**FIXED POINT THEOREMS ON CLOSED SETS
THROUGH ABSTRACT CONES**

by

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Introduction. Let D be a closed subset of a complete metric space (X, ρ) . We seek (i) conditions upon which a map $T : D \rightarrow X$ has a fixed point in D and (ii) the construction of an iterative sequence whose limit is a fixed point in D . If X is a Banach space then a classical approach is to set $G = I - T$ and use a numerical search method to minimize $\|Gx\|$ in D . Another approach, which does not require a Banach space structure, was recently introduced by Caristi and Kirk ([1],[2]). They prove that a metrically inward contractor map T has a fixed point. Both methods assume conditions which guarantee that for arbitrary x in D there exists y in D such that $\rho(y, Ty) < \rho(x, Tx)$. This condition is the basis of our study.

In this paper we consider, as in [3] - [5], a generalized norm, i.e., a norm which takes values in an abstract cone κ and a comparison map $\psi : \kappa \rightarrow \kappa$, which enjoys certain properties in common with the map $u \rightarrow \alpha u$ where $0 \leq \alpha < 1$, but is not necessarily linear. Our main assumption is that there exist, for arbitrary x in D , a $y = f(x)$ in D , such that

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$$(1.1) \quad \rho(y, Ty) \leq \psi(\rho(x, Tx))$$

or, in terms of $G = I - T$,

$$||Gy|| \leq \psi(||Gx||).$$

We require that T is continuous but the function f need not be continuous to obtain the existence of a fixed point. However, if f is continuous then for arbitrary x_0 in D , the sequence of iterates $\{f^n(x_0)\}$ converges to a fixed point z and one obtains bounds on the errors $\rho(f^n(x_0), z)$, $n = 1, 2, \dots$.

In the special case $D = X$ and T is a nonlinear contractor in the sense that

$$(1.3) \quad \rho(Tx, Ty) \leq \psi(\rho(x, y)), \quad x, y \in X,$$

then one can take $f(x) = Tx$ to obtain an iterative sequence which converges to a fixed point. In Section 4, we make use of this observation to expand on results in [3] - [5] dealing with generalizations of the contractive mapping principle (see Section 4).

Our main results on fixed points or zeros on closed subsets are given in Section 3.

The comparison map ψ is assumed monotone (nondecreasing) and it is assumed that the series

$$(1.4) \quad \sum_{n=0}^{\infty} \psi^n(u) \text{ is convergent for each } u \text{ in } \kappa.$$

In the special cases of linear positive maps ψ on finite dimensional

cones, sufficient conditions for (1.4) are known [6]. Section 2 deals with extensions of those results to nonlinear maps on more general cones.

2. Comparison maps and cones: Let E be a real Banach space. A set $\kappa \subset E$ is called a cone if (i) κ is closed; (ii) if $u, v \in \kappa$, then $\alpha u + \beta v \in \kappa$ for all $\alpha, \beta \geq 0$; (iii) of each pair of vectors $u, -u$ at least one does not belong to κ , provided $u \neq \theta$, where θ is the zero of the space E . We say that $u \geq v$ if and only if $u - v \in \kappa$. A set $M \subset E$ is called κ -bounded if there exists an element z such that $x \leq z$ ($x \in M$). If in the set N of all upper bounds, there is a unique element z_0 such that $z_0 \leq z$ for all $z \in N$, then z_0 is called the (least upper bound) $\sup$ of the set M . The (greatest lower bound) $\inf$ is defined in a similar fashion. The cone κ is called strongly minihedral if every κ -bounded set has a $\sup$. Every bounded subset M in a strongly minihedral cone has an $\inf$. In fact, if β and γ , respectively, denote the $\sup$ of M and the $\sup$ of $\{\beta - m \mid m \in M\}$, then the $\inf$ of M is $\beta - \gamma$. The cone κ is called minihedral if every pair of elements in E has a $\sup$. The cone is called regular if every nondecreasing sequence x_n ($x_1 \leq x_2 \leq \dots \leq x_n \leq \dots$) which is κ -bounded converges with respect to the norm. The following proposition combines some results from Krasnoselskii's book [7].

Proposition 2.1: In separable spaces a regular minihedral cone is strongly minihedral and the $\sup$ of a set M is the limit of a nondecreasing sequence of vectors of the form $\sup\{x_1, x_2, \dots, x_n\}$ where the x_i 's are in M . A similar statement holds for the $\inf$.

Some examples of regular, strongly minihedral cones are the cones in finite dimensional spaces and the cones κ_p of nonnegative functions in L_p (the space of functions which are p th summable on a bounded set).

The strict inequality $x < y$ means $y - x$ belongs to the interior (relative to κ) of κ .

Let ψ be a map from a subset L of a cone κ into itself. ψ is said to be *monotone* on L if $\psi(u) \geq \psi(v)$ whenever $u \geq v$. Suppose L is closed. The map ψ is called *upper semicontinuous from the right, uscr* if whenever $\{u_n\}$ and $\{\psi(u_n)\}$ are both convergent, nonincreasing sequences in L then $\lim \psi(u_n) \leq \psi(\lim u_n)$. Similarly, ψ is called *lower semicontinuous from the left, lscr* if whenever $\{u_n\}$ and $\{\psi(u_n)\}$ are both convergent, nondecreasing sequences in L then $\lim \psi(u_n) \geq \psi(\lim u_n)$.

Let v and w be in κ with $v \leq w$. The *conic segment* formed by v and w is the subset of κ , $\{u : v \leq u \leq w\}$.

The first part of the following result was proved in [5]. It is a generalization of the Bellman-Gronwall-Reid inequality. The second part is obtained in an analogous fashion.

Proposition 2.2: Let $\langle \theta, u_0 \rangle$ be a segment in a regular cone and

let ψ be a monotone map from $\langle \theta, u_0 \rangle$ into itself. Then

(1) If ψ is uscr, then the sequence of iterates $\{\psi^n(u_0)\}$ is nonincreasing and converges to a fixed point w of ψ . Moreover if $v \leq \psi(v)$, then $v \leq w$. In particular, w is the maximal fixed point of ψ in $\langle \theta, u_0 \rangle$.

(11) If ψ is lscl then the sequence of iterates $\{\psi^n(\theta)\}$ is non-decreasing and converges to a fixed point z of ψ . Moreover, if $v \geq \psi(v)$ then $v \geq z$. In particular, z is the minimal fixed point of ψ on $\langle \theta, u_0 \rangle$.

We now turn our attention to the consideration of condition (1.4).

Lemma 2.1: Let κ be a regular cone and $\psi : \kappa \rightarrow \kappa$ be monotone and either uscr or lscl. Suppose:

(2.1) For every v in κ the equation $\psi(u) + v = u$ has at most one solution in κ .

(2.2) For every v in κ there exists $u \geq v$ such that $\psi(u) + v \leq u$

Then condition (1.4) is satisfied for ψ .

Proof: In view of condition (2.2) we may restrict attention to those u in κ such that $\psi(u) \leq u$. For such a u we have, for each integer $n > 0$, $s_n - \psi(s_n) = u - \psi^{n+1}(u) \geq \theta$ where $s_n = \sum_{i=0}^n \psi^{(i)}(u)$.

By condition (2.2) there exists $w \geq u$ such that the map $\phi : \kappa \rightarrow \kappa$ defined by $\phi(q) = \psi(q) + p$, where $p = u - \psi^{n+1}(u)$, satisfies $\phi(w) \leq w$. Since ϕ is monotone it maps the segment $\langle \theta, w \rangle$ into itself. It then follows from Proposition 2.2 that ϕ has a fixed point in $\langle \theta, w \rangle$. Since s_n is also a fixed point of ϕ , we deduce from condition (2.1) that $s_n \leq w$. Thus the nondecreasing sequence $\{s_n\}$ is bounded and must converge by regularity of the cone.

A map ψ from a cone into itself is said to be *subadditive* if the property $\psi(u + v) \leq \psi(u) + \psi(v)$ is satisfied on the cone.

Lemma 2.2: Let κ be a regular cone and suppose $\psi : \kappa \rightarrow \kappa$ is monotone and subadditive. Assume there exists $u_0 > \theta$ such that:

(i) $u_0 > \psi(u_0)$, (ii) $\lim_{n \rightarrow \infty} (u_n/n) = \theta$, (iii) θ is the unique fixed point of ψ in the segment $\langle \theta, u_0 \rangle$. Then the following is true.

- (a) ψ is continuous.
- (b) Conditions (2.1) and (2.2) are satisfied.
- (c) Condition (1.4) is satisfied and the map

$$\gamma(u) = \sum_{n=0}^{\infty} \psi^n(u) \text{ is monotone, subadditive and}$$

continuous.

Proof: (a) Monotonicity and subadditivity imply $\psi(u) \leq \psi(u + v) \leq \psi(u) + \psi(v)$ for u and v in κ . Thus $\theta \leq \psi(u + v) - \psi(v) \leq \psi(u)$. If ψ is continuous at θ then we see, from the latter inequality, that ψ is necessarily continuous on all of κ . To see that ψ is continuous at θ , observe, from condition (ii) and $u_0 > 0$, that for each positive integer m there exists a positive integer $j = j(m)$ such that $\psi(u_0/j) \leq u_0/m$. Now u_0/j bounds a neighborhood of θ in κ , say N . Thus for $u \in N$, $\psi(u) \leq \psi(u_0/j) \leq u_0/m$.

(b) Consider (2.2) first. Let v in κ be specified and choose an integer n sufficiently large such that $v \leq n(u_0 - \psi(u_0))$ and set $w = nu_0$. Subadditivity gives $\psi(nu_0) \leq nu_0$. Thus $\psi(w) + v \leq n\psi(u_0) + n(u_0 - \psi(u_0)) = nu_0 = w$.

Now consider (2.1). Let v in κ be specified and choose $w \geq v$ such that $\psi(w) + v \leq w$. Observe that the map $\phi : \kappa \rightarrow \kappa$

given by $\phi(u) = \psi(u) + v$ is monotone and maps the segment $\langle \theta, w \rangle$ into itself. Since ϕ is continuous, it follows from Proposition 2.2 (ii) that ϕ has a fixed point z and $z = \lim \phi^n(\theta)$. Let p be any fixed point of ϕ in κ . Since $p \geq \theta$ and ϕ is monotone $p = \phi^n(z) \geq \phi^n(\theta)$ for every integer $n \geq 0$. Thus $p \geq z$. Let $m = p - z$. Subadditivity implies $\psi(p) \leq \psi(m) + \psi(z)$. Thus $\psi(m) \geq \psi(p) - \psi(z) = \phi(p) - \phi(z) = p - z = m$. From condition (2.2), which we have already verified, there exists $w \geq m$ such that $\psi(w) \leq w$. It follows from Proposition 2.2 that $m \leq q$, the maximal solution of ψ on the segment $\langle \theta, w \rangle$. By hypotheses, $q = \theta$. Thus $p = z$ and z is the unique solution of $\psi(u) + v = u$ on κ .

(c) The existence of γ follows from (b) and Lemma 2.1. Clearly γ is monotone and subadditive. To see that γ is also continuous let a positive real number ε be specified. Choose an integer

n_0 sufficiently large so that $\sum_{j=n_0+1}^{\infty} \psi^j(u_0) \leq \varepsilon u_0/z$. Let N be

a neighborhood of θ in κ such that for u in N $u \leq u_0$ and $\sum_{j=0}^{n_0} \psi^j(u) \leq \varepsilon u_0/z$. It follows that for u in N , $\gamma(u) \leq \varepsilon u_0$.

In the case of a linear map $\psi : \kappa \rightarrow \kappa$ where κ is a finite dimensional cone, the hypotheses of the above result can be significantly weakened. In this case κ is necessarily regular and ψ is necessarily continuous and monotone. In addition, the uniqueness condition is unnecessary. Recall in the proof condition (2.2) is obtained before the uniqueness condition is used. From condition (2.2) one obtains the existence of a solution to $\psi(u) + v = u$ for

arbitrary v in κ . To see this let $u_0 \geq v$ such $\psi(u_0) + v \leq u_0$ and apply Proposition 2.2 to obtain a fixed point of the map $\phi(u) = \psi(u) + v$ on the segment $\langle \theta, u_0 \rangle$. Thus the map $I_\kappa - \psi$, where I_κ is the identity map on κ , is onto, hence one to one.

Observe also that the condition (2.2) implies $I_\kappa - \psi$ is onto, thus if $v > \theta$ there exists u such that $u - \psi(u) = v > \theta$.

In the case of linear maps on finite dimensional cones Lemma 2.2 has the following form.

Lemma 2.3: Let $\psi : \kappa \rightarrow \kappa$ be a linear map on a cone κ in finite dimensional space. Then there exists u in κ such that $u > \psi(u)$ iff κ has an interior point and condition (2.2) is satisfied.

Lemma 2.3 is known in the cases of cones consisting of vectors with nonnegative components where one has also several other sets of equivalent conditions of this type (see [6] and references cited therein).

Let X be a set and κ be a cone. A function $\rho : X \times X \rightarrow \kappa$ is said to be a κ -metric on X if ρ has the following properties: (i) $\rho(x, y) = \rho(y, x) \geq \theta$, (ii) $\rho(x, y) = \theta$ iff $x = y$, (iii) $\rho(x, y) = \rho(x, z) + \rho(z, y)$.

Convergence in a κ -metric space is more advantageous than convergence with respect to a scalar norm. See remark in [4, p.28].

We say that a κ -metric space (X, ρ) is *convex* if for given x, y in X and u in κ with $u \leq \rho(x, y)$ there exists z in X such that $\rho(x, z) + \rho(z, y) = \rho(x, y)$ and $\rho(x, z) \leq u$. In the special case $\kappa = R^+$, the nonnegative real line it is known that convexity is equivalent to the seemingly weaker condition: given

any two points $x \neq y$ there exists a point z , $z \neq x$, $z \neq y$, such that $\rho(x,y) = \rho(x,z) + \rho(z,y)$. Whether or not this result can be extended to more general cones is an open question. We note, however, that in some applications (see references in [4]) in which κ metric spaces are used, our definition of convex is satisfied.

Let κ be a regular cone. In [3] - [5] a comparison map ψ from a conic segment $\langle \theta, u_0 \rangle$ into itself is used which has the following properties. *Conditions A:*

(A1) ψ is monotone and uscr.

(A2) $\psi(u_0) < u_0$

(A3) $u = \theta$ is the unique fixed point of ψ on $\langle \theta, u_0 \rangle$.

The following lemma shows that in convex κ -metric space, where the cone is regular and strongly minihedral, the existence of a comparison map satisfying conditions A (above) implies the existence of another comparison map having more properties.

Lemma 2.4: Let κ be a regular, strongly minihedral cone, X and Y κ -metric spaces with X convex. Suppose ψ is a map, with properties (A), which compares a function $T : X \rightarrow Y$ i.e.

$$(2.3) \quad \rho(Tx, Ty) \leq \psi(\rho(x, y)) \quad \text{when} \quad \rho(x, y) \leq u_0$$

Then there is a map $\phi : \kappa \rightarrow \kappa$ which satisfies the hypotheses of Lemma 2.2 and

$$(2.4) \quad \rho(Tx, Ty) \leq \phi(\rho(x, y)) \quad \text{for all } x, y \text{ in } X$$

Moreover, $\phi(u) \leq \psi(u)$ for $\theta \leq u \leq u_0$.

Proof: For arbitrary u in κ , define the set $\Gamma(u) = \{\rho(Tx, Ty) : \rho(x, y) \leq u\}$. Since $\Gamma(u)$ is κ -bounded by $\psi(u)$ and the cone is strongly minihedral, $\phi(u) = \sup(\Gamma(u))$ exists and $\phi(u) \leq \psi(u)$, provided $u \leq u_0$. To see that ϕ is defined on all of κ we argue as follows.

We first show by induction that ϕ is defined on nu_0 , $n = 1, 2, \dots$. Suppose ϕ is defined on nu . Then if $\rho(x, y) \leq (n+1)u_0$ there exists, by convexity, z in X such that $\rho(x, z) \leq u_0$, and $\rho(z, y) \leq nu_0$. Thus $\rho(Tx, Ty) \leq \rho(Tx, Tz) + \rho(Tz, Ty) \leq \phi(u_0) + \phi(nu_0)$. It follows that $\Gamma((n+1)u_0)$ is κ -bounded and thus $\phi((n+1)u_0)$ is defined.

Since u_0 is an interior point, the segments $\langle \theta, nu_0 \rangle$, $n = 1, 2, \dots$, cover κ . Let u be arbitrary in κ and let an integer n , be chosen sufficiently large so that $u \leq nu_0$. Then $\Gamma(u)$ is κ -bounded by $\phi(nu_0)$ and so $\phi(u)$ is defined. Thus ϕ is defined on all of κ .

Clearly (2.4) is satisfied and ϕ is monotone.

For subadditivity, consider $\rho(x, y) = u = v + w$. Then, by convexity, there exists z in X such that $\rho(x, z) = v$ and $\rho(z, y) = w$. The inequality $\rho(Tx, Ty) \leq \rho(Tx, Tz) + \rho(Tz, Ty)$ gives $\phi(u) \leq \phi(v) + \phi(w)$.

Since ψ is uscr and $\psi(\theta) = \theta$, $\lim_{n \rightarrow \infty} \psi(u_0/n) = \theta$. Because

$$\psi(u_0/n) \geq \phi(u_0/n), \quad \lim_{n \rightarrow \infty} \phi(u_0/n) = \theta.$$

It remains to verify the uniqueness condition. Suppose u is a fixed point of ϕ in $\langle \theta, u_0 \rangle$. Then $\psi(u) \geq \phi(u) = u$. Proposition 2.2 implies $u \leq \theta$. Since $u \in \kappa$, $u = \theta$.

3. Fixed points on closed subsets: The following observations concerning the relationship between condition (1.1) and fixed points, are based, in part, on the work of Caristi and Kirk ([1], [2], [8]).

Lemma 3.1: Let X be a κ -metric space where κ is a regular cone. Then a function $f : X \rightarrow X$ admits a majorant is a map $\omega : X \rightarrow \kappa$ such that

$$(3.1) \quad \rho(x, f(x)) \leq \omega(x) - \omega(f(x)), \quad x \in X$$

iff the series

$$(3.2) \quad \sum_{n=0}^{\infty} \rho(f^n(x), f^{n+1}(x)) \text{ converges, } x \in X.$$

Proof: Suppose (3.1). Let x_0 be arbitrary in X and define

$$x_n = f^n(x_0) \quad \text{condition (3.1) implies that } s_n = \sum_{i=0}^n \rho(x_i, x_{i+1}) \leq$$

$\omega(x_0)$, $n = 1, 2, \dots$. Thus $\{s_n\}$ is increasing and κ -bounded, hence, convergent.

Suppose (3.2). Let

$$(3.3) \quad \omega(x) = \sum_{i=0}^{\infty} \rho(f^i(x), f^{i+1}(x)), \quad x \in X$$

Clearly (3.1) is satisfied.

The above argument shows that if the set of majorants of f is non-empty, then it contains a least member given by (3.3).

Lemma 3.2: Let X be a complete κ -metric space where κ is a regular cone. Suppose $f : X \rightarrow X$ is continuous and admits a majorant ϕ . Then for arbitrary x_0 in X , the sequence of iterates $\{f^n(x_0)\}$ converges to a fixed point z of f . Moreover, for each integer $p > 0$,

$$(3.4) \quad \rho(f^p(x_0), z) \leq \phi(f^n(x_0)).$$

Proof: Let x_0 be arbitrary in X . Apply Lemma 3.1 to show that $f^n(x_0)$ is a Cauchy sequence and let z be its limit. By the continuity of f , z is a fixed point of f .

Suppose p and q are positive integers with $p < q$. Then for arbitrary x_0 in X , $\rho(f^p(x_0), f^q(x_0)) \leq \sum_{n=p}^q \rho(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=p}^q (\phi(f^n(x_0)) - \phi(f^{n+1}(x_0))) = \phi(f^p(x_0)) - \phi(f^{q+1}(x_0)) \leq \phi(f^p(x_0))$.

Letting $q \rightarrow \infty$ gives the inequality (3.4).

Caristi and Kirk have shown, in the case $\kappa = R^+$, that condition 3.1 implies a fixed point for an f , which is not necessarily continuous provided ϕ is lower semicontinuous. The following extends Kirk's proof [8] of Caristi's theorem [1] to abstract cones. Observe that in view of Proposition 2.1, the $\liminf$, and hence also, the notion of lower semicontinuity is defined in a regular minihedral cone in a separable space.

Lemma 3.3: Let x be a complete κ -metric space where κ is a regular, minihedral cone, with a non-empty interior in a separable space. Suppose $f: X \rightarrow X$ admits a majorant ω which is lower semicontinuous. Then for arbitrary x_0 in X , there is a fixed point z of f such that

$$(3.5) \quad \omega(z) \leq \omega(z_0)$$

Proof: For $x, y \in X$ define the relation

$$(3.6) \quad x \leq y \text{ iff } \rho(x, y) \leq \omega(x) - \omega(y)$$

Then $(x, \leq)$ is a partially ordered set. Fix x_0 in X and use Zorn's lemma to obtain a maximal (relative to set inclusion) totally ordered subset of E of X containing x_0 . Assume $E = \{x(\alpha)\}_{\alpha \in I}$ where I is totally ordered and

$$x(\alpha) \leq x(\beta) \text{ iff } \alpha \leq \beta. \quad (\alpha, \beta \in I)$$

Since κ is strongly minihedral (see Proposition 2.1) the set $\Omega = \{\omega(x(\alpha))\}_{\alpha \in I}$ has a greatest lower bound, r , in κ . Moreover, $x(\alpha) \leq x(\beta)$ implies $\omega(x(\alpha)) \geq \omega(x(\beta))$, and therefore Ω is totally ordered. It follows from proposition 2.1 that there exists a sequence $x(\alpha_n)$ in E such that $\omega(\alpha_n) \rightarrow r$ as $n \rightarrow \infty$ and, for each β in I , there exists α_n such that $x(\beta) \leq x(\alpha_n)$. Let u_0 be an interior point in κ and let $\epsilon > 0$. There exists $\alpha_N \in I$ such that

$$\alpha_n \geq \alpha_N \rightarrow r \leq \omega(x(\alpha_n)) \leq r + \epsilon u_0.$$

Therefore if $\alpha_n \geq \alpha_m \geq \alpha_N$ then

$$\rho(x(\alpha_n), x(\alpha_m)) \leq \omega(x(\alpha_n)) - \omega(x(\alpha_m)) \leq \epsilon u_0$$

and this proves that the sequence $\{x(\alpha_n)\}$ is a Cauchy sequence in X . By completeness there exists z in X such that $x(\alpha_n) \rightarrow z$ as $n \rightarrow \infty$. Since ω is lower semicontinuous, $\omega(z) \leq r$. Also for $\alpha_n \geq \alpha_m$,

$$\rho(x(\alpha_n), x(\alpha_m)) \leq \omega(x(\alpha_n)) - \omega(x(\alpha_m))$$

and letting $m \rightarrow \infty$,

$$\rho(x(\alpha_n), z) \leq \omega(x(\alpha_n)) - r \leq \omega(x(\alpha_n)) - \omega(z)$$

yielding $x_{\alpha_n} \leq z$, $n = 1, 2, \dots$. But then $x_\alpha \leq z$, $\alpha \in I$. Since E is maximal, $z \in E$. But also

$$\rho(z, f(z)) \leq \omega(z) - \omega(f(z))$$

so it follows that

$$x_\alpha \leq z \leq f(z), \quad \alpha \in I$$

and by maximality, $f(z) \in E$. Therefore $f(z) \leq z$ and it follows that $f(z) = z$.

Since $\omega(x_0) \in \Omega$ and $\omega(z) \leq r = \inf \Omega$, $\omega(z) \leq \omega(x_0)$.

Our main result is the following.

Theorem 3.1: Let X be a complete κ -metric space where κ is a regular cone, D is a closed subset of X , and $T : D \rightarrow X$ is continuous. Suppose there is a function $f : D \rightarrow D$ and a monotone map $\psi : \kappa \rightarrow \kappa$ which satisfies condition (1.4). Suppose further the map

$\gamma : \kappa \rightarrow \kappa$, $\gamma(u) = \sum_{n=0}^{\infty} \psi^n(u)$, is continuous (see Lemmas 2.1, 2.2.

2.3) and for each x in D :

$$(3.6) \quad \rho(x, f(x)) \leq \rho(x, Tx),$$

$$(3.7) \quad x = f(x) \text{ iff } x = Tx,$$

$$(3.8) \quad \rho(f(x), Tf(x)) \leq \psi(\rho(x, Tx)).$$

Assume at least one of the conditions:

(3.9) f is continuous

(3.10) K is minihedral, has non-empty interior, and lies in a separable space. f is lower semi-continuous.

Then for arbitrary x_0 in X , T has a fixed point z such that $\omega(z) \leq \omega(x_0)$ where

$$(3.11) \quad \omega(x) = \gamma(\rho(x, Tx)), \quad x \in X.$$

If (3.9) is satisfied then, for arbitrary x_0 in D , the sequence of iterates $\{f^n(x_0)\}$ converges to a fixed point z of T and

$$(3.12) \quad \rho(f^n(x_0), z) \leq \omega(f^n(x_0))$$

Proof: For each x in X , $\rho(x, f(x)) \leq \rho(x, Tx) = \omega(x)$

$$- \sum_{n=0}^{\infty} \psi^n(\rho(x, Tx)) \leq \omega(x) - \sum_{n=0}^{\infty} \psi^n(\rho(f(x), Tf(x))) = \omega(x) - \omega(f(x)).$$

Thus the continuous map ω is a majorant for f . If condition (3.9) is true, apply Lemma 3.2, or if condition (3.10) is true, apply Lemma 3.3. In either case we obtain a fixed point of f which, by condition (3.7), is also a fixed point of T . In the case of (3.9), Lemma 3.2 gives convergence of the sequence of iterates $\{f^n(x_0)\}$, where x_0 is arbitrary in D , to a fixed point of both f and T , and the error estimate (3.12).

It is interesting to compare the above result with a Theorem in [2] where the assumption is made that T is a metrically inward con-

tractor. This hypothesis implies the existence of a function $f : D \rightarrow D$ satisfying: condition (3.7), $\rho(x, Tx) = \rho(x, f(x)) + \rho(f(x), Tx)$, (hence $\rho(x, f(x)) \leq \rho(x, Tx)$) and $\rho(f(x), Tf(x)) < \rho(x, Tx)$. In this theorem, $\kappa = R^+$ and the hypothesis is, in some ways, more restrictive than that of Theorem 3.1.

The following corollary of Theorem 3.1 suggests a numerical procedure for calculating zeros of operators.

A κ -normed linear space [4] is a linear space X equipped with a cone-valued norm, i.e. for arbitrary vectors x, y and scalars α :

(i) $||x|| \in \kappa$, a cone, (ii) $||\alpha x|| = |\alpha| ||x||$, (iii) $||x + y|| \leq ||x|| + ||y||$, and (iv) $||x|| = \theta$ iff x is the zero in X . A κ -Banach space is a complete κ -normed linear space.

Corollary 3.1: Let X be a κ -Banach space where the cone κ is regular. Let D be a closed subset of X and $G : D \rightarrow X$ a continuous function. Let $\psi : \kappa \rightarrow \kappa$ be monotone, subadditive and let there exist $u_0 > \theta$ such that: (i) $u_0 > \psi(u_0)$, (ii) $\lim_{n \rightarrow \infty} (u_0/n) = \theta$, (iii) θ is the unique fixed point of ψ in the segment $\langle \theta, u_0 \rangle$.

Suppose for arbitrary x in D a y in x can be found in accordance with the following conditions:

- (a) y depends continuously on x .
- (b) $||y - x|| \leq ||Gx||$ and $y = x$ iff $||Gx|| = \theta$
- (c) $||Gy|| \leq \psi(||Gx||)$

Then for arbitrary x_0 in D , the sequence $\{x_n\}$, where x_{n+1} is the y corresponding to x_n , converges to a vector z such that $||Gz|| = \theta$ and

$$\|x_n - z\| \leq \psi(\|Gx_n\|), \quad n = 1, 2, \dots$$

Proof: Apply Theorem 3.1 $T = I - G$ where I is the identity on X .

Lemma 2.2 gives the necessary requirements on ψ .

4. Nonlinear contractions: If T is a contractor map then T satisfies an inequality of the type (1.3) where $\psi(u) = \alpha u$ maps $R^+ \rightarrow R^+$ and $0 < \alpha < 1$. The map ψ satisfies condition (4.1) (below)

and the map $(1 - \psi)^{-1} = \sum_{n=0}^{\infty} \psi^n(u)$ is continuous and ψ is monotone.

The following result, which is a corollary of Theorem 3.1, generalizes the contraction mapping principle.

Theorem 4.1: Let X be a κ -metric space, where κ is a regular cone, $T : X \rightarrow X$ a continuous function, $\psi : \kappa \rightarrow \kappa$ a monotone map

such that $\gamma = \sum_{n=1}^{\infty} \psi^n$ is continuous. Let the inequality

$$(4.1) \quad \rho(Tx, Ty) \leq \psi(\rho(x, y)), \quad x, y \in X$$

be satisfied in X . Then for arbitrary x_0 in X , the sequence of iterates $\{T^n x_0\}$ converges to a fixed point z of T and

$$(4.2) \quad \rho(T^n x_0, z) \leq \omega(T^n x_0)$$

where ω is the map given by (3.11).

Proof: Apply Theorem 3.1 with $f = T$.

Theorems resembling the above appear in [3] - [5] but with different assumptions on ψ . One set of such assumptions are the con-

ditions (A) which are hypothesized in Lemma 2.4.

Theorem 4.2: Let X be a convex κ -metric space where κ is a regular, strongly minihedral cone. Suppose $\psi : \kappa \rightarrow \kappa$ is monotone and satisfies conditions (A). Let $T : X \rightarrow X$ be continuous and suppose condition (4.1) is satisfied. Then for arbitrary x_0 in X , the sequence of iterates $\{T^n x_0\}$ converges to a fixed point z of T . Moreover, z is the unique fixed point of T .

Proof: Apply Lemmas 2.4 and 2.2 to obtain a map $\phi \leq \psi$, which has the properties (a), (b), and (c) listed in Lemma 2.2 and which compares T . Then apply Theorem 4.1 to obtain the convergence of $\{T^n x_0\}$ to a fixed point and the error bound (4.2) for arbitrary x_0 in X .

For uniqueness suppose $x = Tx$ and $y = Ty$. Set $v = \rho(x, y)$. Then $v = \rho(Tx, Ty) \leq \phi(\rho(x, y))$. Thus $v = \rho(x, y)$ satisfies $v \leq \phi(v)$. Apply condition (2.2) to obtain $u_0 \geq v$ such that $\phi(u_0) \leq u_0$. Then apply Proposition 2.2 to ϕ on the segment $\langle \theta, u_0 \rangle$ to show that v lies below the maximal fixed point of ϕ in $\langle \theta, u_0 \rangle$. But θ is the only fixed point of ϕ . Thus $v = \theta$ and $x = y$.

In the case of a κ -Banach space one can draw additional conclusions.

Corollary 4.1: Let X be a convex κ -Banach space, $T : X \rightarrow X$, and $\psi : \kappa \rightarrow \kappa$. Suppose the cone κ , the function T , and the map ψ have the properties specified in the hypothesis of Theorem 4.2. Then for each y in X , the equation

$$(4.3) \quad Tx + y = z$$

is uniquely solvable. Moreover, if x_0 is any vector in X the

sequence of iterates $\{T_y^n x_0\}$, where

$$T_y x = Tx + y$$

converges to the solution z of (4.3).

Proof: Apply Theorem 4.2 to T_y . Observe that $\|T_y a - T_y b\| \leq \|Ta - Tb\| \leq \psi \|a - b\|$ for arbitrary a, b in X .

Remark: The above result can also be proved by setting $G_y x = X - Tx - y$ for x in X and applying Corollary 3.1 to G_y .

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REFERENCES

1. J. Caristi, Fixed Point Theorems for Mappings Satisfying Inwardness Conditions, University of Iowa Ph.D. Dissertation, 1975.
2. J. Caristi and W. A. Kirk, Geometric Fixed Point Theory and Inwardness Conditions, preprint.
3. J. Eisenfeld and V. Lakshmikantham, Comparison Principle and Nonlinear Contractions in Abstract Spaces, J. Math. Anal. Appl. 49(1975), 504-511.
4. J. Eisenfeld and V. Lakshmikantham, Fixed Point Theorems Through Abstract Cones, J. Math. Anal. Appl. 52(1975), 25-35.
5. J. Eisenfeld and V. Lakshmikantham, Remarks on Nonlinear Contractions in Abstract Spaces, Technical Report No. 25, University of Texas at Arlington.
6. R. S. Varga, M-Matrix Theory and Recent Results in Numerical Linear Algebra, to appear in Sparse Matrix Computations.
7. M. A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.
8. W. A. Kirk, Caristi's Fixed Point Theorem and Metric Convexity, preprint.