COVARIANT COMPUTATION IN
SPECIAL RELATIVISTIC DYNAMICS

by

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TR #298

DATE DUE

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ABSTRACT

The motion of a particle in a special relativistic framework is studied numerically. Difference equations which are covariant are developed first. It is then shown how computations can be done in both the lab and rocket frames so that the numerical results are related by the Lorentz transformation. Finally, assuming the usual force formula for harmonic oscillation, it is shown that the dynamical equation is necessarily nonlinear and trajectories are studied numerically for various initial data.

1. Introduction Most popular numerical methods, like Runge-Kutta, multistep, and Taylor expansion methods, when applied to approximate solutions of second order, nonlinear, Newtonian initial value problems, do not preserve fundamental physical invariants, like energy and momentum, when these are present. However, specialized implicit methods are now available which do preserve such invariants [1]. If one considers second order, nonlinear, special relativistic initial value problems, it would be important to have a numerical method which, when applied on identical computers in the lab and rocket frames, would yield numerical results which are connected by the Lorentz transformation. In this paper, we show how this can be done in a way that is consistent with recent models of discrete space-time [2-7].

2. Relativistic Considerations Consider two one-dimensional, inertial frames, coincident at time $t=0$, denoted by X and X' . Let X' be in motion in the x direction with speed u relative to the X frame. The X frame is called the lab frame and the X' frame the rocket frame. All the usual assumptions about lab and rocket frames [8 - 12] are appropriated here. In particular, we have the Axiom of Relativity, that is, that all laws of physics and all physical constants, including the speed of light c , are the same in every initial reference frame. The Lorentz transformation connecting events (x,t) in the lab frame and (x',t') in the rocket frame are

$$(2.1a) \quad x' = \frac{c(x - ut)}{(c^2 - u^2)^{\frac{1}{2}}} \quad t' = \frac{c^2 t - ux}{c(c^2 - u^2)^{\frac{1}{2}}}$$

or, equivalently,

$$(2.1b) \quad x = \frac{c(x' + ut')}{(c^2 - u^2)^{\frac{1}{2}}} \quad , \quad t = \frac{c^2 t' + ux'}{c(c^2 - u^2)^{\frac{1}{2}}} \quad .$$

Next, recall that all velocities to be considered will satisfy $|v| < c$. Finally, recall that the motion P of a particle of mass m , acted upon by a force F , is given in the lab frame by

$$(2.2) \quad F = \frac{d}{dt}(mv), \quad m = \frac{m_o}{\sqrt{1 - (\frac{v}{c})^2}}$$

and, covariantly, in the rocket frame by

$$(2.3) \quad F' = \frac{d}{dt'}(m'v'), \quad m' = \frac{m_o}{\sqrt{1 - \left(\frac{v'}{c}\right)^2}}.$$

3. Numerical Methodology In addition to identical clocks, we assume now that the observers in the lab and rocket frames have identical computers. In the lab, let $\Delta t > 0$ and $t_k = k\Delta t, k=0,1,\dots$. At time t_k , let P be at x_k in the lab. Then, in the rocket, P will be at x'_k at time t'_k , where

$$(3.1) \quad x'_k = \frac{c(x_k - ut_k)}{(c^2 - u^2)^{\frac{1}{2}}}, \quad t'_k = \frac{c^2 t_k - ux_k}{c(c^2 - u^2)^{\frac{1}{2}}},$$

or equivalently,

$$(3.2) \quad x_k = \frac{c(x'_k + ut'_k)}{(c^2 - u^2)^{\frac{1}{2}}}, \quad t_k = \frac{c^2 t'_k + ux'_k}{c(c^2 - u^2)^{\frac{1}{2}}}.$$

The validity of (3.1) and (3.2) follows because they are merely special cases of (2.1a) and (2.1b), that is, they result from the particular choices $x = x_k, t = t_k$.

The concepts of velocity and acceleration are now introduced as follows. At t_k in the lab, set

$$(3.3) \quad v_k = \frac{\Delta x_k}{\Delta t_k} = \frac{x_{k+1} - x_k}{t_{k+1} - t_k}, \quad a_k = \frac{\Delta v_k}{\Delta t_k} = \frac{v_{k+1} - v_k}{t_{k+1} - t_k}$$

At t'_k in the rocket, then, let

$$(3.4) \quad v'_k = \frac{\Delta x'_k}{\Delta t'_k} = \frac{x'_{k+1} - x'_k}{t'_{k+1} - t'_k}, \quad a'_k = \frac{\Delta v'_k}{\Delta t'_k} = \frac{v'_{k+1} - v'_k}{t'_{k+1} - t'_k}.$$

Direct substitution of (3.1) and (3.2) into (3.3) and (3.4) yields

$$(3.5) \quad v'_k = \frac{c^2(v_k - u)}{c^2 - uv_k}, \quad v_k = \frac{c^2(v'_k + u)}{c^2 + uv'_k}$$

$$(3.6) \quad a'_k = \frac{c^3(c^2 - u^2)^{\frac{3}{2}}}{(c^2 - uv_{k+1})(c^2 - uv_k)^2} a_k, \quad a_k = \frac{c^3(c^2 - u^2)^{\frac{3}{2}}}{(c^2 + uv_{k+1}')(c^2 + uv'_k)^2} a'_k.$$

In the limit, (3.3)-(3.6) converge to their correct continuum counterparts.

We turn finally to (2.2), (2.3). At t_k the lab, we approximate (2.2) by the arithmetic formula

$$(3.7) \quad F_k = \frac{m(t_k)c^2}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{\frac{1}{2}}} a_k, \quad m(t_k) = \frac{cm_o}{\sqrt{c^2 - v_k^2}}$$

We now have a basic theorem.

Theorem 1 If in the rocket

$$(3.8) \quad m'(t_k') = \frac{cm_o}{\sqrt{c^2 - (v_k')^2}},$$

and if $F_k' = F_k$, then under the Lorentz transformation (3.7) is covariant, that is, it maps into

$$(3.9) \quad F_k' = \frac{m'(t_k')c^2}{[(c^2 - (v_k')^2)(c^2 - (v_{k+1}')^2)]^{\frac{1}{2}}} a_k'$$

Proof . The proof follows by direct substitution of (3.3) into the right-hand-side of (3.7) and algebraic simplification using the mass formulas.

Once force F is given, trajectory computation in the lab proceeds as follows. First normalize so that $m_o = c = 1$. Then (3.3) and (3.7) are rewritten in the forms

$$(3.10) \quad x_{k+1} = x_k + (t_{k+1} - t_k)v_k, \quad k = 0, 1, 2, \dots$$

$$(3.11) \quad v_{k+1} = v_k + (t_{k+1} - t_k)(1 - v_k^2)(1 - v_{k+1}^2)^{\frac{1}{2}} F_k, \quad k = 0, 1, 2, \dots$$

Unfortunately, (3.11) is implicit in v_{k+1} . This impediment is, however, easily removed by solving for v_{k+1} explicitly. This is done by writing (3.11) as

$$v_{k+1} - v_k = (t_{k+1} - t_k)(1 - v_k^2)(1 - v_{k+1}^2)^{\frac{1}{2}} F_k,$$

squaring both sides, and solving the resulting quadratic equation for v_{k+1} . Of the two resulting roots, one is extraneous and the unique solution is

$$(3.12) \quad v_{k+1} = \frac{v_k + (t_{k+1} - t_k)(1 - v_k^2)F_k \sqrt{1 - v_k^2 + (t_{k+1} - t_k)^2(1 - v_k^2)^2 F_k^2}}{1 + (t_{k+1} - t_k)^2(1 - v_k^2)^2 F_k^2}$$

In the limit, (3.7) and (3.9) converge to equations which are entirely equivalent to (2.2) and (2.3).

Thus, in the lab, motion of a particle P , from given initial data, is generated uniquely by (3.10) and (3.12), $k=0,1,2,\dots$. In the rocket, the would be calculated from

$$(3.13) \quad x_{k+1}' = x_k' + (t_{k+1}' - t_k')v_k'$$

$$(3.14) \quad v_{k+1}' = \frac{v_k' + (t_{k+1}' - t_k')(1 - (v_k')^2)F_k' \sqrt{1 - (v_k')^2 + (t_{k+1}' - t_k')^2(1 - (v_k')^2)^2 (F_k')^2}}{1 + (t_{k+1}' - t_k')^2(1 - (v_k')^2)^2 (F_k')^2}$$

We now come to our major theorem.

Theorem 2 Computations in the lab and the rocket frames of the motion of a particle P of mass m acted upon by a given force are related by the Lorentz transformation.

Proof Assume that the initial data x_o, v_o, x_o', v_o' are related by the Lorentz transformation. Hence

$$(3.15) \quad x_o' = \frac{x_o - ut_o}{(1 - u^2)^{\frac{1}{2}}}$$

$$(3.16) \quad v_o' = \frac{v_o - u}{1 - uv_o}$$

Now, by (3.13)

$$(3.17) \quad x_1' = x_o' + (t_1' - t_o')v_o'.$$

Suppose that under the Lorentz transformation x_1' maps into $\bar{x}_1$. Then

$$(3.18) \quad x_1' = \frac{\bar{x}_1 - ut_1}{(1 - u^2)^{\frac{1}{2}}}$$

We need only show that $\bar{x}_1 = x_1$.

Hence, (3.15)-(3.18) and

$$t_o' = \frac{t_o - ux_o}{(1-u^2)^{\frac{1}{2}}}, \quad t_1' = \frac{t_1 - u\bar{x}_1}{(1-u^2)^{\frac{1}{2}}}$$

yield

$$\begin{aligned} \frac{\bar{x}_1 - ut_1}{(1-u^2)^{\frac{1}{2}}} &= x_o' + v_o'(t_1' - t_o') \\ &= \frac{x_o - ut_o}{(1-u^2)^{\frac{1}{2}}} + \left(\frac{v_o - u}{1 - uv_o} \right) (t_1 - t_o + ux_o - u\bar{x}_1). \end{aligned}$$

Hence

$$\bar{x}_1 - ut_1 = x_o - ut_o + \left(\frac{v_o - u}{1 - uv_o} \right) (t_1 - t_o + uv_o - u\bar{x}_1),$$

or

$$(\bar{x}_1 - ut_1)(1 - uv_o) = (x_o - ut_o)(1 - uv_o) + (v_o - u)(t_1 - t_o + ux_o - u\bar{x}_1),$$

which simplifies to

$$(\bar{x}_1 - x_o)(1 - u^2) = v_o(1 - u^2)(t_1 - t_o)$$

Thus,

$$\bar{x}_1 = x_o + (t_1 - t_o)v_o$$

so that, by (3.10), $\bar{x}_1 = x_1$ and, by induction it follows that x_{k+1}' and x_{k+1} are related by the Lorentz transformation.

Let us show next that v_{k+1}' and v_{k+1} are related by the Lorentz transformation. For $k=0$, (3.14) implies

$$(3.19) \quad v_1' = \frac{v_o' + (t_1' - t_o')(1 - (v_o')^2)F_o' \sqrt{1 - (v_o')^2 + (t_1' - t_o')^2(1 - (v_o')^2)^2(F_o')^2}}{1 + (t_1' - t_o')^2(1 - (v_o')^2)^2(F_o')^2}$$

Substitution of

$$t_1' = \frac{t_1 - ux_1}{(1 - u^2)^{\frac{1}{2}}}, \quad t_0' = \frac{t_0 - ux_0}{(1 - u^2)^{\frac{1}{2}}}, \quad x_0' = \frac{x_0 - ut_0}{(1 - u^2)^{\frac{1}{2}}}, \quad v_0' = \frac{v_0 - u}{1 - uv_0}$$

and

$$F_0' = F_0 = \frac{v_1 - v_0}{(t_1 - t_0)(1 - v_0^2)(1 - v_1^2)^{\frac{1}{2}}}$$

into (3.19) yields, after simplification,

$$(3.20) \quad v_1' = \frac{(v_1 - u)[(1 - v_0v_1)(1 - uv_0) + (v_1 - v_0)(u - v_0)]}{(1 - uv_1)[(1 - v_0v_1)(1 - uv_0) + (v_1 - v_0)(u - v_0)]}$$

The theorem will follow then from (3.20), since the terms in the brackets are identical, provided these terms are not zero. However, this is the case and we will show it next, but recall that all velocities under consideration are less than the speed of light. Hence, since $|u| < 1$ and $|v| < 1$, then

$$\begin{aligned} (1 - u^2)(1 - v^2) &> 0 \\ (u^2 - 1)(1 - v^2) &< 0 \\ u^2 + v^2 - 1 - u^2v^2 &< 0 \\ u^2 + v^2 &< 1 + u^2v^2 \\ u^2 - 2uv + v^2 &< 1 + u^2 - 2uv \\ (u - v)^2 &< (1 - uv)^2 \end{aligned}$$

so that

$$|u - v| \leq |1 - uv| = 1 - uv$$

Hence

$$|v_1 - v_0| < 1 - v_0v_1, \quad |u - v_0| < 1 - uv_0$$

so that

$$(1 - v_0v_1)(1 - uv_0) > |v_1 - v_0| \cdot |u - v_0| \geq -(v_1 - v_0)(u - v_0).$$

Thus

$$(1 - v_0v_1)(1 - uv_0) + (v_1 - v_0)(u - v_0) > 0,$$

so that the terms in the brackets in (3.20) cancel.

4. A Relativistic Harmonic Oscillator As in Newtonian mechanics, we now define a harmonic oscillator in the lab frame to be a particle P of mass m upon which acts a force $F = -K^2x$. For simplicity K is normalized to unity.

Assume then that $x(0) = x_0 = 0$ and $v(0) = v_0$. With $\Delta t = 0.0001$, the motion of P was generated using (3.10) and (3.11) for 30,000 times for each of the cases $v_0 = 0.001, 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9$.

Fig. 1 shows the amplitude and period of the first complete oscillation for the case $v_0 = 0.001$. For such a relatively low velocity, the oscillator should behave like a Newtonian oscillator and, indeed, this is the case, with the amplitude being 0.001 and, to two decimal places, the period being 6.28 ($\sim 2\pi$). Subsequent motion of this oscillator continues to show almost no change in amplitude or period. At the other extreme, Fig. 2 shows the motion for $v_0 = 0.9$, which is relatively close to the speed of light. To two decimal places, the amplitude of the first oscillation is 1.61 while the period is 8.88. These results are distinctly non-Newtonian, and, to thirty thousand time steps, these results remain constant to two decimal places but do show small increments in the third decimal place. Finally, in Fig. 3 is shown how the amplitude of a relativistic oscillator deviates from that of a Newtonian oscillator with increasing v_0 .

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