underline{\theta}) = : F(\underline{\beta}; \hat{\underline{q}}_N^*), \quad \underline{\theta} \equiv \underline{\beta}, \quad \underline{\theta}_T \equiv \underline{\beta}_T, \quad \nu = \dim(\underline{\beta}),$$

$$\tilde{\epsilon}_N = : \epsilon_N = Y_N - F(\underline{\beta}_T; \hat{q}_N^*),$$

$$\tilde{B} = \text{Blockdiag}\{\underbrace{I_{m-1}, \dots, I_{m-1}}_{r-1 \text{ blocks}}, \underbrace{0 \cdot I_{m-1}, \dots, 0 \cdot I_{m-1}}_{m \text{ blocks}}\},$$

and $h_N = n_N^{-1/2}$. (Here the convergences in (2.20) and (2.21) hold, under (2.4), as consequences of Theorem 2.1.) In Lemma 2.3 below it is shown that these quantities (as defined in the present setting) satisfy the conditions (ii) and (iii) of Theorem 2.2. Thus, for any fixed p.d. weight matrix Q , any consistent minimizer of $S_N(\underline{\beta})$ in (2.19) (which via differentiability will necessarily satisfy its condition (iv)) will be asymptotically normal per Theorem 2.2. The consistency of $\hat{\underline{\beta}}_Q$ is also established in Lemma 2.3.

Lemma 2.3 Under conditions (A1) - (A5) and model (2.4),

(a) $\tilde{B}_N \tilde{\epsilon}_N \xrightarrow{L} N_s(0, \sum \epsilon)$ as $N \rightarrow \infty$, where

$$\sum \epsilon = \dot{\xi}(W^*) L \Gamma_W L^T \dot{\xi}^T(W^*),$$

$$L = \underline{D} \otimes I_{m-1}, \underline{D} = \text{Diag}(\lambda_1^{-\frac{1}{2}}, \dots, \lambda_r^{-\frac{1}{2}}, \underline{1}_m^T),$$

Γ_W and W^* are as in Theorem 2.1, and

$$\dot{\xi}(W^*) = \begin{bmatrix} A - \frac{\partial}{\partial \hat{q}_N^*} F_1(\beta_T; q^*) & 0 \\ \hline 0 & I_{m(m-1)} \end{bmatrix}.$$

(b) $\text{rank}\{\tilde{B} \frac{\partial F}{\partial \underline{\beta}}(\beta_T; q^*)\} = \nu = m(m-1)$.

(c) $\hat{\underline{\beta}}_{QN} \xrightarrow{P} \underline{\beta}_T$ as $N \rightarrow \infty$.

Thus by Theorem 2.2 we have the main result of this section.

Theorem 2.4 Under conditions (A1)-(A5) and model (2.4), for any p.d. weight matrix $\underline{Q}$, $\hat{\underline{\beta}}_{QN}$ minimizing (2.19) satisfies

$$n_N^{1/2}(\hat{\underline{\beta}}_{QN} - \underline{\beta}_T) \xrightarrow{L} N(\underline{0}, \Gamma_{\beta}(\underline{\beta}_T; \underline{Q})),$$

where, with $\underline{\dot{F}} = \underline{\dot{F}}(\underline{\beta}_T; \underline{Q})^*$ and $\underline{M}(\underline{Q}) = \underline{\dot{F}}^T \underline{\ddot{B}}^T \underline{Q} \underline{\ddot{B}} \underline{\dot{F}}$,

$$\Gamma_{\beta}(\underline{\beta}_T; \underline{Q}) = \underline{M}^{-1}(\underline{Q}) \underline{\dot{F}}^T \underline{\ddot{B}}^T \underline{Q} \sum_{\epsilon} \underline{Q} \underline{\ddot{B}} \underline{\dot{F}} \underline{M}^{-1}(\underline{Q}).$$

Corollary 2.5 With $\underline{Q} = \sum_{\epsilon}^{-1}$ (or a consistent estimate of it), the asymptotically optimal gls estimate $\hat{\underline{\beta}}_N$, defined as $\hat{\underline{\beta}}_{QN}$ with this choice of $\underline{Q}$, satisfies

$$n_N^{1/2}(\hat{\underline{\beta}}_N - \underline{\beta}_T) \xrightarrow{L} N(\underline{0}, \Gamma_{\beta}(\underline{\beta}_T; \sum_{\epsilon}^{-1})), \text{ where}$$

$$\Gamma_{\beta}(\underline{\beta}_T; \sum_{\epsilon}^{-1}) = \{\underline{\dot{F}}^T \underline{\ddot{B}} \sum_{\epsilon}^{-1} \underline{\ddot{B}} \underline{\dot{F}}\}^{-1} = \{\underline{\dot{F}}_1^T \sum_{11}^* \underline{\dot{F}}_1\}^{-1},$$

$$\sum_{\epsilon}^{-1} = \begin{bmatrix} \sum_{11}^* & \sum_{12}^* \\ \sum_{12}^{*T} & \sum_{22}^* \end{bmatrix}, \quad \sum_{11}^* \text{ is } v \times v \text{ and } \underline{\dot{F}}^T = [\underline{\dot{F}}_1^T, \underline{\dot{F}}_2^T].$$

Corollary 2.6. With $\underline{Q} = \underline{I}$, the ordinary least squares (ols) estimator $\hat{\underline{\beta}}_{IN}$ satisfies

$$n_N^{1/2}(\hat{\underline{\beta}}_{IN} - \underline{\beta}_T) \xrightarrow{L} N(\underline{0}, \Gamma_{\beta}(\underline{\beta}_T; \underline{I})), \text{ where}$$

$$\Gamma_{\beta}(\underline{\beta}_T; \underline{I}) = \underline{M}^{-1}(\underline{I}) \underline{\dot{F}}^T \underline{\ddot{B}}^T \sum_{\epsilon} \underline{\ddot{B}} \underline{\dot{F}} \underline{M}^{-1}(\underline{I}) = (\underline{\dot{F}}_1^T \underline{\dot{F}}_1)^{-1} \underline{\dot{F}}_1^T \sum_{11} \underline{\dot{F}}_1 (\underline{\dot{F}}_1^T \underline{\dot{F}}_1)^{-1},$$

$$\sum_{\epsilon} = \begin{bmatrix} \sum_{11} & \sum_{12} \\ \sum_{12}^T & \sum_{22} \end{bmatrix} \text{ and } \sum_{11} \text{ is } v \times v.$$

It is remarkable in Corollary 2.6 that the asymptotic covariance matrix of the ols estimator $\hat{\underline{\beta}}_{IN}$ is the same as it would be if the "recapture" data were left out of the $\underline{Y}$ -vector in (2.5)! (The residual variance matrix $\sum_{11}$ involves only the $\underline{Y}_1$ component of $\underline{Y}$). Of course this covariance

matrix applies only as the matrix $\tilde{B}_N \rightarrow \tilde{B}$ (i.e. as the ratio $g_N/n_N \rightarrow 0$), which, for all the n_t 's being equal, is the same as the sampling fraction (SF) $n_t/N \rightarrow 0$. In fact, the Monte Carlo results in Section 2.8 show that for SF 's of 5%, 10% or 20%, the *ols* estimate $\hat{\beta}_{IN}$ dramatically out-performs the "no-recaptures" estimate. This may be attributed to the SF being too far from zero for the limit result of Corollary 2.6 to apply. What's more, we found that the covariance matrix of $\hat{\beta}_{IN}$ is much better estimated by replacing $\tilde{B}$ by $\tilde{B}_N$ in Corollary 2.6 than by using the final expression which omits $\tilde{B}$. Interestingly, the asymptotic covariance matrix of the optimal *gls* estimator $\hat{\beta}_N$ in Corollary 2.5 depends on the recapture process only through the matrix $\sum_{11}^*$, which by a partitioned matrix inverse formula satisfies

$$\sum_{11}^* = (\sum_{11} - \sum_{12} \sum_{22}^{-1} \sum_{12}^T)^{-1}.$$

Of course, $\sum_{11}^*$ is the inverse of the conditional covariance matrix of the prediction error by the marginal component Y_1 , given the recapture data Y_2 . Although this observation may have some intuitive merit, it was again found that a better estimate of the covariance matrix of $\hat{\beta}_N$ was obtained by replacing $\tilde{B}$ by $\tilde{B}_N$ in Corollary 2.5.

2.5.4. Estimating $\sum_{\epsilon}$

The asymptotically optimal weight matrix $\sum_{\epsilon}$ is given in Lemma 2.3 as

$\sum_{\epsilon} = \dot{\xi}(W^*) L \Gamma_W L^T \dot{\xi}^T(W^*)$. Here Γ_W is the asymptotic covariance matrix of $B_N\{W_N - W^*\}$. For estimating $\sum_{\epsilon}$ we shall therefore replace Γ_W by the matrix $B_N \text{Var}(W_N) B_N$, which itself must be estimated. Natural estimates of L and $\dot{\xi}(w^*)$ are obtained by replacing λ_t by n_t/n_N in L , and by replacing q^* by $\hat{q}_N^*$ and β_T by $\hat{\beta}_{IN}$ (the consistent *ols* estimate) in $\dot{\xi}(w^*)$.

Now the matrix $\text{Var}(W_N)$ has the form

$$\text{Var}(W_N) = \begin{bmatrix} \text{var}(\hat{q}_N^*) & \text{cov}(\hat{q}_N^*, Y_{2N}) \\ \text{cov}^T(\hat{q}_N^*, Y_{2N}) & \text{var}(Y_{2N}) \end{bmatrix}.$$

The matrix $\text{var}(\hat{\underline{q}}_N^*)$ is given following (2.17) with a consistent estimate provided by (2.17).

The derivations of $\text{Var}(\underline{Y}_{2N})$ and $\text{cov}(\hat{\underline{q}}_N^*, \underline{Y}_{2N})$ are long and tedious but straight-forward, so only the basic ideas and final results are given here, along with the basic idea for obtaining consistent estimates. Complete details are in HHE.

$\text{Var}(\underline{Y}_2) = \text{Var} \begin{bmatrix} \hat{\underline{p}}_1 \\ \dots \\ \hat{\underline{p}}_m \end{bmatrix}$ is a matrix of $(m-1) \times (m-1)$ blocks, where the (i, j) block is $\text{cov}(\hat{\underline{p}}_i, \hat{\underline{p}}_j)$, $1 \leq i, j \leq m$.

Now for $1 \leq i \leq m$, $\hat{\underline{p}}_i = \underline{R}_i / D_i$ is a ratio of random quantities. If we define $\underline{g}: \mathbf{R}^m \rightarrow \mathbf{R}^{m-1}$ by $\underline{g}(x_1, \dots, x_m) = [g_1(x), \dots, g_{m-1}(x)]^T = [x_1/x_m, \dots, x_{m-1}/x_m]^T$, then $\hat{\underline{p}}_i = \underline{g}(\underline{R}_i, D_i)$, so we can use a delta method approximation. Writing $\underline{J}_g$ as the Jacobian of $\underline{g}$ and setting $\underline{\mu}_i = E[\underline{R}_i^T, D_i]$, we have

$$\text{Cov}(\hat{\underline{p}}_i, \hat{\underline{p}}_j) = E\{[\hat{\underline{p}}_i - E\hat{\underline{p}}_i][\hat{\underline{p}}_j - E\hat{\underline{p}}_j]^T\} \approx E\{[\underline{g}(\underline{R}_i, D_i) - \underline{g}(\underline{\mu}_i)][\underline{g}(\underline{R}_j, D_j) - \underline{g}(\underline{\mu}_j)]^T\}$$

$$\approx E\left\{ \underline{J}_g(\underline{\mu}_i) \begin{bmatrix} \underline{R}_i - E\underline{R}_i \\ D_i - ED_i \end{bmatrix} \right\} \left\{ \underline{J}_g(\underline{\mu}_j) \begin{bmatrix} \underline{R}_j - E\underline{R}_j \\ D_j - ED_j \end{bmatrix} \right\}^T = \underline{J}_g(\underline{\mu}_i) \underline{W}_{ij} \underline{J}_g^T(\underline{\mu}_j),$$

where

$$\underline{W}_{ij} = \begin{bmatrix} \text{cov}(\underline{R}_i, \underline{R}_j) & \text{cov}(\underline{R}_i, D_j) \\ \text{cov}(D_i, \underline{R}_j) & \text{cov}(D_i, D_j) \end{bmatrix}. \quad (2.22)$$

Thus, our approximation to $\text{Var}(\underline{Y}_2)$ has (i, j) block $\underline{J}_g(\underline{\mu}_i) \underline{W}_{ij} \underline{J}_g^T(\underline{\mu}_j)$. We need only compute $\underline{\mu}_i$ and $\underline{W}_{ij}$.

In this direction let $\underline{c}_{ilt} = Z_{lti} Z_{l, t+1}$, so that

$$\underline{R}_i = \sum_{t=1}^{r-1} g_N^{-1} \sum_{l=1}^N \underline{c}_{ilt} I_{lt} I_{l, t+1}. \quad \text{Then straight-forward calculation gives}$$

$$E(\underline{R}_i) = \sum_t g_{Nt}^{-1} \sum_l \underline{c}_{ilt} E(I_{lt} I_{l,t+1}) = \sum_t g_{Nt}^{-1} \sum_l \underline{c}_{ilt} e_t = \sum_{t=1}^{r-1} \bar{\underline{c}}_t^{(N)}$$

where $e_t \equiv f_t f_{t+1}$ and $\bar{\underline{c}}_t^{(N)} = \frac{1}{N} \sum_{l=1}^N \underline{c}_{ilt}$, and

$$\text{cov}(\underline{R}_i, \underline{R}_j) = E[\underline{R}_i - E\underline{R}_i][\underline{R}_j - E\underline{R}_j]^T = \sum_l \sum_t \sum_{l'} \sum_{t'} \underline{c}_{ilt} \underline{c}_{jl't'}^T M_{lt, l't'},$$

where

$$g_{Nt} g_{Nt'} M_{lt, l't'} =$$

$$l = l', t = t': e_t(1 - e_t)$$

$$l \neq l', t = t': \frac{n_t(n_t - 1)}{N(N - 1)} - \frac{n_{t+1}(n_{t+1} - 1)}{N(N - 1)} - e_t^2$$

$$|t - t'| > 1, \text{ all } l, l': 0$$

$$t' = t + 1, l = l': \frac{n_t}{N} - \frac{n_{t+1}}{N} - \frac{n_{t+2}}{N} - e_t e_{t+1}$$

$$t' = t + 1, l \neq l': \frac{n_t}{N} - \frac{n_{t+1}(n_{t+1} - 1)}{N(N - 1)} - \frac{n_{t+2}}{N} - e_t e_{t+1}$$

$$t' = t - 1, l = l': \frac{n_t}{N} - \frac{n_{t+1}}{N} - \frac{n_{t-1}}{N} - e_t e_{t-1}$$

$$t' = t - 1, l \neq l': \frac{n_t(n_t - 1)}{N(N - 1)} - \frac{n_{t+1}}{N} - \frac{n_{t-1}}{N} - e_t e_{t-1}.$$

Similarly, defining $\underline{D} = [D_1, \dots, D_m]^T$ and $\bar{\underline{Z}}_t^{o(N)} = \frac{1}{N} \sum_{l=1}^N \underline{Z}_{lt}^o$,

$$\underline{D} = \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N \underline{Z}_{lt}^o I_{lt} I_{l,t+1}, \text{ so}$$

$$E(\underline{D}) = \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N \underline{Z}_{lt}^o E[I_{lt} I_{l,t+1}] = \sum_t g_{Nt}^{-1} \sum_l \underline{Z}_{lt}^o e_t = \sum_{t=1}^{r-1} \bar{\underline{Z}}_t^{o(N)}$$

and $\text{cov}(D_i, D_j)$ is the (i, j) element of

$$Var(\underline{D}) = E\{[\underline{D} - E\underline{D}][\underline{D} - E\underline{D}]^T\} = \sum_t \sum_l \sum_{t'} \sum_{l'} \underline{Z}_{lt}^0 \underline{Z}_{l't'}^{0T} M_{lt, l't'}.$$

Finally, the matrices $cov(D_i, \underline{R}_j)$ are submatrices of

$$cov(\underline{D}, \underline{R}_j) = E\{[\underline{D} - E\underline{D}][\underline{R}_j - E\underline{R}_j]^T\} = \sum_t \sum_l \sum_{t'} \sum_{l'} \underline{Z}_{lt}^0 \underline{Z}_{jl't'}^{0T} M_{lt, l't'}.$$

From (2.8) and (2.9), the $(r-1)(m-1) \times m(m-1)$ matrix

$$cov(\underline{Y}_1, \underline{Y}_2) = cov(\underline{A}\hat{\underline{q}}^*, \underline{Y}_2) = \underline{A} cov(\hat{\underline{q}}^*, \underline{Y}_2)$$

and

$$cov(\hat{\underline{q}}^*, \underline{Y}_2) = E\{[\hat{\underline{q}}^* - E\hat{\underline{q}}^*][\underline{Y}_2 - E\underline{Y}_2]^T\}$$

has block structure with (t, i) block $(1 \leq t \leq r, 1 \leq i \leq m)$

$$\underline{B}_{ti} = : cov(\hat{\underline{q}}_t, \hat{\underline{p}}_i) = E\{[\hat{\underline{q}}_t - E\hat{\underline{q}}_t][\hat{\underline{p}}_i - E\hat{\underline{p}}_i]^T\} \approx E\{[\hat{\underline{q}}_t - E\hat{\underline{q}}_t][g(\underline{R}_i, D_i) - g(\underline{\mu}_i)]^T\}$$

$$\approx E\left\{[\hat{\underline{q}}_t - E\hat{\underline{q}}_t] \left\{ \underline{J}_g(\underline{\mu}_i) \begin{bmatrix} \underline{R}_i - E\underline{R}_i \\ D_i - ED_i \end{bmatrix} \right\}^T\right\}$$

$$= E\{[\hat{\underline{q}}_t - E\hat{\underline{q}}_t][(\underline{R}_i - E\underline{R}_i)^T, D_i - ED_i]\} \underline{J}_g^T(\underline{\mu}_i)$$

$$= [cov(\hat{\underline{q}}_t, \underline{R}_i) : cov(\hat{\underline{q}}_t, D_i)] \underline{J}_g^T(\underline{\mu}_i) = [\underline{H}_{ti}^{(1)} \quad : \quad \underline{H}_{ti}^{(2)}] \underline{J}_g^T(\underline{\mu}_i), \text{ say.} \quad (2.23)$$

$$\text{Now since } \hat{\underline{q}}_t = [\underline{I}_{m-1} : \underline{O}_{m-1}] \hat{\underline{q}}_t^0 = : \underline{B} \hat{\underline{q}}_t^0,$$

$$\underline{H}_{ti}^{(1)} = cov(\underline{B} \hat{\underline{q}}_t^0, \underline{R}_i) = \underline{B} cov(\hat{\underline{q}}_t^0, \underline{R}_i) = \underline{B} \underline{G}_{ti}^0,$$

say, where for fixed $t \in \{1, \dots, r-1\}$,

$$\underline{G}_{ti}^0 = : E\{[\hat{\underline{q}}_t^0 - E\hat{\underline{q}}_t^0][\underline{R}_i - E\underline{R}_i]^T\}$$

$$= \frac{1}{n_t} \sum_{l=1}^N \sum_{l'=1}^N \{g_{Nt}^{-1} E_{tll'}^* \underline{Z}_{lt}^0 \underline{e}_{il't}^T + g_{N,t-1}^{-1} I(t-1 \geq 1) E_{tll'}^{**} \underline{Z}_{lt}^0 \underline{e}_{il',t-1}^T\}.$$

where

$$E_{tll'}^* = \begin{cases} \text{if } l=l': e_t(1-f_t) \\ \text{if } l \neq l': e_t \left[\frac{n_t - N}{N(N-1)} \right] \end{cases}$$

and

$$E_{tll'}^{**} = \begin{cases} \text{if } l=l': e_{t-1}(1-f_t) \\ \text{if } l \neq l': e_{t-1} \left[\frac{n_t - N}{N(N-1)} \right]. \end{cases}$$

Now define $\underline{e}_{mi}$ to be the column vector in $\mathbb{R}^m$ with entries all zero except position i , which is one. Then for $\underline{H}_{ti}^{(2)}$ in (2.23), we have $\underline{H}_{ti}^{(2)} =$

$$\begin{aligned} \text{cov}(\hat{\underline{q}}_t, D_i) &= \text{cov}(\underline{B} \hat{\underline{q}}_t^0, \underline{e}_{mi}^T \underline{D}) \\ &= \underline{B} \text{cov}(\hat{\underline{q}}_t^0, \underline{D}) \underline{e}_{mi} \text{ and} \end{aligned}$$

$$\text{cov}(\hat{\underline{q}}_t^0, \underline{D}) = E\{[\hat{\underline{q}}_t^0 - E\hat{\underline{q}}_t^0][\underline{D} - E\underline{D}]^T\}$$

$$= \frac{1}{n_t} \sum_l \sum_{l'} \{g_{Nt}^{-1} E_{tll'}^* \underline{Z}_{lt}^0 \underline{Z}_{l't}^{0T} + g_{N,t-1}^{-1} I(t-1 \geq 1) E_{tll'}^{**} \underline{Z}_{lt}^0 \underline{Z}_{l',t-1}^{0T}\}.$$

Consistent estimation of $\sum \epsilon$ comes down to consistent estimation of $\text{Var}(\underline{W}_N)$, which reduces to such for $\text{Var}(\underline{Y}_{2N})$ and $\text{cov}(\hat{\underline{q}}_N^*, \underline{Y}_{2N})$ since $\text{Var}(\hat{\underline{q}}_N^*)$ is consistently estimated by (2.17). $\text{Var}(\underline{Y}_{2N})$ and $\text{cov}(\hat{\underline{q}}_N^*, \underline{Y}_{2N})$ can be consistently estimated, under model (2.4), via the following approach. The basic idea is to express the variance formulae in terms of certain population fractions which, under model (2.4), have representations in terms of the marginal probabilities q_{ti} and the transition probabilities p_{ij} . Then $\hat{q}_{ti}$ and the consistent *ols* estimate $\hat{p}_{ij} = p_{ij}(\hat{\underline{\beta}}_I)$ may be plugged into the resulting expressions to yield consistent estimates.

To estimate $\text{Var}(\underline{Y}_2)$ it is clearly enough to estimate $\underline{J}_g(\underline{\mu}_i)$ and $\underline{W}_{ij}$

given by (2.22). A consistent estimate of $\underline{J}_g(\underline{\mu}_i)$ is $\underline{J}_g(\hat{\underline{\mu}}_i)$ where $\hat{\underline{\mu}}_i^T = [\hat{R}_i, \hat{D}_i]$ and $\hat{R}_i$ and $\hat{D}_i$ are defined as R_i and D_i with g_{Nt} replaced by $n_{t,t+1}$. For $\underline{W}_{ij}$, all four of the component submatrices depend on the constants $M_{lt,l't'}$ which depend on the sampling fractions f_t at each wave t . Since these are usually known only roughly, a simple approximation $\hat{M}_{lt,l't'}$ to $M_{lt,l't'}$ is obtained by replacing (ala assumption (A3)) terms like $(n_t-1)/(N-1)$ by f_t , so that

$$g_{Nt} g_{Nt'} \hat{M}_{lt,l't'} = \begin{array}{lll} l = l', & t = t': & e_t(1 - e_t) \\ l \neq l', & t = t': & 0 \\ \text{all } l, l', & |t - t'| > 1: & 0 \\ l = l', & t' = t + 1: & e_t(f_{t+2} - e_{t+1}) \\ l \neq l', & t' = t + 1: & 0 \\ l = l', & t' = t - 1: & e_t(f_{t-1} - e_{t-1}) \\ l \neq l', & t' = t - 1: & 0 \end{array}$$

The upper left submatrix of $\underline{W}_{ij}$, with $M_{lt,l't'}$ replaced by $\hat{M}_{lt,l't'}$, is given by

$$\begin{aligned} & \sum_{l=1}^N \sum_{t=1}^{r-1} \sum_{l'=1}^N \sum_{t'=1}^{r-1} \underline{e}_{ilt} \underline{e}_{jl't'}^T \hat{M}_{lt,l't'} \\ &= \sum_{t=1}^{r-1} \sum_{t'=1}^{r-1} \sum_{l=1}^N \hat{M}_{lt,l't'} \underline{e}_{ilt} \underline{e}_{jl't'}^T \\ &= \sum_{t=1}^{r-1} \sum_{l=1}^N \hat{M}_{lt,lt} \underline{e}_{ilt} \underline{e}_{jlt}^T + \sum_{t \neq t'} \sum_{l=1}^N \hat{M}_{lt,lt'} \underline{e}_{ilt} \underline{e}_{jl't'}^T \\ &= \sum_{t=1}^{r-1} \sum_{l=1}^N \hat{M}_{lt,lt} \underline{e}_{ilt} \underline{e}_{jlt}^T \end{aligned} \tag{2.24}$$

$$+ \sum_{t=1}^{r-2} \sum_{l=1}^N \hat{M}_{lt,l,t+1} \{ \underline{e}_{ilt} \underline{e}_{jl,t+1}^T + \underline{e}_{il,t+1} \underline{e}_{jlt}^T \}, \tag{2.25}$$

the third equality following since $\hat{M}_{lt,lt'} = \hat{M}_{lt',lt}$ for $t \neq t'$ and $\hat{M}_{lt,lt'} = 0$ if $|t - t'| > 1$. Toward consistent estimates of (2.24) and (2.25) by the above-

noted scheme, note that $\hat{M}_{lt,lt'}$ does not depend on l , so define $d_{Ntt'} = N\hat{M}_{lt,lt'}$. Then (2.24) equals (writing δ_{ij} as the Kronecker delta)

$$\begin{aligned} & \sum_{t=1}^{r-1} d_{Ntt} \left\{ \frac{1}{N} \sum_{l=1}^N Z_{lti} Z_{ltj} Z_{l,t+1} Z_{l,t+1}^T \right\} \\ &= \delta_{ij} \sum_{t=1}^{r-1} d_{Ntt} \left\{ \frac{1}{N} \sum_{l=1}^N Z_{lti} Z_{l,t+1} Z_{l,t+1}^T \right\} \\ &= \delta_{ij} \sum_{t=1}^{r-1} d_{Ntt} \text{Diag}\{\tau_{i1}^t, \dots, \tau_{i,m-1}^t\}, \end{aligned}$$

where τ_{ik}^t = fraction of the population in stage i at wave t and in stage k at wave $t+1$. Under the Markov model (2.4), $\tau_{ik}^t = q_{ti} p_{ik}$, so the above-noted method gives a consistent estimate of (2.24). Similarly (2.25) may be shown to equal $\sum_{t=1}^{r-2} d_{Nt,t+1} \{ \underline{\varepsilon}_{m-1,j} [\alpha_{ij1}^t, \dots, \alpha_{ij,m-1}^t]^T + [\alpha_{ji1}^t, \dots, \alpha_{ji,m-1}^t]^T \underline{\varepsilon}_{m-1,i}^T \}$, where α_{ikj}^t = fraction of the population in stage i at wave t , in stage k at wave $t+1$ and in stage j at wave $t+2$. Under the Markov model (2.4), $\alpha_{ikj}^t = q_{ti} p_{ik} p_{kj}$, so the proposed scheme provides a consistent estimate of (2.25).

The corresponding representations of $\text{cov}(\underline{R}_i, D_j)$ and $\text{cov}(D_i, D_j)$ in (2.22) are

$$\begin{aligned} \text{cov}(\underline{R}_i, D_j) &\approx \delta_{ij} \sum_{t=1}^{r-1} d_{Ntt} [\tau_{i1}^t, \dots, \tau_{i,m-1}^t]^T \\ &+ \sum_{t=1}^{r-2} d_{Nt,t+1} \left\{ \begin{bmatrix} \alpha_{1i1}^t & \dots & \alpha_{mi1}^t \\ \vdots & & \vdots \\ \alpha_{1i,m-1}^t & & \alpha_{mi,m-1}^t \end{bmatrix} + \begin{bmatrix} \tau_{i1}^t & 0 & 0 \\ & \ddots & \vdots \\ 0 & \tau_{i,m-1}^t & 0 \end{bmatrix} \right\} \underline{\varepsilon}_{mj}, \\ \text{cov}(D_i, D_j) &\approx \delta_{ij} \sum_{t=1}^{r-1} d_{Ntt} q_{ti} + 2 \sum_{t=1}^{r-2} d_{Nt,t+1} \tau_{ij}^t. \end{aligned}$$

These results provide the consistent estimate of $\underline{W}_{ij}$ in (2.22), and hence for $\text{Var}(\underline{Y}_2)$.

To obtain a consistent estimate of $\text{cov}(\underline{Y}_1, \underline{Y}_2)$, it is enough (by (2.23) and the ensuing computations) to estimate $\underline{G}_{ti}^o$ and $\text{cov}(\hat{\underline{q}}_t^o, \underline{D})$. But these may be expressed (with approximations $\hat{E}_{ill'}^* = \delta_{ill'} e_t(1-f_t)$ and $\hat{E}_{ill'}^{**} = \delta_{ill'} e_{t-1}(1-f_t)$ substituted for $E_{ill'}^*$ and $E_{ill'}^{**}$) as

$$G_{ti}^o \approx n_t^{-1}(1-f_t) \left\{ I(\leq r-1) \underline{e}_{mi} q_{ti} [p_{i1}, \dots, p_{i,m-1}] \right. \\ \left. + I(t \geq 2) q_{t-1,i} \begin{bmatrix} p_{i1} & \dots & 0 \\ 0 & & p_{i,m-1} \\ \hline 0 & \dots & 0 \end{bmatrix} \right\},$$

$$\text{cov}(\underline{\hat{q}}_t^o, \underline{D}) = n_t^{-1}(1-f_t) \left\{ \text{Diag}(\underline{q}_t^o) + I(t \geq 2) \underline{P}^T \text{Diag}(\underline{q}_{t-1}^o) \right\}.$$

So the proposed method works here as well.

Unfortunately, $\hat{\Sigma}_\epsilon$ may not be p.d. in general, due to the piece-meal construction of $\hat{\text{Var}}(\underline{W}_N)$. This causes problems in implementing our method on a nonlinear regression package which only does *ols* calculations, since one needs to obtain the square root of $\hat{\Sigma}_\epsilon^{-1}$ to do generalized least squares via *ols* in the transformed model. We remedied this problem by replacing $\hat{\Sigma}_\epsilon$ by the p.d. matrix $\tilde{\Sigma}_\epsilon = (\hat{\Sigma}_\epsilon^T \hat{\Sigma}_\epsilon)^{1/2}$, where the square root is the eigen-square-root. This seems justifiable since $\tilde{\Sigma}_\epsilon$ has the same p-limit as does $\hat{\Sigma}_\epsilon$.

2.6. Lack of Fit Testing

It follows from Corollary 2.5 and standard theory that, with $Q = \tilde{\Sigma}_\epsilon^{-1}$ and $\hat{\underline{\beta}}_{GLS}$ denoting the corresponding estimator of $\underline{\beta}$, $X_{LOF}^2 = n_N S_N(\hat{\underline{\beta}}_{GLS})$ has approximately the χ_{s-v}^2 distribution under model (2.4), where $s = \dim(Y) = (r+m-1)(m-1)$ and $v = \dim(\underline{\beta}) = m(m-1)$. This statistic is, of course, just the value of the error sum of squares from the generalized least squares fit of (2.5) using $Q = \tilde{\Sigma}_\epsilon^{-1}$, so it is easily obtained.

2.7. Some Remarks on Computing

Computation of the asymptotically optimal $\hat{\underline{\beta}}_N$ and its covariance matrix in Corollary 2.5 requires

- 1) the *ols* estimate $\hat{\underline{\beta}}_{IN}$
- 2) the estimated optimal weight matrix $\tilde{\Sigma}_\epsilon$.

The *ols* estimate $\hat{\underline{\beta}}_I$ can be obtained by any standard nonlinear least squares program. We used SAS's NLIN procedure. The estimate $\tilde{\Sigma}_\epsilon$ requires

considerable pre-processing of the basic data estimates. We used SAS's IML procedure for this. Given $\sum \epsilon$, we transformed model (2.5) by multiplication by the inverse of the eigen-square-root of $\sum \epsilon$, and then fit the resulting model using NLIN. By this method, the estimate given by NLIN equals $\hat{\beta}_{GLS}$, and the covariance estimate computed by NLIN equals the estimated covariance matrix in Corollary 2.5. Also, the lack of fit statistic X^2_{LOF} comes out naturally as the error sum of squares. The IML procedure is then used following NLIN to compute the GLS estimate $\hat{p}^{(GLS)}$ of the p_{ij} 's via (2.10), along with their delta method covariance estimate. A SAS program to do these computations is available from the first author.

2.8. Monte Carlo Study

To verify the asymptotic results and to investigate the sensitivity of the performance of the estimates to the sampling fraction, a Monte Carlo study was performed. The design parameters were: $m=4$ categories for Z and $r=8$ sampling waves, initial distribution $\underline{q}_1^0 = (.80, .10, .05, .05)^T$ and transition matrix

$$P = \begin{bmatrix} .80 & .10 & .05 & .05 \\ .05 & .50 & .40 & .05 \\ .05 & .05 & .50 & .40 \\ .05 & .05 & .20 & .70 \end{bmatrix}.$$

Table 2.3(a) shows the results of 100 Monte Carlo replications for population size $N=1000$ and constant (from wave to wave) sampling fractions of 10% and 20%. Table 2.3(b) shows similar results for $N=10,000$ and sampling fractions of 5% and 10%. Displayed in these tables are the *gls* and *ols* results for estimating P using the haphazard recapture data, and also (in the column headed "No Recap") the results of *ols* estimation (analogous to Kabfleisch & Lawless (1984)) which does not use the recapture data. The columns headed "Ave Est" and "MC se" are the average and standard deviation of the 100 Monte Carlo estimates. The column headed "Est se" is the square root of the average variance estimate, obtained from the asymptotic variance matrix of $\hat{\beta}$ for the given method and subsequent application of the delta method.

Looking at the Monte Carlo results, one sees that the *gls* estimator only modestly out-performs the *ols* estimator (of course this depends on q_1^0 and P), but both of these estimators dramatically improve on the no-recaptures estimator. Also, the estimated standard errors are sufficiently close to the Monte Carlo standard error estimates to be reliably used for standard "normal theory" inferences.

Table 2.3 goes here

Table 2.3. Monte Carlo Results for Estimating the Transition Matrix

) $N = 1,000$

		sf=10%						sf=20%									
		GLS			OLS			No recap.		GLS			OLS			No Recap.	
Param	True Value	Ave Est	MC s.e.*	Est s.e.*	Ave Est.	MC s.e.*	Est. s.e.*	Ave Est.	MC s.e.*	Ave. Est.	mc s.e.*	Est. s.e.*	Ave. Est.	MC s.e.*	Est s.e.*	Ave Est.	mc s.e.*
p_{11}	.80	.80	2.9	3.1	.78	3.3	3.8	.74	6.7	.80	1.5	1.6	.80	1.9	2.0	.76	5.4
p_{12}	.10	.10	2.7	2.9	.10	2.9	3.7	.14	3.8	.10	1.5	1.5	.10	1.6	1.8	.14	3.3
p_{13}	.05	.05	2.9	2.7	.06	3.1	3.9	.06	4.8	.05	1.5	1.4	.06	1.7	1.9	.07	4.2
p_{21}	.05	.06	7.2	9.1	.07	8.1	11.1	.27	35.4	.05	3.8	3.6	.05	4.1	4.0	.25	28.6
p_{22}	.50	.44	11.9	13.3	.46	13.8	19.1	.09	15.1	.49	6.5	6.2	.50	7.3	8.2	.11	16.0
p_{23}	.40	.44	14.1	13.8	.39	14.8	18.7	.43	33.2	.42	6.5	6.7	.40	7.4	8.1	.37	33.1
p_{31}	.05	.05	5.6	6.7	.07	6.2	9.0	.07	12.3	.05	2.7	2.9	.05	3.0	3.3	.05	9.6
p_{32}	.05	.06	6.1	6.7	.06	5.4	8.1	.15	16.8	.05	2.7	2.8	.05	2.9	3.1	.19	17.3
p_{33}	.50	.45	12.2	11.7	.46	11.5	13.9	.25	24.7	.50	5.9	5.3	.49	6.1	5.9	.25	24.4
p_{41}	.05	.06	5.3	5.9	.07	6.2	7.9	.06	9.4	.05	2.4	2.7	.06	3.0	3.2	.04	6.0
p_{42}	.05	.07	5.4	5.7	.06	5.1	7.0	.12	11.4	.05	2.4	2.6	.05	2.7	2.9	.09	10.9
p_{43}	.20	.23	10.2	9.4	.22	9.9	11.3	.37	22.5	.20	4.9	4.4	.20	5.0	4.7	.41	23.8

times 100

b) $N = 10,000$

<i>sf</i> =5%											<i>sf</i> =10%							
GLS					OLS			No Recap			GLS		OLS			No Recap		
aram	True Value	Ave Est	MC s.e.*	Est s.e.*	Ave Est	MC s.e.*	Est s.e.*	Ave Est	MC s.e.*	Ave Est	MC s.e.*	Est s.e.*	Ave Est	MC s.e.*	Est s.e.*	Ave Est	MC s.e.*	
11	.80	.80	1.4	1.6	.80	1.8	1.9	.77	4.3	.80	.95	.88	.80	1.1	1.1	.78	4.2	
12	.10	.11	1.6	1.5	.10	1.8	1.9	.14	2.7	.10	.84	.84	.10	1.0	1.0	.14	2.5	
13	.05	.05	1.5	1.5	.05	1.9	1.9	.05	3.8	.05	.70	.81	.05	.95	1.1	.05	3.7	
21	.05	.06	4.7	4.8	.06	5.1	5.7	.20	22.8	.05	1.9	2.3	.05	2.1	2.4	.17	21.3	
22	.50	.47	8.6	7.8	.48	10.1	10.3	.13	17.1	.49	4.2	3.9	.50	4.7	4.9	.14	16.6	
23	.40	.42	8.8	8.0	.40	9.5	10.3	.46	32.0	.41	4.3	4.0	.40	4.9	4.9	.52	29.9	
31	.05	.06	3.5	3.5	.06	4.1	4.5	.04	7.4	.05	1.8	1.7	.05	2.0	1.9	.06	10.0	
32	.05	.05	3.3	3.6	.05	3.5	4.2	.18	14.2	.05	1.8	1.8	.05	2.0	1.9	.22	15.1	
23	.50	.48	6.8	6.9	.48	7.1	7.6	.32	29.1	.50	3.4	3.4	.50	3.5	3.6	.25	26.4	
41	.05	.05	3.4	3.0	.06	3.8	3.9	.04	5.3	.05	1.5	1.6	.05	1.7	1.8	.03	5.1	
42	.05	.05	3.0	3.1	.05	3.3	3.6	.08	10.8	.05	1.6	1.6	.05	1.7	1.8	.05	7.1	
43	.20	.21	4.9	5.5	.21	5.1	6.0	.32	21.7	.20	3.2	2.7	.20	3.2	2.9	.36	20.1	

: times 100

3. Example Application

3.1 Background and Data

The data shown in Tables 3.1 and 3.2 were collected as part of a CDC-sponsored study of the potential benefits of an intervention program attempting to modify sex- and drug-related behavior toward preventing the spread of the AIDS virus. Behavior was measured in terms of a composite index of several attitude/action criteria. (see Di Clemente et al (1985)). This index is here termed "stage", and has four possible values (the higher the stage, the better the behavior):

Stage = 1, means pre-contemplative (i.e. not thinking about changing behavior)

= 2, means contemplating making a change in behavior

= 3, means ready to make a change in behavior

= 4, means maintaining change in behavior.

This component of the study, which targeted self-identified intravenous drug users, was conducted in two geographically disjoint study areas, each consisting of several square city blocks in Long Beach, California. One area, designated the "Intervention" area, was subjected to a variety of intervention efforts, ranging from billboards, pamphlets and posters to visits by social workers conducting various educational discussions. The other area, designated as the "Comparison area", underwent no study-sponsored intervention. The comparison area served as a control, to allow an assessment of the benefits of the intervention, over and above the "background" AIDS - related information at large in (and hopefully consistent across) both study areas.

The study was conducted over a period of about two years, and consisted of repeated surveys (or waves) in each study area at roughly six-week intervals. Each survey wave was conducted by one or more interviewers standing on street corners. The survey design called for interviewing every fifth person who was willing to be interviewed. Since the survey was rather long and necessarily dealt with sensitive personal information, a \$5.00 payment was made to respondents as an incentive to participate. Efforts were made to prevent people from participating more than once in each survey wave. However, although they were not intentionally sought

out, people occasionally were selected for interview in consecutive waves, possibly due to the monetary reward for participating. These haphazard recaptures provide the transition data shown in Tables 3.1 (b) and 3.2 (b), which was used to augment the aggregate data shown in part (a) of these tables.

Tables 3.1 and 3.2 go here

3.2. Analysis Results

Table 3.3 displays the transition probability estimates based on the *gls* estimate of β (recall Corollary 2.5) based on $r=8$ waves and $m=4$ response categories. Standard recapture estimates of the sampling fraction were about 20%, so the estimated standard errors were deemed trustworthy in view of our Monte Carlo results. The lack of fit statistics (each on 21 *df*) were 30.2 and 35.7, respectively, for the intervention and comparison areas, suggesting that the Markov model fit reasonably well.

Table 3.3. Transition Probability Estimates from
Long Beach AIDS - Prevention Study

<i>i</i>	<i>j</i>	Intervention Area		Comparison Area	
		$\hat{p}_{ij}^{(GLS)}$	<i>s.e.</i>	$\hat{p}_{ij}^{(GLS)}$	<i>s.e.</i>
1	1	.23	.10	.46	.06
1	2	.06	.07	.08	.04
1	3	.65	.11	.42	.07
1	4	.07	.08	.04	.02
2	1	.01	.02	.27	.13
2	2	.54	.12	.28	.11
2	3	.20	.15	.29	.13
2	4	.25	.13	.16	.08
3	1	.10	.03	.17	.04
3	2	.02	.02	.06	.03
3	3	.70	.04	.76	.05
3	4	.18	.03	.00	.04
4	1	.05	.04	.00	.01
4	2	.03	.04	.02	.08
4	3	.37	.07	.52	.17
4	4	.55	.06	.45	.16

Looking at Table 3.3, one sees that the standard errors are still unacceptably large for some *i,j*. However, for evaluating the efficacy of

the intervention, the transition matrix $\underline{P}=(p_{ij})$, which measures the one-step (here one-wave) transition behavior, is not of primary interest. Rather, the eventual (or stationary) distribution is of more use. Standard Markov chain theory says that if such a stationary distribution exists, then it may be obtained as $\lim_{k \rightarrow \infty} \underline{P}^k$. We found that for our estimated matrices in Table 3.3 the powers stabilized by $k=8$, so we took the rows of $\underline{P}^8$ as the eventual distribution. Thus, we obtained estimates of the eventual distribution for each study area, with standard errors obtained by the delta method from the covariance matrices of the estimates in Table 3.3. These are displayed in Table 3.4, along with the initial distribution estimates from Tables 3.1(a) and 3.2(a).

Table 3.4. Initial and Eventual Distribution Estimates
for Long Beach AIDS - Prevention Study (s.e.'s in paren)

Stage	Intervention Area		Comparison Area	
	Initial Prob.	Eventual Prob.	Initial Prob.	Eventual Prob.
1	.14(.03)	.09(.01)	.32(.04)	.24(.02)
2	.14(.03)	.05(.01)	.10(.02)	.09(.02)
3	.42(.04)	.58(.02)	.43(.04)	.63(.03)
4	.30(.04)	.28(.02)	.15(.03)	.04(.03)

Note that due to the contractive nature of the map $\underline{P} \rightarrow \underline{P}^8$ for stochastic matrices $\underline{P}$, the standard errors of the eventual estimates are acceptably small. Also recall that by Markov theory, the eventual distribution is independent of the initial distribution, so that although the initial distributions differ substantially, the eventual distributions are nonetheless comparable for assessment of the impact of the intervention if it were to be maintained indefinitely. A standard Wald test of the equality of these eventual distributions gives a χ^2_3 statistic of 67.6

($p=0$), and examination of these distributions indicates that the intervention area eventually will have about 24% more of the population in stage 4 than will the comparison area.

ACKNOWLEDGEMENT: The authors thank Martin A. krepcho, Ph.D., Anne C. Freeman, M.S.P.H. and Elvin Magee, M.S. of the Dallas County Health Department for suggesting this research. They also thank the Center for Disease Control for permission to use the data of section 3.

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Table 3.1 Data For Intervention Area

(a) Aggregate Data

wave	n_t	Stage				Marginal fraction $\hat{q}_{ti}$			
		1	2	3	4	i=1	2	3	4
t = 1	145	20	20	61	44	.14	.14	.42	.30
2	114	16	15	50	33	.14	.13	.44	.29
3	123	16	17	59	31	.13	.14	.48	.25
4	106	8	7	43	48	.07	.07	.41	.45
5	103	13	7	57	26	.13	.07	.55	.25
6	104	17	7	55	25	.16	.07	.53	.24
7	112	3	3	80	26	.03	.03	.71	.23
8	120	14	6	69	31	.12	.04	.58	.26

(b) Transition Data from Haphazard Recaptures

t	wave pair	$n_{t,t+1}$	n_{11}	n_{12}	n_{13}	n_{14}	n_{21}	n_{22}	n_{23}	n_{24}	n_{31}	n_{32}	n_{33}	n_{34}	n_{41}	n_{42}	n_{43}	n_{44}
1	(1,2)	23	0	0	3	0	0	0	0	0	2	1	5	5	0	0	2	5
2	(2,3)	16	0	1	1	0	0	1	2	1	1	0	4	1	0	0	2	2
3	(3,4)	21	1	0	0	1	0	0	2	0	1	0	10	1	0	0	1	4
4	(4,5)	17	1	0	0	0	0	0	0	0	1	0	7	1	0	0	3	4
5	(5,6)	14	3	0	1	0	0	1	0	0	0	0	4	1	0	0	1	3
6	(6,7)	15	0	0	3	0	0	0	1	0	0	0	3	1	0	0	5	2
7	(7,8)	21	0	0	0	0	0	0	1	0	2	1	9	2	0	0	2	4
	$\hat{p}_{ij}$		.37	.06	.50	.07	.00	.24	.66	.10	.10	.03	.69	.18	.00	.00	.41	.59

Table 3.2. Data for Comparison Area

(a) Aggregate Data

wave	n_t	Stage Marginal fraction $\hat{q}_{ti}$							
		1	2	3	4	$i=1$	2	3	4
$t=1$	148	47	15	63	23	.32	.10	.43	.15
2	103	16	12	62	13	.16	.12	.60	.12
3	131	31	14	65	21	.24	.11	.50	.16
4	118	25	18	59	16	.21	.15	.50	.14
5	114	31	13	55	15	.27	.11	.48	.13
6	127	29	16	72	10	.23	.13	.57	.08
7	152	37	17	77	21	.24	.11	.51	.14
8	147	25	5	110	7	.17	.03	.75	.05

(b) Transition Data from Haphazard Recaptures

t wave pair	$n_{t,t+1}$	n_{11}	n_{12}	n_{13}	n_{14}	n_{21}	n_{22}	n_{23}	n_{24}	n_{31}	n_{32}	n_{33}	n_{34}	n_{41}	n_{42}	n_{43}	n_{44}
1 (1,2)	14	1	1	1	1	2	0	0	1	1	0	5	0	0	0	0	1
2 (2,3)	16	4	0	2	0	0	1	0	0	3	1	5	0	0	0	0	0
3 (3,4)	22	4	2	3	0	2	1	1	0	2	0	4	1	0	0	2	0
4 (4,5)	23	5	0	2	0	0	0	2	0	4	0	8	1	0	0	1	0
5 (5,6)	20	6	0	4	0	1	0	0	0	2	1	4	0	0	0	1	1
6 (6,7)	21	2	1	1	0	0	1	1	0	2	2	7	2	0	0	1	1
7 (7,8)	17	9	1	1	0	1	1	0	0	2	1	1	0	0	0	0	0
$\hat{p}_{ij}$		.60	.09	.29	.02	.40	.25	.28	.07	.27	.08	.59	.06	.00	.00	.64	.36

Appendix

We sketch here the proofs of the main theorems. Complete details are given in the technical report HHE.

Proof of Theorem 2.1

Since $\underline{Y}_{2N}$ is a vector of ratios (see (2.9)), consider the expanded vector

$$\underline{W}_N^* = [\hat{\underline{q}}_N^{*T}, \underline{R}_{N1}^T, \dots, \underline{R}_{Nm}^T, D_{N1}, \dots, D_{Nm}]^T$$

where $\underline{R}_{Ni}$ and D_{Ni} are defined in section 2.5.2. The asymptotic normality of $\underline{W}_N^*$ implies $\underline{W}_N$ by the delta method, so we show the former.

Lemma A.1. Under model (2.4) and assumptions (A1)-(A4),

$$\underline{B}_N^* (\underline{W}_N^* - E \underline{W}_N^*) \xrightarrow{L} N(\underline{0}, \underline{\Gamma}_{\omega^*}) \text{ as } N \rightarrow \infty,$$

where $\underline{B}_N^* = \text{Block diag } \{n_1^{1/2} \underline{I}_{m-1}, \dots, n_r^{1/2} \underline{I}_{m-1}, \underbrace{g_N^{1/2} \underline{I}_{m-1}, \dots, g_N^{1/2} \underline{I}_{m-1}}_{m \text{ times}},$

$g_N^{1/2} \underline{I}_m\}$, $g_N = \sum_{t=1}^{r-1} n_t n_{t+1} / N$ and $\underline{\Gamma}_{\omega^*}$ is some positive definite covariance

matrix of no direct interest.

Proof of Lemma A.1. The idea is to show that any linear compound $\underline{\lambda}^T \underline{W}_N^*$ is asympt univariate normal. In this direction, recall that $\hat{\underline{q}}_N^* = [\hat{\underline{q}}_{1N}, \dots, \hat{\underline{q}}_{rN}^T]^T$, with $\hat{\underline{q}}_{tN}$ given by

and so is a vector of linear statistics in the random sample indicators I_{lt} . Asymptotic n of such statistics under our assumptions (A3) are well known; see White (1988), Hajek (1960 (1965) or Madow (1948). However, observe that

$$\underline{R}_{Ni} = \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N Z_{lti} Z_{l,t+1} I_{lt} I_{l,t+1} \quad \text{and} \quad D_{Ni} = \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N Z_{lti} I_{lt} I_{l,t+1}$$

are nonlinear statistics in the indicators I_{lt} , which apparently require a modified apprr limit distribution theory. Fortunately, the method of Hajek (1962, 1968) for approximat statistics by sums of independent random variables can be extended to these statistics a the required result.

Hajek's method begins with the observation that statistics in the sampling indicators the same distributions as certain rank statistics. Specifically, for each $1 \leq t \leq r$ defin $U(0,1)$ variables $V_{Nl}^{(t)}$, $l=1, \dots, N$ and let $\underline{R}_N^{(t)} = (R_{N1}^{(t)}, \dots, R_{NN}^{(t)})$ denote the vector of ranks of

$V_{N1}^{(t)}, \dots, V_{NN}^{(t)}$. Then the random vector

$\underline{I}_{Nt}^* = (I(R_{N1}^{(t)} \leq n_t), \dots, I(R_{NN}^{(t)} \leq n_t))$ has the same joint distribution as does

$\underline{I}_{Nt} = (I_{1t}, \dots, I_{Nt})$ (conditional on $\sum_{l=1}^N I_{lt} = n_t$). Further, if (as we

assume) $\underline{I}_{N1}, \dots, \underline{I}_{Nr}$ are independent, and if $\{V_{Nl}^{(t)}\}$ are independent across $t=1, \dots, r$, then the $\underline{I}_N^* = (\underline{I}_{N1}^*, \dots, \underline{I}_{Nr}^*)$ and $\underline{I}_N = (\underline{I}_{N1}, \dots, \underline{I}_{Nr})$ have the same distribution, as do $h(\underline{I}_N^*)$ and $h(\underline{I}_N)$ measurable map h . The implication of this here is that for the limit distribution of $\underline{W}_N^*$, assume that the component statistics are defined in terms of $\underline{I}_N^*$ rather than $\underline{I}_N$ (as currently given).

The next step in Hajek's method is to approximate the resulting rank statistics in independent random variables. For linear statistics like the $\hat{q}_t$ (with $\underline{I}_{Nt}^*$ replacing $\underline{I}_N$ the above discussion), we have Hajek's basic lemma, given here as reformulated by White (1 adapted to the present context.

Lemma A.2. For each $t \in \{1, \dots, r\}$ and any scalar array $\{X_{Nkt} : 1 \leq k \leq N\}$ satisfying (as $N \rightarrow \infty$)

$$(i) \quad n_t^{1/2} \bar{X}_{Nt} \rightarrow 0, \quad \bar{X}_{Nt} =: \frac{1}{N} \sum_{k=1}^N X_{Nkt}; \text{ and}$$

$$(ii) \quad n_t^{1/2} S_{Nt}^2 \rightarrow 0, \quad S_{Nt}^2 =: \frac{1}{N-1} \sum_{k=1}^N (X_{Nkt} - \bar{X}_{Nt})^2,$$

we have $E(U_{Nt} - U_{Nt}^*)^2 \rightarrow 0$, where

$$U_{Nt} = \sum_{k=1}^N X_{Nkt} I(R_{Nk}^{(t)} \leq n_t), \quad U_{Nt}^* = \sum_{k=1}^N X_{Nkt} I(V_{Nk}^{(t)} \leq n_t/N).$$

Remark. Lemma A.2. reduces the limit distribution problem for linear statistics U_{Nt} to U_{Nt}^* , which is a linear combination of the independent random variables $\beta_{Nkt}^* =: I(V_{Nk}^{(t)} \leq n_t/N)$, is amenable to the central limit theorem for triangular arrays. The following result, p HHE, extends Lemma A.2 to statistics of the form $\underline{R}_{Ni}$ and D_{Ni} . Define $\beta_{Nkt} =: I(R_{Nk}^{(t)} \leq n_t)$.

Lemma A.3. If the arrays $\{X_{Nkt} : 1 \leq k \leq N\}, t=1, \dots, r$ each satisfy conditions (i) and (ii) of Lemma A.2 and if, for $1 \leq t \leq r-1$,

$$\xi_{Nt} = \sum_{k=1}^N X_{Nkt} \beta_{Nkt} \beta_{Nk, t+1},$$

$$\xi_{Nt}^* = \sum_{k=1}^N X_{Nkt} \beta_{Nkt}^* \beta_{Nk,t+1}^*,$$

then as $N \rightarrow \infty$,

$$(a) E(\xi_{Nt} - \xi_{Nt}^*)^2 \rightarrow 0, \quad 1 \leq t \leq r-1; \text{ and}$$

$$(b) E\left\{\sum_{t=1}^{r-1} \xi_{Nt} - \sum_{t=1}^{r-1} \xi_{Nt}^*\right\}^2 \rightarrow 0.$$

To apply these results to $\underline{W}_N^*$ (with ranks substituted), we first need to decompose component statistics appropriately. We have, for $1 \leq t \leq r$,

$$\hat{\underline{q}}_{Nt} = : n_t^{1/2} \{ \hat{\underline{q}}_{Nt} - E \hat{\underline{q}}_{Nt} \} = n_t^{-1/2} \sum_{l=1}^N (\underline{Z}_{lt} - \bar{\underline{Z}}_t^{(N)}) \beta_{Nlt} = : \sum_{l=1}^N {}_1\underline{X}_{Nlt} \beta_{Nlt}, \text{ where}$$

$${}_1\underline{X}_{Nlt} = n_t^{-1/2} (\underline{Z}_{lt} - \bar{\underline{Z}}_t^{(N)}) \text{ and } \bar{\underline{Z}}_t^{(N)} = : \frac{1}{N} \sum_{l=1}^N \underline{Z}_{lt}. \text{ Further, for } 1 \leq i \leq m,$$

$$\hat{\underline{r}}_{Ni} = : g_N^{1/2} \{ \underline{R}_{Ni} - E \underline{R}_{Ni} \}$$

$$\begin{aligned} &= g_N^{1/2} \left\{ \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^n \underline{e}_{lti} \beta_{Nlt} \beta_{Nl,t+1} - \sum_{t=1}^{r-1} \bar{\underline{e}}_{it}^{(N)} \right\} \\ &= g_N^{1/2} \left\{ \sum_{t=1}^{r-1} g_{Nt}^{-1} \sum_{l=1}^N \underline{e}_{lti} \beta_{Nlt} \beta_{Nl,t+1} - \sum_{t=1}^{r-1} g_{Nt}^{-1} n_{t,t+1} \bar{\underline{e}}_{it}^{(N)} \right\} \\ &\quad + g_N^{1/2} \left\{ \sum_{t=1}^{r-1} g_{Nt}^{-1} n_{t,t+1} \bar{\underline{e}}_{it}^{(N)} - \sum_{t=1}^{r-1} \bar{\underline{e}}_{it} \right\} \\ &= \sum_{t=1}^r \sum_{l=1}^N g_N^{1/2} g_{Nt}^{-1} \{ \underline{e}_{ilt} - \bar{\underline{e}}_{it}^{(N)} \} \beta_{Nlt} \beta_{Nl,t+1} \end{aligned}$$

$$\begin{aligned}
& + g_N^{1/2} \sum_{t=1}^{r-1} \frac{\bar{\epsilon}_{it}^{(N)}}{g_{Nt}} \left\{ \sum_{l=1}^N \beta_{Nlt} \beta_{Nl,t+1} - n_t n_{t+1} / N \right\} \\
& =: \sum_{t=1}^{r-1} \sum_{l=1}^N {}_2X_{Nlt}^{(i)} \beta_{Nlt} \beta_{Nl,t+1} \quad (\text{call this } {}_1\hat{\epsilon}_{Ni}) \\
& + \sum_{t=1}^{r-1} {}_2d_{Nti} \sum_{l=1}^N \{ \beta_{Nlt} \beta_{Nl,t+1} - E(\beta_{Nlt} \beta_{Nl,t+1}) \} \quad (\text{call this } {}_2\hat{\epsilon}_{Ni})
\end{aligned}$$

where ${}_2X_{Nlt}^{(i)} =: g_N^{1/2} g_{Nt}^{-1} \{ \underline{\epsilon}_{ilt} - \bar{\underline{\epsilon}}_{it}^{(N)} \}$, $\bar{\underline{\epsilon}}_{it}^{(N)} =: \frac{1}{N} \sum_{l=1}^N \underline{\epsilon}_{ilt}$

and ${}_2d_{Nti} = g_N^{1/2} \bar{\underline{\epsilon}}_{it}^{(N)} / g_{Nt}$. Similarly, for $1 \leq i \leq m$,

$$\hat{\delta}_{Ni} =: g_N^{1/2} \{ D_{Ni} - E D_{Ni} \}$$

$$\begin{aligned}
& = \sum_{t=1}^{r-1} \sum_{l=1}^N {}_3X_{Nlt}^{(i)} \beta_{Nlt} \beta_{Nl,t+1} \quad (\text{call this } {}_1\hat{\delta}_{Ni}) \\
& + \sum_{t=1}^{r-1} {}_3d_{Nti} \sum_{l=1}^N \{ \beta_{Nlt} \beta_{Nl,t+1} - E(\beta_{Nlt} \beta_{Nl,t+1}) \} \quad (\text{call this } {}_2\hat{\delta}_{Ni})
\end{aligned}$$

where ${}_3X_{Nlt}^{(i)} =: g_N^{1/2} g_{Nt}^{-1} \{ Z_{lti} - \bar{Z}_{it}^{(N)} \}$, $\bar{Z}_{it}^{(N)} =: \frac{1}{N} \sum_{l=1}^N Z_{ilt}$

and ${}_3d_{Nti} =: g_N^{1/2} \bar{Z}_{it}^{(N)} / g_{Nt}$.

The next step is to obtain approximations to $\hat{\alpha}_{Nt}$, $\hat{\tau}_{Ni}$ and $\hat{\delta}_{Ni}$ in terms of the ind variables β_{Nlt}^* . In this direction, we have first that

$$E \| \hat{\alpha}_{Nt} - \hat{\alpha}_{Nt}^* \|^2 \rightarrow 0, \quad E \| {}_1\hat{\tau}_{Ni} - {}_1\hat{\tau}_{Ni}^* \|^2 \rightarrow 0, \quad E \| {}_1\hat{\delta}_{Ni} - {}_1\hat{\delta}_{Ni}^* \|^2 \rightarrow 0,$$

by Lemma A.2, Lemma A.3 and Lemma A.3, respectively, where $\hat{\alpha}_{Nt,1}^*$, $\hat{\tau}_{Ni}^*$ and $\hat{\delta}_{Ni}^*$ are the same corresponding "unstarred" quantities, but with β_{Nlt} replaced by β_{Nlt}^* . Results (A.6) verifying that each of the arrays

${}_1\hat{X}_{Nlt}$, ${}_2\hat{X}_{Nlt}$ and ${}_3\hat{X}_{Nlt}$ satisfies conditions (i) and (ii) of Lemma A.2. This is done in HHE.

The approximation is completed if we can find suitable approximations for ${}_2\hat{\tau}_{Ni}$ in (A.3) in (A.5). Note that Lemma A.3 does not apply to these quantities. Nevertheless, the result holds. (Proof in HHE)

Lemma A.4. Let

$${}_2\hat{\tau}_{Ni}^* = \sum_{t=1}^{r-1} {}_2d_{Nti} \sum_{l=1}^N \{ \beta_{Nlt}^* \beta_{Nl,t+1}^* - E(\beta_{Nlt}^* \beta_{Nl,t+1}^*) \},$$

$${}_2\hat{\delta}_{Ni}^* = \sum_{t=1}^{r-1} {}_3d_{Nti} \sum_{l=1}^N \{ \beta_{Nlt}^* \beta_{Nl,t+1}^* - E(\beta_{Nlt}^* \beta_{Nl,t+1}^*) \}.$$

Then $E \| {}_2\hat{\tau}_{Ni} - {}_2\hat{\tau}_{Ni}^* \|^2 \rightarrow 0$ and $E \{ {}_2\hat{\delta}_{Ni} - {}_2\hat{\delta}_{Ni}^* \}^2 \rightarrow 0$.

Results (A.6) and (A.7) imply that, for arbitrary λ ,

$$E \{ \lambda^T \bullet B_N^* [W_N^* - E W_N^*] - \lambda^T \tilde{W}_N \}^2 \rightarrow 0, \text{ where}$$

$$\tilde{W}_N^T = [\hat{\alpha}_{N1}^{*T}, \dots, \hat{\alpha}_{Nr}^{*T}, ({}_1\hat{\tau}_{N1}^* + {}_2\hat{\tau}_{N1}^*)^T, \dots, ({}_1\hat{\tau}_{Nm}^* + {}_2\hat{\tau}_{Nm}^*)^T, {}_1\hat{\delta}_{Ni}^* + {}_2\hat{\delta}_{N1}^*, \dots, {}_1\hat{\delta}_{Nm}^* + {}_2\hat{\delta}_{Nm}^*].$$

The asymptotic normality of $\lambda^T \tilde{W}_N$ follows from the central limit theorem for triangular independent random variables. (See the proof in HHE). Together with (A.8), this proves Lem