ON THE THEORY OF TERMINAL VALUE PROBLEMS
FOR ORDINARY DIFFERENTIAL EQUATIONS

by

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1. INTRODUCTION.

It is well known [2] that the comparison principle for initial value problems of ordinary differential equations is a very useful tool in the study of qualitative and quantitative theory. Recently, attempts have been made to study the corresponding comparison principle for terminal value problems (TVP for short), [1,3,4]. Since the theory related to TVP's is much more complicated than that of initial value problems, the development of the theory corresponding to these problems has met with difficulties. For example, one of the difficulties arises because the existence of a solution of the TVP $u' = f(t,u)$, $u(\infty) = u_0$ need not imply that the TVP $u' = f(t,u) \pm \frac{1}{n}$, $u(\infty) = u_0 \pm \frac{1}{n}$ has a solution.

In this paper, we shall first discuss the theory of terminal differential inequalities, existence of extremal solutions of TVP's and the corresponding comparison principle. Our approach is parallel to that of initial value problems and the appropriate modifications incorporated appear as a natural setting for this theory. We shall also discuss existence of solutions of TVP in a sector extending the method of upper and lower solutions and develop monotone iterative techniques to obtain multiple solutions of TVP in a sector.

2. DIFFERENTIAL INEQUALITIES.

We begin by proving a fundamental result on differential inequalities relative to terminal conditions. This we shall do once for all for non-strict inequalities.

Theorem 1. Let $f \in C[R_+ \times R, R]$, $u, v \in C[R_+, R]$ such that $u(\infty), v(\infty)$ exist and

- i) $D_+ u(t) \leq f(t, u(t)), t \in [0, \infty).$
- ii) $D_+ v(t) \geq f(t, v(t)), t \in [0, \infty).$
- iii) $f(t, x)$ is nondecreasing in x for each t .

If $v(\infty) \leq u(\infty)$, then $v(t) \leq u(t)$ for $t \in [0, \infty).$

Proof. At first we prove the theorem for strict inequalities. Suppose that the conclusion is not true, then there exists a $t_1 \in [0, \infty)$ such that $v(t_1) = u(t_1)$ and $v(t) < u(t)$, for $t \in (t_1, \infty)$. For sufficiently small $h > 0$, we have

$$v(t_1+h) < u(t_1+h)$$

and consequently

$$\frac{v(t_1+h) - v(t_1)}{h} < \frac{u(t_1+h) - u(t_1)}{h}$$

which leads to

$$D_+ v(t_1) \leq D_+ u(t_1).$$

This implies, in view of (i) and (ii), $f(t_1, v(t_1)) < f(t_1, u(t_1))$, which is a contradiction. Hence,

$$v(t) < u(t) \quad \text{for } t \in [0, \infty).$$

Now define for $\epsilon > 0$ arbitrary,

$$\tilde{v}(t) = v(t) - \epsilon(1 + e^{-t}).$$

Then $\tilde{v}(t) < v(t)$ for $t \in [0, \infty)$ and $\tilde{v}(\infty) < v(\infty)$. Hence, using (iii), we get

$$D_+ \tilde{v}(t) = D_+ v(t) + \epsilon e^{-t} \geq f(t, v(t)) + \epsilon e^{-t} \geq f(t, \tilde{v}(t)) + \epsilon e^{-t} > f(t, \tilde{v}(t)).$$

It then follows by the earlier argument that

$$\tilde{v}(t) < u(t).$$

Now, letting $\epsilon \rightarrow 0$, we get $v(t) \leq u(t)$, $t \in [0, \infty)$. The proof is complete.

Remark. A result analogous to Theorem 1 relative to initial conditions is well known [2] and requires one sided Lipschitz condition. In Theorem 1, we have assumed monotonic condition on f instead of Lipschitz condition. The following example shows that the result of Theorem 1 may not be valid if f satisfies a Lipschitz condition and not monotonic condition.

Consider the example $x' = -x$. The functions $v(t) = e^{-t/2}$ and $u(t) = e^{-2t}$ satisfy the assumptions of Theorem 1. However, $v(t) \not\leq u(t)$.

The above example illustrates the peculiarities of the theory of differential inequalities relative to terminal values compared to the analogous theory relative to initial values.

3. EXISTENCE OF EXTREMAL SOLUTIONS.

We need the following known result [2] for our discussion.

Theorem 2. Let $f: R_+ \times R \rightarrow R$, be continuous,

$$x' = f(t, x), \quad x(\infty) = x_\infty \quad (3.1)$$

and $|f(t, x)| \leq g(t, |x|)$, where $g \in C[R_+ \times R_+, R_+]$ and $g(t, u)$ is monotone nondecreasing in u for each $t \in R_+$. Assume that all solutions $u(t)$ of

$$u' = g(t, u), \quad u(t_0) = u_0 > 0,$$

are bounded on $[t_0, \infty)$. Then (3.1) has a solution on some interval $[a, \infty)$.

We are now in a position to prove the existence of extremal solutions of the terminal value problem (3.1).

Theorem 3. Assume that $f \in C[R_+ \times R, R]$ and $|f(t, x)| \leq \lambda(t)|x|$ where $\lambda \in L^1[0, \infty)$. Then there exist maximal and minimal solutions for the terminal value problem (3.1) on some interval $[a, \infty)$.

Proof. We shall only prove the existence of the maximal solution since the proof is similar for the minimal solution with minor modifications.

Consider the terminal value problem

$$x'_n(t) = f(t, x_n(t)) - \frac{1}{n} e^{-t}, \quad x_n(\infty) = x_\infty + \frac{1}{n}, \quad (3.2)$$

for each n . Then we have

$$|f(t, x) - \frac{1}{n} e^{-t}| \leq \lambda(t)|x| + e^{-t}.$$

Thus, $g(t,u) = \lambda(t)u + e^{-t}$ and hence $g(t,u)$ satisfies the assumptions of Theorem 2. Consequently, by Theorem 2 the terminal value problem (3.1) has a solution $x_n(t)$ on some interval $[a, \infty)$. For $n = 1, 2$ we have

$$\begin{aligned} x_1'(t) &= f(t, x_1(t)) - e^{-t} < f(t, x_1(t)) - \frac{1}{2}e^{-t}, & x_1(\infty) &= x_\infty + 1 \\ x_2'(t) &= f(t, x_2(t)) - \frac{1}{2}e^{-t}, & x_2(\infty) &= x_\infty + \frac{1}{2} \end{aligned}$$

and

$$x_1(\infty) > x_2(\infty).$$

Now an application of Theorem 1 for strict inequalities yields

$$x_1(t) > x_2(t), \quad t \in [a, \infty).$$

If we continue the same procedure, we obtain

$$x_1(t) > x_2(t) > \dots > x_n(t) > \dots, \quad t \in [a, \infty).$$

On the other hand, it is easy to see using Theorem 1 that for each n , $x_n(t) > x(t)$, $t \in [a, \infty)$, where $x(t)$ is any solution of (3.1). Therefore, the monotone sequence $\{x_n(t)\}_{n=1}^{\infty}$ converges to $r(t)$. We shall show that $r(t)$ is continuous, and moreover, it is the maximal solution of (3.1).

For each n , define $F_n(t) = f(t, x_n(t))$, then $\{F_n(t)\}_{n=1}^{\infty}$ is a sequence of continuous functions. As a result,

$$\lim_{n \rightarrow \infty} F_n(t) = \lim_{n \rightarrow \infty} f(t, x_n(t)) = f(t, r(t)) = F(t).$$

Further, note that $F(t)$ is measurable on $[a, \infty)$ and also, in view of

the fact $|x_n(t)| \leq u(t)$ on $[0, \infty)$, we have

$$|F_n(t)| = |f(t, x_n(t))| \leq \lambda(t) |x_n(t)| \leq \lambda(t) u(t),$$

where $u(t)$ is the solution of $u' = g(t, u) = \lambda(t)u + e^{-t}$.

Since $\int_t^\infty \lambda(s) ds < \infty$, so also $\int_t^\infty f(s, r(s)) ds$ exists. Now using

Lebesgue dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \int_t^\infty F_n(s) ds = \int_t^\infty F(s) ds,$$

or

$$\lim_{n \rightarrow \infty} \int_t^\infty f(s, x_n(s)) ds = \int_t^\infty f(s, r(s)) ds$$

which exists. Since each $x_n(t)$ is a solution of

$$x_n(t) = x_\infty + \frac{1}{n} + \frac{1}{n} e^{-t} - \int_t^\infty f(s, x_n(s)) ds,$$

taking limit as $n \rightarrow \infty$, we have

$$r(t) = x_\infty - \int_t^\infty f(s, r(s)) ds, \quad t \in [a, \infty). \quad (3.3)$$

since a function is an indefinite integral if and only if it is absolutely continuous, the relation (3.3) yields that $r(t)$ is continuous. Hence,

$r(t)$ is a solution of (3.1) on $[a, \infty)$. We shall now show that $r(t)$ is the desired maximal solution of (3.1) on $[a, \infty)$. Let $x(t)$ be any solution of (3.1). Clearly

$$x(\infty) < x_n(\infty),$$

and

$$x' = f(t, x) > f(t, x) - \frac{1}{n} e^{-t},$$

$$x'_n(t) = f(t, x_n) - \frac{1}{n} e^{-t}.$$

Using Theorem 1 for strict inequalities we have $x(t) < x_n(t)$ for each n . Letting $n \rightarrow \infty$ we get

$$x(t) \leq \lim_{n \rightarrow \infty} x_n(t) = r(t),$$

or

$$x(t) \leq r(t) \quad \text{for } t \in [a, \infty).$$

Thus, $r(t)$ is the maximal solution of (3.1) and the proof is complete.

Having Theorem 3 at our disposal, it is easy to prove the following comparison theorem for terminal value problem.

Theorem 4. Let the hypothesis of Theorem 3 hold. Suppose further that $v \in C^1[R_+, R]$ such that $v(\infty)$ exists and $v' \geq f(t, v)$. If $v(\infty) \leq x_\infty$, then $v(t) \leq r(t)$ $t \in [a, \infty)$, where $r(t)$ is the maximal solution of (3.1) existing on $[a, \infty)$.

Proof. Let $x_n(t)$ be a solution of

$$x'_n(t) = f(t, x_n(t)) - \frac{1}{n} e^{-t}, \quad x_n(\infty) = x_\infty + \frac{1}{n},$$

which exists on some interval $[a, \infty)$. Then

$$x'_n(t) < f(t, x_n(t)),$$

and

$$v(\infty) < x_{\infty} + \frac{1}{n}.$$

Hence, using Theorem 1, we have $v(t) < x_n(t)$, $t \in [a, \infty)$. Letting $n \rightarrow \infty$, we get the desired conclusion.

4. METHOD OF UPPER, LOWER SOLUTIONS.

We shall next consider an existence result for terminal value problem in a closed set, namely, a conical segment. The technique depends on the method of upper and lower solutions.

Theorem 5. Let $v, w \in C^1[R_+, R]$ such that $v(t) \leq w(t)$ in R_+ . Let $D = \{(t, z) \mid t \in R_+ \text{ and } v(t) \leq z \leq w(t)\}$ and $f: D \rightarrow R$ be continuous.

Suppose that

$$i) \quad v'(t) \geq f(t, v(t)),$$

$$ii) \quad w'(t) \leq f(t, w(t)),$$

and $|f(t, x)| \leq \lambda(t)$ on D where $\lambda \in L^1$. If $v(\infty) \leq u(\infty) \leq w(\infty)$, then the terminal value problem

$$u' = f(t, u), \quad u(\infty) = u_\infty, \quad (4.1)$$

has a solution which satisfies

$$v(t) \leq u(t) \leq w(t) \quad \text{on } [a, \infty) \quad \text{for some } a \geq 0.$$

Proof. Consider $P: R_+ \times R \rightarrow R$ defined by

$$P(t, x) = \max\{v(t), \min\{x, w(t)\}\}.$$

Then $f(t, P(t, x))$ defines a continuous extension of f to $R_+ \times R$ which is also bounded by $\lambda(t)$, since f is bounded by $\lambda(t)$. Therefore, by Theorem 2 the terminal value problem

$$u' = f(t, P(t, u)), \quad u(\infty) = u_\infty \quad (4.2)$$

has a solution on $[a, \infty) \subset \mathbb{R}_+$. We show that u is in D and therefore, a solution of (4.1). For $\varepsilon > 0$, consider

$$\tilde{w}(t) = w(t) + \varepsilon(1+e^{-t}), \quad (4.3)$$

$$\tilde{v}(t) = v(t) - \varepsilon(1+e^{-t}). \quad (4.4)$$

Then $\tilde{w}(t) > w(t)$, $\tilde{v}(t) < v(t)$ and $\tilde{v}(\infty) < u_\infty < \tilde{w}(\infty)$. We claim that $\tilde{v}(t) < u(t) < \tilde{w}(t)$ on $[a, \infty)$. If it is not true, then there exists $t_1 \in [a, \infty)$ such that $\tilde{v}(t) < u(t) < \tilde{w}(t)$, $t \in (t_1, \infty)$ and either

$$\tilde{v}(t_1) = u(t_1),$$

or

$$\tilde{w}(t_1) = u(t_1).$$

If $u(t_1) = \tilde{w}(t_1)$, then for sufficiently small $h > 0$

$$\frac{u(t_1+h) - u(t_1)}{h} < \frac{\tilde{w}(t_1+h) - \tilde{w}(t_1)}{h}$$

which implies

$$u'(t_1) \leq \tilde{w}'(t_1).$$

Hence, from (4.3) and (ii), we get, in view of the fact that

$$w(t_1) = P(t_1, u(t_1)),$$

$$\begin{aligned} f(t_1, P(t_1, u(t_1))) &= u'(t_1) \leq \tilde{w}'(t_1) = w'(t_1) - \varepsilon e^{-t_1} \leq f(t_1, w(t_1)) - \varepsilon e^{-t_1}, \\ &= f(t_1, P(t_1, u(t_1))) - \varepsilon e^{-t_1} < f(t_1, P(t_1, u(t_1))). \end{aligned}$$

Which is a contradiction. If $\tilde{v}(t_1) = u(t_1)$, then we arrive at a similar contradiction. Thus, it follows that

$$\tilde{v}(t) < u(t) < \tilde{w}(t) \quad \text{on } [a, \infty).$$

Now, letting $\varepsilon \rightarrow 0$, we get

$$v(t) \leq u(t) \leq w(t) \quad \text{on } [a, \infty).$$

It, therefore, follows that $u(t)$ is actually a solution of the terminal value problem (4.1) and hence the proof of the theorem is complete.

5. MONOTONE ITERATIVE TECHNIQUE.

Finally, we shall develop a monotone iterative technique for the problem (4.1) which yields monotone sequences through the solutions of linear terminal value problems such that these sequences converge uniformly to maximal and minimal solutions of the terminal value problem (4.1). This constructive method offers another way of proving existence of extremal solutions in addition to obtaining solutions in closed sets.

Theorem 6. Let the hypothesis of Theorem 3 hold. Suppose further that $v_0, w_0 \in C^1[R_+, R]$ with $v_0(t) \leq w_0(t)$ on R_+ , $v_0(\infty), w_0(\infty)$ exist and

- i) $v_0' \geq f(t, v_0)$,
- ii) $w_0' \leq f(t, w_0)$, for $t \in R_+$.

Assume that

$$f(t, x) - f(t, y) \leq M(t)[x - y], \quad (4.5)$$

whenever $v_0(t) \leq y \leq x \leq w_0(t)$ on R_+ and $M \in L^1$. Then there exist monotone sequences $\{v_n\}, \{w_n\}$ such that $v_n \rightarrow v, w_n \rightarrow w$ as $n \rightarrow \infty$ uniformly and monotonically on $[a, \infty)$ for some $a \geq 0$ and that v and w are the minimal and the maximal solutions of (4.1) respectively.

Proof. For any $\eta \in C[R_+, R]$ such that $\eta(\infty)$ exists and $v_0 \leq \eta \leq w_0$ on R_+ , we consider the terminal value problem

$$u' = F(t, u), \quad u(\infty) = x_\infty \quad (4.6)$$

where $F(t, u) = f(t, \eta(t)) + M(t)(u - \eta)$ and $v_0^{(\infty)} \leq x_\infty \leq w_0^{(\infty)}$. Now

$$\begin{aligned} |F(t, u)| &\leq |f(t, \eta(t))| + M(t)|u - \eta(t)| \\ &\leq (\lambda(t) + M(t))|\eta(t)| + M(t)|u|. \end{aligned}$$

Setting $g(t, r) = M(t)r + \sigma(t)$, where $\sigma(t) = (\lambda(t) + M(t))|\eta(t)|$ we see that g satisfies the assumptions of Theorem 2, since the solutions $r(t)$ of $r' = g(t, r)$, $r(t_0) = r_0$ are bounded. Hence, (4.6) has a solution u on $[a, \infty)$ for $a \geq 0$. Also, since F is linear on u , the solution is unique. Define a mapping A by $A\eta = u$. This mapping will be used to define the sequences $\{v_n\}$, $\{w_n\}$. Let us prove that

(a) $v_0 \leq Av_0$, $w_0 \geq Aw_0$;

(b) A is monotone operator on the segment

$$[v_0, w_0] = \{u \in R; v_0(t) \leq u \leq w_0(t), t \in R_+\}.$$

To prove (a), set $Av_0 = v_1$ where v_1 is the unique solution of (4.6) with $\eta = v_0$. Setting $\phi = v_1 - v_0$, we see that

$$\phi' = v_1' - v_0' \leq f(t, v_0) + M(t)[v_1 - v_0] - f(t, v_0) = M(t)\phi,$$

and $\phi(\infty) \geq 0$. This shows that $\phi(t) \geq \phi(\infty)e^{-\int_t^\infty M(s)ds} \geq 0$ and hence $v_0 \leq v_1$ on some interval $[a, \infty)$ or equivalently $v_0 \leq Av_0$. Similarly, we can prove that $w_0 \geq Aw_0$.

To prove (b), let $\eta_1, \eta_2 \in [v_0, w_0]$ such that $\eta_1 \leq \eta_2$. Suppose that $u_1 = A\eta_1$ and $u_2 = A\eta_2$. Set $\phi = u_2 - u_1$ so that

$$\phi' = f(t, \eta_2) + M(t)(u_2 - \eta_2) - f(t, \eta_1) - M(t)(u_1 - \eta_1).$$

Now, using (4.5), we get

$$\phi' \leq M(t)[\eta_2 - \eta_1] + M(t)[(u_2 - u_1) - (\eta_2 - \eta_1)]$$

or

$$\phi' \leq M(t)\phi, \quad \text{and} \quad \phi(\infty) = 0$$

As before, this implies that $A\eta_1 \leq A\eta_2$ proving (b). We can now define the sequences $v_n = Av_{n-1}$, $w_n = Aw_{n-1}$ and conclude from the previous argument that

$$v_0 \leq v_1 \leq v_2 \leq \dots \leq v_n \leq w_n \leq \dots \leq w_2 \leq w_1 \leq w_0 \quad \text{on} \quad [a, \infty).$$

It then follows that $\lim_{n \rightarrow \infty} v_n = v$ and $\lim_{n \rightarrow \infty} w_n = w$ on $[a, \infty)$. Using the same argument as in Theorem 3, we can show that $v(t)$ and $w(t)$ are continuous. It is easy to show that v and w are solutions of (4.1) in view of the fact that v_n, w_n satisfy

$$v'_n = f(t, v_{n-1}) + M(t)[v_n - v_{n-1}], \quad v_n(\infty) = x_\infty,$$

$$w'_n = f(t, w_{n-1}) + M(t)[w_n - w_{n-1}], \quad w_n(\infty) = x_\infty.$$

To prove that v, w are minimal and maximal solutions of (4.1), we need to show that if x is any solution of (4.1) such that $v_0 \leq x \leq w_0$ on $[a, \infty)$, then $v_0 \leq v \leq x \leq w \leq w_0$ on $[a, \infty)$. To do this, suppose that for some n , $v_n \leq x \leq w_n$ on $[a, \infty)$ and set $\phi = x - v_{n+1}$ so that

$$\begin{aligned} \phi' &= f(t, x) - f(t, v_n) - M(t)[v_{n+1} - v_n] \\ &\leq M(t)[x - v_n] - M(t)[v_{n+1} - v_n] = M(t)[x - v_{n+1}] \end{aligned}$$

or

$$\phi' \leq M(t)\phi, \phi(\infty) = 0.$$

Thus, $\phi(t) \geq 0$, which implies $x \geq v_{n+1}$ on $[a, \infty)$. Similarly, $x \leq w_{n+1}$ on $[a, \infty)$ and hence $v_{n+1} \leq x \leq w_{n+1}$ on $[a, \infty)$. Since $v_0 \leq x \leq w_0$ on $[a, \infty)$, this proves by induction that $v_n \leq x \leq w_n$ on $[a, \infty)$ for all n . Taking limit as $n \rightarrow \infty$, we conclude $v \leq x \leq w$ on $[a, \infty)$ and the proof is complete.

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