ESTIMATION OF VARIANCE IN BIVARIATE NORMAL DISTRIBUTION AFTER PRELIMINARY TEST OF HOMOGENEITY

by

JUAN MANUEL ROMERO PADILLA

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2014

To my family

My wife Andrea, my daughter Yaretzi and my son Manuel

Acknowledgements

I would like to express my gratitude to my advisor Dr. Chien-Pai Han for all his help around the way and for being patient and understanding at difficult times. I would also say thanks to the members of my committee Dr. Andrzej Korzeniowski, Dr. Shan Sun-Mitchell, Dr. Doyle. L. Hawkins and Dr. Benito Chen-Charpentier for taking the time of reading my dissertation

Thanks to the department of mathematics of UTA and finally thanks to the working people of Mexico for the financial support that I got from the CONACYT.

July 15, 2014

Abstract

Estimation of Variance in Bivariate Normal Distribution After

Preliminary Test of Homogeneity

Juan Manuel Romero Padilla, PhD

The University of Texas at Arlington, 2014

Supervising Professor: Chien Pai Han

A preliminary test estimator of variance in the bivariate normal distribution is proposed after Pitman-Morgan test of homogeneity of two variances. We propose one estimator of σ_x^2 after preliminary test of two tails and another one for one tail test. The biases and mean square errors of both estimators are derived. The relative efficiency (*RE*) of the preliminary test estimator is studied. Computations and 3D graphs of RE for different parameters are analyzed. In order to get the maximum *RE*, recommendations of the significance level for the preliminary test are given for various sample sizes by using the max-min criterion.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Introduction

Comparing the variances of random variables arises in a variety of situations and in some procedures, the homogeneity of variances is frequently assumed. If we have two independent samples available for estimating the variance and we do not know whether the two samples are from populations with the same variance, then usually a preliminary test of the equality of two population variances is carried out. When the test is not significant the samples are pooled to obtain a pooled estimator, otherwise the individual sample variance is used. When the variables are independent Bancroft (1944) studied the estimation for the variance after a preliminary test, additional discussion are given by Bancroft and Han (1983) and Toyoda and Wallace (1975). We will consider the correlated case. In chapter 3 a preliminary test estimator of variance in the bivariate normal distribution is proposed after Pitman-Morgan test of homogeneity of two variances. We propose one estimator of σ_x^2 after preliminary test of two tails and another one for one tail test. The biases and mean square errors (MSE) of both estimators are derived

In order to obtain the variance and MSE we use a nice theorem to get the product moments of an arbitrary order for the Wishart distribution that was derived by Joarder (2006). The theorem given by Joarder involves an infinite series that does not include an important term. With the missing term, the theorem does not work for some product moments. In Chapter 2 we made the necessary adjustment to find a more general result that allows us to get the moments of any order, including fractions. Using the corrected theorem, we derive the biases and MSEs of the preliminary test estimators in Chapter 3

In Chapter 4 we discuss the relative efficiency and recommendation of significance level for the preliminary test for the estimator proposed in the present work. Finally in Chapter 5 some conclusions are stated

1.2 Literature Review

Statistical inference procedure incorporating preliminary tests was first considered by Bancroft (1944). Estimations of mean after preliminary test of significance, have been studied by Mosteller (1948), Bennet (1952) and Han and Bancroft (1968). Johnson et al (1977) provide a pooling methodology for regression in prediction. Han (1978) studied preliminary test estimators of variance components.

When samples are independent, estimation of the variance after preliminary test, have been studied by Bancroft (1944), Srivatava (1966) and Sisodia and Rai (1991). The relative efficiency and significance level of the preliminary test estimator was studied by Bancroft and Han (1983) and Toyota and Wallace (1975).

The estimation following a preliminary test of hypothesis are widely used in practice to improve efficiency of estimators. A closely related procedure is the interval estimation following a rejection of a preliminary test. Meeks and D'agostino (1983) proposed a method to find the conditional confidence interval (CCI) for the mean in the normal distribution. Chiou and Han (1994, 1995) got a conditional interval for the location and scale parameter for the exponential distribution, Mahdi (2003,2004) got a CCI for the shape and scale parameters for the Weibull distribution, further information could be seen in Madhi (2000) and Chiou and Han (1998).

Saleh (2006) provide a balanced description of the theory of preliminary test, he start with the two sample problem of pooling means in a general set up then raise the level of discussion form chapter to chapter in his book. He also states that the preliminary test estimators are the precursor of the Stein-type estimators and a careful look at the preliminary test estimator reveals that a simple replacement of the indicator function by a multiple of the reciprocal of the test-statistics, define the Stein-type estimators. This procedure was called in his book as the preliminary test approach to shrinkage estimators or quasi-empirical Bayes approach.

Furthermore, many authors have proposed test for homogeneity of variances when the variables are correlated. The most important one was proposed by Pitman (1939) and Morgan (1939). They made a transformation to the original variables to get two new variables and the test of equality of two variances is equivalent to test independence of the new variables. In the normal multivariate case, to test homogeneity of variances, Han (1968) proposed four tests, Harris (1985) proposed four methods for large sample size and Cohen (1986) applied Pitman-Morgan method for each pair of variances

We now give some examples to show the importance of estimation of variance. In the estimations of component of variance, one method that provides a technique to estimate the variance of random factors is the ANOVA method (e.g. Henderson, 1953; Rash and Masata, 2006). Another example is the estimation of risk of extinction or decline of a population that requires estimation of variability of vital rates such as survival and fecundity (Akcakaya, 2002). In quantitative genetics the heritability is defined as the ratio of genetic variance over environmental variance (Falconer and Mackey 1996, Ch 10). As a final example, estimations of inflation are important in economic, uncertainty in tomorrow's price impairs the efficiency of today's allocations decisions, It is mentioned (Engle, 1983), that higher rates of inflation are generally associated with higher variability of inflation. In general improve the estimation of variance is important, we need to do our analysis with the best estimator available.

Chapter 2

Product Moments of Wishart Distribution

2.1 Wishart Distribution

Let $X_1, \dots, X_N$ be iid $N_p(\mu, \Sigma)$ where Σ is positive defined, the Wishart matrix $n > p$ is $A = \sum_{j=1}^{n}(X_j - \bar{X})(X_j - \bar{X})^T$, A is said to have a Wishart distribution with parameters p, n=N-1 and $\Sigma(pxp)$, $A \sim W_p(n, \Sigma)$, if the probability density function (pdf) is given by (e.g. Anderson 2003)

$$f(A) = \frac{|A|^{\frac{(n-p-1)}{2}} exp\left(-\frac{1}{2} tr \Sigma^{-1} \mathrm{A}\right)}{2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{2}} |\Sigma|^{\frac{n}{2}} \prod_{i=1}^{p} \Gamma\left(\frac{1}{2}(n+1-i)\right)} \qquad (2.1)$$

with n > p and A positive definite.

For p=2 we have the bivariate case with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

where $a_{11} = nS_1^2$, $a_{22} = nS_2^2$, $a_{12} = nS_{12} = nrS_1S_2$, $r = \frac{S_{12}}{S_1 S_2} = \frac{a_{12}}{(a_1 a_2)^{1/2}}$

and

$$f(a_{11}, a_{22}, a_{12}) = \frac{(1-\rho^2)^{-\frac{n}{2}}(\sigma_1\sigma_2)^{-n}}{2^n \pi^{1/2} \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} (a_{11}a_{22} - a_{12}^2)^{\frac{n-3}{2}}$$

$$\times exp\left(-\frac{1}{2}\left[\frac{a_{11}}{(1-\rho^2)\sigma_1^2} + \frac{a_{22}}{(1-\rho^2)\sigma_2^2} + \frac{2\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right]\right) \qquad (2.2)$$

2.2 Product Moments

We use the joint distribution of S_1^2, S_2^2 and *r* given by (see Joarder 2006)

$$f(S_1^2, S_2^2, r) = \frac{n^n(1-\rho^2)^{-\frac{n}{2}}(\sigma_1\sigma_2)^{-n}}{4\pi\Gamma(n-1)} S_1^{n-2} S_2^{n-2} (1-r^2)^{\frac{n-3}{2}}$$

$$\times\, exp\left(-\frac{1}{2}\left[\frac{nS_1^2}{(1-\rho^2)\sigma_1^2} + \frac{nS_2^2}{(1-\rho^2)\sigma_2^2} + \frac{2\rho n r S_1 S_2}{(1-\rho^2)\sigma_1\sigma_2}\right]\right) \quad (2.3)$$

The product moments, $E(S_1^{2a}S_2^{2b}r^c)$, for any *a, b, c* are given by

$$E\left(S_1^{2a}S_2^{2b}r^c\right) = \int_{-1}^{1}\int_0^{\infty}\int_0^{\infty} S_1^{2a}S_2^{2b}r^c f(S_1^2, S_2^2, r)dS_1^2 dS_2^2 dr \quad (2.4)$$

$$= \frac{n^n(1-\rho^2)^{-\frac{n}{2}}(\sigma_1\sigma_2)^{-n}}{4\pi\Gamma(n-1)}$$

$$\times \sum_{k=0}^{\infty} \frac{(n\rho)^k}{k!(1-\rho^2)^k(\sigma_1\sigma_2)^k} \int_0^{\infty} S_1^{n+k+2a-2}\, exp\left(-\frac{nS_1^2}{2(1-\rho^2)\sigma_1^2}\right) dS_1^2$$

$$\times \int_0^{\infty} S_2^{n+k+2b-2}\, exp\left(-\frac{nS_2^2}{2(1-\rho^2)\sigma_2^2}\right) dS_2^2$$

$$\times \left[\int_{-1}^{0} r^{k+c}\,(1-r^2)^{(n-3)/2}dr + \int_0^1 r^{k+c}\,(1-r^2)^{(n-3)/2}dr\right]$$

To obtain the first two integrals of equation (2.4) we use the property

$$\int_0^{\infty} t^x e^{-yt} dt = \frac{\Gamma(x+1)}{y^{x+1}}$$

Hence

$$\int_0^{\infty} S_1^{n+k+2a-2}\, exp\left(-\frac{nS_1^2}{2(1-\rho^2)\sigma_1^2}\right) dS_1^2 \int_0^{\infty} S_2^{n+k+2b-2}\, exp\left(-\frac{nS_2^2}{2(1-\rho^2)\sigma_2^2}\right) dS_2^2$$

$$= \Gamma\left(\frac{k+n}{2}+a\right)\Gamma\left(\frac{k+n}{2}+b\right)\left[\left(\frac{n}{2(1-\rho^2)\sigma_1^2}\right)^{\frac{n+k}{2}+a}\left(\frac{n}{2(1-\rho^2)\sigma_2^2}\right)^{\frac{n+k}{2}+b}\right]^{-1}$$

$$= \Gamma\left(\frac{k+n}{2}+a\right)\Gamma\left(\frac{k+n}{2}+b\right)\frac{2^{n+k+a+b}(1-\rho^2)^{n+k+a+b}\sigma_1^{n+k+2a}\sigma_2^{n+k+2b}}{n^{n+k+a+b}}$$

To obtain the last two integrals of equation (2.4) we use the incomplete beta function for the two cases, $r \geq 0$ and $r < 0$. When $r \geq 0$ we have

$$\left[\int_{-1}^{0} r^{k+c}\,(1-r^2)^{\frac{n-3}{2}}dr + \int_{0}^{1} r^{k+c}\,(1-r^2)^{\frac{n-3}{2}}dr\right]$$

$$= [(-1)^k + 1]\int_0^1 r^{k+c}\,(1-r^2)^{\frac{n-3}{2}}dr = [(-1)^k + 1]\frac{1}{2}\int_0^1 w^{\frac{k+c-1}{2}}\,(1-w)^{\frac{n-3}{2}}dw$$

$$= \frac{1}{2}[(-1)^k + 1]B\left(\frac{k+c+1}{2},\frac{n-1}{2}\right)$$

When $r < 0$ we notice that

$(-r)^{k+c} = (-1)^{k+c}r^{k+c}$ and $[(-1)^k + 1](-1)^k = [(-1)^k + 1]$

Therefore we define

$$d = \begin{cases} (-1)^c & if\ r < 0 \\ 1 & if\ r \geq 0 \end{cases}$$

and

$$\left[\int_{-\lambda^{1/2}}^{0} r^{k+c}\,(1-r^2)^{\frac{n-3}{2}}dr + \int_{0}^{\lambda^{1/2}} r^{k+c}\,(1-r^2)^{\frac{n-3}{2}}dr\right]$$

$$= \frac{1}{2}d[(-1)^k + 1]B\left(\frac{k+c+1}{2},\frac{n-1}{2}\right)$$

The product moments are given by the expression

$$E[S_1^{2a}S_2^{2b}r^c] = \frac{2^{a+b}(1-\rho^2)^{a+b+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^{a+b}}\sigma_1^{2a}\sigma_2^{2b} \tag{2.5}$$

$$\times \sum_{k=0}^{\infty}\frac{1}{2}d[(-1)^k + 1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+a\right)\Gamma\left(\frac{k+n}{2}+b\right)\Gamma\left(\frac{k+c+1}{2}\right)\Gamma^{-1}\left(\frac{k+n+c}{2}\right)$$

The difference with respect to the result obtained by Joarder (2006) is the term

$$\frac{1}{2}d[(-1)^k + 1]$$

The generalized hypergeometric function is given by (e.g. Bailey, 1964)

$${}_pF_q\left(a_1, \dots, a_p: b_1, \dots, b_q; z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!}$$

Next we will use the following results

$$(a)_0 = 1;\ (a)_k = (a+k-1)(a+k-2)\dots(a+1)a = \frac{\Gamma(a+k)}{\Gamma(a)}$$

$$\Gamma\left(\frac{1}{2}+k\right) = \frac{\sqrt{\pi}(2k-1)!}{2^{2k-1}(k-1)!} = \frac{\sqrt{\pi}(2k)!}{2^{2k}k!}$$

and redefine $j = 2k$ to express $E[S_1^{2a}S_2^{2b}r^c]$ in terms of hypergeometric function as, from (2.5)

$$E[S_1^{2a}S_2^{2b}r^c] = \frac{2^{a+b}(1-\rho^2)^{a+b+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^{a+b}}\sigma_1^{2a}\sigma_2^{2b}d$$

$$\times \sum_{j=0}^{\infty}\frac{\sqrt{\pi}(\rho)^{2j}}{j!}\Gamma\left(\frac{n}{2}+j+a\right)\Gamma\left(\frac{n}{2}+j+b\right)$$

$$\times\Gamma\left(\frac{c+1}{2}+j\right)\Gamma^{-1}\left(\frac{n+c}{2}+j\right)\Gamma^{-1}\left(\frac{1}{2}+j\right)$$

$$= \frac{2^{a+b}(1-\rho^2)^{a+b+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^{a+b}}d\sigma_1^{2a}\sigma_2^{2b}\Gamma\left(\frac{n}{2}+a\right)\Gamma\left(\frac{n}{2}+b\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma^{-1}\left(\frac{n+c}{2}\right)$$

$$\times\sum_{k=0}^{\infty}\frac{(\rho^2)^j}{j!}\left(\frac{n}{2}+a\right)_j\left(\frac{n}{2}+b\right)_j\left(\frac{c+1}{2}\right)_j\left[\left(\frac{n+c}{2}\right)_j\left(\frac{1}{2}\right)_j\right]^{-1}$$

$$E[S_1^{2a}S_2^{2b}r^c] = \frac{2^{a+b}(1-\rho^2)^{a+b+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^{a+b}}d\sigma_1^{2a}\sigma_2^{2b}\Gamma\left(\frac{n}{2}+a\right)\Gamma\left(\frac{n}{2}+b\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma^{-1}\left(\frac{n+c}{2}\right)$$

$$\times_3 F_2\left(\frac{n+2a}{2},\frac{n+2b}{2},\frac{c+1}{2};\frac{n+c}{2},\frac{1}{2};\rho^2\right) \qquad (2.6)$$

Next we rewrite the Theorem 3.1 of Joarder (2006) which needs adjustment too; the change is given in the expression $b_{k,m}$ defined below.

Define

$$c^{\{k\}} = c(c-1)\dots(c-k+1),\ c^{\{0\}} = 1$$

and

$$b_{k,m} = \frac{1}{2}[(-1)^k + 1]\frac{(2)^k}{k!}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+n}{2}\right), n>0,\ k>0$$

Then let

$$L(n,\rho) = \sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-\frac{n}{2}}$$

we have

i) $\sum_{k=0}^{\infty} \rho^k b_{k,n} = L(n,\rho)$ (2.7)

ii) $\sum_{k=0}^{\infty} k\rho^k b_{k,n} = n\rho^2(1-\rho^2)^{-1}L(n,\rho)$ (2.8)

iii) $\sum_{k=0}^{\infty} k^{\{2\}}\rho^k b_{k,n} = (n(n+1)\rho^4 + n\rho^2)(1-\rho^2)^{-2}L(n,\rho)$ (2.9)

iv) $\sum_{k=0}^{\infty} k^{\{3\}}\rho^k b_{k,n} = w_{\{3\}}(n,\rho)L(n,\rho)$ (2.10)

v) $\sum_{k=0}^{\infty} k^{\{4\}}\rho^k b_{k,n} = w_{\{4\}}(n,\rho)L(n,\rho)$ (2.11)

vi) $\sum_{k=0}^{\infty} k^2\rho^k b_{k,n} = (n^2\rho^4 + 2n\rho^2)(1-\rho^2)^{-2}L(n,\rho)$ (2.12)

vii) $\sum_{k=0}^{\infty} k^3\rho^k b_{k,n} = w_3(n,\rho)L(n,\rho)$ (2.13)

viii) $\sum_{k=0}^{\infty} k^4\rho^k b_{k,n} = w_4(n,\rho)L(n,\rho)$ (2.14)

where

$$w_{\{3\}}(n,\rho) = \left((n^3 + 3n^2 + 2n)\rho^6 + (3n^2 + 6n)\rho^4\right)(1-\rho^2)^{-3}$$

$$w_{\{4\}}(n,\rho) = n\left((n^3 + 6n^2 + 11n + 6)\rho^8 + (6n^2 + 30n + 36)\rho^6 + (3n + 6)\rho^4\right)(1-\rho^2)^{-4}$$

$$w_3(n,\rho) = (4n\rho^2 + (6n^2 + 4n)\rho^4 + n^3\rho^6)(1-\rho^2)^{-3}$$

$$w_4(n,\rho) = \big((n^4 + 18n^2 + 12n)\rho^8 + (12n^3 - 20n^2 + 8n)\rho^6 + (46n^2 - 4n)\rho^4$$
$$+ 32m\rho^2\big)(1-\rho^2)^{-4}$$

The proofs are the same as that of Joarder (2006), we note that

$$\sum_{k=0}^{\infty} \rho^k\, b_{k,m} = \sum_{k=0}^{\infty} \frac{1}{2}[(-1)^k + 1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+n}{2}\right)$$
$$= \sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{(2k)!}\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{n}{2}\right)$$
$$= \sqrt{\pi}\sum_{k=0}^{\infty} \frac{(\rho)^{2k}}{(k)!}\Gamma\left(k+\frac{n}{2}\right) = \sqrt{\pi}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{\infty} \frac{(\rho^2)^k}{(k)!}\left(\frac{n}{2}\right)_k$$
$$= \sqrt{\pi}\Gamma\left(\frac{n}{2}\right) \quad {}_1F_0\left(\frac{n}{2};;\rho^2\right)$$

So

$$(1-\rho^2)^{-\frac{n}{2}} = \quad {}_1F_0\left(\frac{n}{2};;\rho^2\right)$$

Other results are

$$\sum_{k=0}^{\infty} \frac{1}{2}[(-1)^k + 1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{k+1}{2}\right)$$
$$= \left(\frac{n}{2}\right)\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-n/2} + \left(\frac{1}{2}\right)\, n\rho^2\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-\frac{n}{2}-1}$$
$$= \left(\frac{n}{2}\right)\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-\frac{n}{2}}\{1+\rho^2(1-\rho^2)^{-1}\}$$
$$= \frac{n\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{2(1-\rho^2)^{1+n/2}}$$
$$= \sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{(2k)!}\Gamma\left(\frac{n+2}{2}+k\right)\Gamma\left(\frac{1}{2}+k\right)$$
$$= \sqrt{\pi}\Gamma\left(\frac{n+2}{2}\right) \quad {}_1F_0\left(\frac{n+2}{2};;\rho^2\right) \tag{2.15}$$

and

$$\frac{2(1-\rho^2)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}\sum_{k=0}^{\infty}\left(\frac{k+n}{2}\right)\frac{1}{2}[(-1)^k+1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k+1}{2}\right)$$

$$=\frac{2(1-\rho^2)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}\left\{\left(\frac{n}{2}\right)\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-n/2}+\left(\frac{1}{2}\right)n\rho^2\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-\frac{n}{2}-1}\right\}$$

$$=\frac{2(1-\rho^2)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}\left(\frac{n}{2}\right)\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)(1-\rho^2)^{-n/2}\{1+\rho^2(1-\rho^2)^{-1}\}=1 \qquad (2.16)$$

Joarder (2006) got several cases of product moments, in all the cases he use integers for *a* and *b*, that is the reason why all the results that he got are correct, but in the cases where *a* and *b* are fractions, there are no closed form, we need the expression given in (2.6). We show some examples

i) $a=\frac{1}{2}, b=\frac{1}{2}, c=1$

$$E[S_1S_2r]=\frac{2(1-\rho^2)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}d\sigma_1\sigma_2\Gamma\left(\frac{n+1}{2}\right)\quad {}_1F_1\left(\frac{n+1}{2};\frac{1}{2};\rho^2\right)$$

ii) $a=\frac{1}{2}, b=\frac{3}{2}, c=1$

$$E[S_1S_2^3r]=\frac{4(1-\rho^2)^{2+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^2}d\sigma_1\sigma_2^3\Gamma\left(\frac{n+3}{2}\right)\quad {}_1F_1\left(\frac{n+3}{2};\frac{1}{2};\rho^2\right)$$

iii) $a=1, b=\frac{1}{2}, c=4$

$$E[S_1^2S_2r^4]=\frac{2^{\frac{3}{2}}(1-\rho^2)^{\frac{3+n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^{\frac{3}{2}}}\sigma_1^2\sigma_2\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{5}{2}\right)\Gamma^{-1}\left(\frac{n+4}{2}\right)$$

$$\times_3F_2\left(\frac{n+2}{2},\frac{n+1}{2},\frac{5}{2};\frac{n+4}{2},\frac{1}{2};\rho^2\right)$$

As an exercise we wrote a program in R version 2.15.1 to get the values of the above expectations for some parameters, the results are given in Table 2.1.

Table 2.1 Values for some expectation, *r* is positive in all the cases, different values for of n, σ_x^2, σ_y^2 and σ_{xy} are considered

	$\sigma_x^2 = 4, \sigma_y^2 = 3.6,$ $\sigma_{xy} = 3$		$\sigma_x^2 = 4, \sigma_y^2 = 3.6,$ $\sigma_{xy} = 3$		$\sigma_x^2 = 4, \sigma_y^2 = 3.6,$ $\sigma_{xy} = 3$	
	n=10	n=40	n=10	n=20	n=10	n=20
$E[S_1S_2r]$	3.162603	3.162278	1.493336	1.339566	5.118579	4.110479
$E[S_1S_2^3r]$	40.4784	35.41751	7.845511	6.324694	100.2661	73.43899
$E[S_1^2S_2r^4]$	9.237892	8.438663	1.747181	1.147746	10.47631	5.747624

Chapter 3

Estimation of Variance After Preliminary Test

3.1 Estimator of σ_x^2 After Preliminary Test of Homogeneity

Let us consider the bivariate normal distribution of X and Y with

$$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}, \quad \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

The usual estimators for μ and Σ are given by

$$\hat{\mu} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \qquad \hat{\Sigma} = \begin{bmatrix} S_x^2 & S_{xy} \\ S_{xy} & S_y^2 \end{bmatrix}, \quad \hat{\rho}_{xy} = r_{xy} = \frac{S_{xy}}{S_x S_y}$$

where

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k, \qquad S_x^2 = \frac{1}{n}\sum_{k=1}^{n} (x_k - \bar{x})^2, \qquad S_{xy} = \frac{1}{n}\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y})$$

We are interested in estimating σ_x^2. It is known that when $\sigma_x^2 = \sigma_y^2$ the pooled estimator is $S_p^2 = \frac{1}{2}\left(S_x^2 + S_y^2\right)$. In some practical situations the experimenter may not know whether $\sigma_x^2 = \sigma_y^2$, in such cases he/she can use a preliminary test to resolve the uncertainty. The preliminary test is $H_0: \sigma_x^2 = \sigma_y^2$ VS $H_1: \sigma_x^2 \neq \sigma_y^2$

Since X and Y are correlated we can use Pitman-Morgan test, see Pitman (1939) and Morgan (1939). Let $U = X + Y$ and $V = X - Y$, testing H_0 is equivalent to test independence between *U* and *V* and use the test statistic

$$T = \frac{r_{uv}\sqrt{n-2}}{\sqrt{1 - r_{uv}^2}} \tag{3.1}$$

where

$$r_{uv} = \frac{\sum_{k=1}^{n}(u_k-\bar{u})(v_k-\bar{v})}{\sqrt{\sum_{k=1}^{n}(u_k-\bar{u})^2\sum_{k=1}^{n}(v_k-\bar{v})^2}} = \frac{S_x^2-S_y^2}{\left\{\left(S_x^2+S_y^2\right)^2-4r_{xy}^2S_x^2S_y^2\right\}^{1/2}} \tag{3.2}$$

Under H_0 T has a t-student distribution with (n-2) df.

Also to test H_0 we can test $H_0: \rho_{uv}^2 = 0$ vs $H_0: \rho_{uv}^2 \neq 0$ and we know that r_{uv}^2 has a beta-distribution $B\left(\frac{1}{2}, \frac{1}{2}n - 1\right)$ under H_0

The estimator of σ_x^2 after preliminary test is defined as

$$\hat{\sigma}_x^2 = \begin{cases} \frac{1}{2}\left(S_x^2 + S_y^2\right) & if\ r_{uv}^2 < \lambda \quad (|r_{uv}| < \lambda^{\frac{1}{2}}) \\ S_x^2 & if\ r_{uv}^2 \geq \lambda \quad (|r_{uv}| \geq \lambda^{\frac{1}{2}}) \end{cases} \tag{3.3}$$

Where λ is a Beta value corresponding to an α- level of significance for the preliminary test, $\lambda = B_{1-\alpha}\left(\frac{1}{2}, \frac{1}{2}n - 1\right)$. When $\lambda = 0, \hat{\sigma}_x^2 = S_x^2$, which is the never pool estimator, when $\lambda = 1, \hat{\sigma}_x^2 = S_p^2 = \frac{1}{2}\left(S_x^2 + S_y^2\right)$, which is the always-pool estimator

3.2 Bias and Mean Square Error

Following Bancroft (1944)

$$E[\hat{\sigma}_x^2] = E\left[\frac{1}{2}\left(S_x^2 + S_y^2\right)|r_{uv}^2 < \lambda\right]P(r_{uv}^2 < \lambda) + E[S_x^2|r_{uv}^2 \geq \lambda]P(r_{uv}^2 \geq \lambda) \tag{3.4}$$

$$\mathrm{E}[(\hat{\sigma}_x^2)^2] = E\left[\left(\frac{1}{2}\left(S_x^2 + S_y^2\right)\right)^2 |r_{uv}^2 < \lambda\right]P(r_{uv}^2 < \lambda) \tag{3.5}$$

$$+E[(S_x^2)^2|r_{uv}^2 \geq \lambda]P(r_{uv}^2 \geq \lambda)$$

$$Var[\hat{\sigma}_x^2] = \mathrm{E}[(\hat{\sigma}_x^2)^2] - (E[\hat{\sigma}_x^2])^2 \tag{3.6}$$

$$MSE[\hat{\sigma}_x^2] = Var[\hat{\sigma}_x^2] + [E[\hat{\sigma}_x^2] - \sigma_x^2]^2 \tag{3.7}$$

From the definition of *u* and *v* we have the following relationships

$$S_u^2 = S_x^2 + S_y^2 + 2S_{xy}; \qquad S_v^2 = S_x^2 + S_y^2 - 2S_{xy}; \qquad S_{uv} = S_x^2 - S_y^2$$

$$\frac{1}{2}\left(S_x^2 + S_y^2\right) = \frac{1}{4}(S_u^2 + S_v^2); \qquad S_x^2 = \frac{1}{4}(S_u^2 + S_v^2) + \frac{1}{2} r_{uv} S_u S_v$$

Hence

$$E\left[\frac{1}{2}\left(S_x^2 + S_y^2\right) | r_{uv}^2 < \lambda\right] = \frac{1}{4} E[S_u^2 | r_{uv}^2 < \lambda] + \frac{1}{4} E[S_v^2 | r_{uv}^2 < \lambda] \tag{3.8}$$

$$E[S_x^2 | r_{uv}^2 \geq \lambda] = E\left[\frac{1}{4}(S_u^2 + S_v^2) | r_{uv}^2 \geq \lambda\right] + \frac{1}{2} E[r_{uv} S_u S_v | r_{uv}^2 \geq \lambda] \tag{3.9}$$

$$= \frac{1}{4} E[S_u^2 | r_{uv}^2 \geq \lambda] + \frac{1}{4} E[S_v^2 | r_{uv}^2 \geq \lambda] + \frac{1}{2} E[r_{uv} S_u S_v | r_{uv}^2 \geq \lambda]$$

$$E\left[\left(\frac{1}{2}\left(S_x^2 + S_y^2\right)\right)^2 | r_{uv}^2 < \lambda\right] = \frac{1}{16} E[(S_u^2 + S_v^2)^2 | r_{uv}^2 < \lambda] \tag{3.10}$$

$$= \frac{1}{16} E[S_u^4 | r_{uv}^2 < \lambda] + \frac{1}{16} E[S_v^4 | r_{uv}^2 < \lambda] + \frac{1}{8} E[S_u^2 S_v^2 | r_{uv}^2 < \lambda]$$

$$E[(S_x^2)^2 | r_{uv}^2 \geq \lambda] = E\left[\left(\frac{1}{4}(S_u^2 + S_v^2) + \frac{1}{2} r_{uv} S_u S_v\right)^2 | r_{uv}^2 \geq \lambda\right] \tag{3.11}$$

$$= \frac{1}{16} E[(S_u^2 + S_v^2)^2 | r_{uv}^2 \geq \lambda] + \frac{1}{4} E[r_{uv}^2 S_u^2 S_v^2 | r_{uv}^2 \geq \lambda]$$

$$+ \frac{1}{4} E[r_{uv} S_u^3 S_v | r_{uv}^2 \geq \lambda] + \frac{1}{4} E[r_{uv} S_u S_v^3 | r_{uv}^2 \geq \lambda]$$

$$= \frac{1}{16} E[S_u^4 | r_{uv}^2 \geq \lambda] + \frac{1}{16} E[S_v^4 | r_{uv}^2 \geq \lambda] + \frac{1}{8} E[S_u^2 S_v^2 | r_{uv}^2 \geq \lambda]$$

$$+ \frac{1}{4} E[r_{uv}^2 S_u^2 S_v^2 | r_{uv}^2 \geq \lambda] + \frac{1}{4} E[r_{uv} S_u^3 S_v | r_{uv}^2 \geq \lambda] + \frac{1}{4} E[r_{uv} S_u S_v^3 | r_{uv}^2 \geq \lambda]$$

In order to get the different expectation we will need the join distribution of S_u^2, S_v^2 and S_{uv} that is given by the Wishart distribution

$$f(S_u^2, S_v^2, S_{uv}) = \frac{(1-\rho^2)^{-\frac{n}{2}}(\sigma_u\sigma_v)^{-n}}{2^n\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}(S_u^2S_v^2 - 2S_{uv}^2)^{\frac{n-3}{2}} \tag{3.12}$$

$$\times \exp\left(-\frac{S_u^2}{2(1-\rho^2)\sigma_u^2} - \frac{S_v^2}{2(1-\rho^2)\sigma_v^2} + \frac{\rho S_{uv}}{(1-\rho^2)\sigma_u\sigma_v}\right)$$

$0 < S_u^2 < \infty$, $0 < S_v^2 < \infty$, $-\infty < S_{uv} < \infty$, $-1 < \rho < 1$, $n > 2$, $\sigma_u > 0$, $\sigma_v > 0$.

Joarder (2006) made a transformation and got

$$f(S_u^2, S_v^2, r_{uv}) = \frac{(1-\rho^2)^{-\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{n}{2\sigma_u\sigma_v}\right)^n (S_uS_v)^{n-2}(1 - r_{uv}^2)^{\frac{n-3}{2}} \tag{3.13}$$

$$\times \exp\left(-\frac{nS_u^2}{2(1-\rho^2)\sigma_u^2} - \frac{nS_v^2}{2(1-\rho^2)\sigma_v^2} + \frac{n\rho r_{uv}S_uS_v}{(1-\rho^2)\sigma_u\sigma_v}\right)$$

Using $\exp\left(\frac{n\rho r_{uv}S_uS_v}{(1-\rho^2)\sigma_u\sigma_v}\right) = \sum_{k=0}^{\infty}\frac{(n\rho)^k(S_uS_v)^k r_{uv}^k}{k!(1-\rho^2)^k(\sigma_u\sigma_v)^k}$

and $2^n = 2^2\Gamma(n-1)\sqrt{\pi}\frac{1}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}$

that comes from the duplication formula

$$\Gamma(z) = \frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}\right) \text{ with } z=n-1$$

we have

$$f(S_u^2, S_v^2, r_{uv}) = \frac{n^n(1-\rho^2)^{-\frac{n}{2}}(\sigma_u\sigma_v)^{-n}}{4\pi\Gamma(n-1)}S_u^{n-2}S_v^{n-2}\exp\left(-\frac{n}{2(1-\rho^2)}\left(\frac{S_u^2}{\sigma_u^2} + \frac{S_v^2}{\sigma_v^2}\right)\right)$$

$$\times \sum_{k=0}^{\infty}\frac{(n\rho)^k(S_uS_v)^k}{k!(1-\rho^2)^k(\sigma_u\sigma_v)^k}r_{uv}^k(1 - r_{uv}^2)^{\frac{n-3}{2}}$$

We obtain the product moments $E[S_u^{2a} S_v^{2b} r_{uv}^{2c} | r_{uv}^2 < \lambda]$ for a finite *a, b, c* as follows. We are using the fact $r_{uv}^2 < \lambda$ implies $|r_{uv}| < \lambda^{1/2}$.

$$E\left[S_u^{2a} S_v^{2b} r_{uv}^{c} \,\middle|\, |r_{uv}| < \lambda^{\frac{1}{2}}\right]$$

$$= \frac{1}{P(r_{uv}^2<\lambda)} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \int_0^\infty \int_0^\infty S_u^{2a} S_v^{2b} r^c f(S_u^2, S_v^2, r_{uv}) dS_u^2 dS_v^2 dr_{uv} \qquad (3.14)$$

$$= \frac{1}{P(r_{uv}^2<\lambda)} \frac{n^n (1-\rho^2)^{-\frac{n}{2}} (\sigma_u \sigma_v)^{-n}}{4\pi\Gamma(n-1)}$$

$$\times \sum_{k=0}^{\infty} \frac{(n\rho)^k}{k!(1-\rho^2)^k (\sigma_u\sigma_v)^k} \int_0^\infty S_u^{n+k+2a-2} \, exp\left(-\frac{nS_u^2}{2(1-\rho^2)\sigma_u^2}\right) dS_u^2$$

$$\times \int_0^\infty S_v^{n+k+2b-2} \, exp\left(-\frac{nS_v^2}{2(1-\rho^2)\sigma_v^2}\right) dS_v^2$$

$$\times \left[\int_{-\lambda^{1/2}}^{0} r_{uv}^{k+c} \, (1-r_{uv}^2)^{(n-3)/2} dr_{uv} + \int_0^{\lambda^{1/2}} r_{uv}^{k+c} \, (1-r_{uv}^2)^{(n-3)/2} dr_{uv}\right]$$

To obtain the first two integrals of equation (3.14) we use the property

$\int_0^\infty t^x e^{-yt} dt = \frac{\Gamma(x+1)}{y^{x+1}}$ and got

$$\int_0^\infty S_u^{n+k+2a-2} \, exp\left(-\frac{nS_u^2}{2(1-\rho^2)\sigma_u^2}\right) dS_u^2 \int_0^\infty S_v^{n+k+2b-2} \, exp\left(-\frac{nS_v^2}{2(1-\rho^2)\sigma_v^2}\right) dS_v^2$$

$$= \Gamma\left(\frac{k+n}{2} + a\right) \Gamma\left(\frac{k+n}{2} + b\right) \left[\left(\frac{n}{2(1-\rho^2)\sigma_u^2}\right)^{\frac{n+k}{2}+a} \left(\frac{n}{2(1-\rho^2)\sigma_v^2}\right)^{\frac{n+k}{2}+b}\right]^{-1}$$

$$= \Gamma\left(\frac{k+n}{2} + a\right) \Gamma\left(\frac{k+n}{2} + b\right) \frac{2^{n+k+a+b} (1-\rho^2)^{n+k+a+b} \sigma_u^{n+k+2a} \sigma_v^{n+k+2b}}{n^{n+k+a+b}}$$

To obtain the last two integrals of equation (3.14) we use the incomplete beta function for the two cases, r_{uv} positive and negative. When $r_{uv} \geq 0$ we have

$$\left[\int_{-\lambda^{1/2}}^{0} r_{uv}^{k+c}\left(1-r_{uv}^{2}\right)^{\frac{n-3}{2}} dr_{uv} + \int_{0}^{\lambda^{1/2}} r_{uv}^{k+c}\left(1-r_{uv}^{2}\right)^{\frac{n-3}{2}} dr_{uv}\right]$$

$$= [(-1)^k + 1] \int_0^{\lambda^{\frac{1}{2}}} r_{uv}^{k+c}\left(1-r_{uv}^{2}\right)^{\frac{n-3}{2}} dr_{uv}$$

$$= [(-1)^k + 1] \frac{1}{2} \int_0^{\lambda} w^{\frac{k+c-1}{2}} (1-w)^{\frac{n-3}{2}} dw$$

$$= \frac{1}{2}[(-1)^k + 1] \mathrm{B}_{\lambda}\left(\frac{k+c+1}{2}, \frac{n-1}{2}\right) B\left(\frac{k+c+1}{2}, \frac{n-1}{2}\right)$$

where $\mathrm{B}_{\lambda}(a,b)$ is the cdf of the Beta distribution and $B(a,b)$ is the Beta function. When $r_{uv} < 0$ we notice that

$(-r_{uv})^{k+c} = (-1)^{k+c} r_{uv}^{k+c}$ and $[(-1)^k + 1](-1)^k = [(-1)^k + 1]$

Therefore we define

$$d = \begin{cases} (-1)^c & if\ r_{uv} < 0 \\ 1 & if\ r_{uv} \geq 0 \end{cases}$$

and

$$\left[\int_{-\lambda^{1/2}}^{0} r_{uv}^{k+c}\left(1-r_{uv}^{2}\right)^{\frac{n-3}{2}} dr + \int_{0}^{\lambda^{1/2}} r_{uv}^{k+c}\left(1-r_{uv}^{2}\right)^{\frac{n-3}{2}} dr\right]$$

$$= \frac{1}{2} d[(-1)^k + 1] \mathrm{B}_{\lambda}\left(\frac{k+c+1}{2}, \frac{n-1}{2}\right) B\left(\frac{k+c+1}{2}, \frac{n-1}{2}\right)$$

The product moments are given by the expression

$$E[S_u^{2a} S_v^{2b} r_{uv}^c | r_{uv}^2 < \lambda] = \frac{1}{P(r_{uv}^2 < \lambda)} \frac{2^{a+b}\left(1-\rho^2\right)^{a+b+\frac{n}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right) n^{a+b}} \sigma_u^{2a} \sigma_v^{2b} d \tag{3.15}$$

$$\times \sum_{k=0}^{\infty} \frac{1}{2}[(-1)^k + 1] \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+n}{2} + a\right) \Gamma\left(\frac{k+n}{2} + b\right)$$

$$\times \Gamma\left(\frac{k+c+1}{2}\right) \Gamma^{-1}\left(\frac{k+n+c}{2}\right) \mathrm{B}_{\lambda}\left(\frac{k+c+1}{2}, \frac{n-1}{2}\right)$$

In the same way we can find $E[S_u^{2a} S_v^{2b} r_{uv}^c | r_{uv}^2 \geq \lambda]$ as

$$E[S_u^{2a}S_v^{2b}r_{uv}^c|\, r_{uv}^2 \ge \lambda] = \frac{1}{P(r_{uv}^2\ge\lambda)}\frac{2^{a+b}(1-\rho^2)^{a+b+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^{a+b}}\sigma_u^{2a}\sigma_v^{2b}d \tag{3.16}$$

$$\times \sum_{k=0}^{\infty}\frac{1}{2}[(-1)^k+1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+a\right)\Gamma\left(\frac{k+n}{2}+b\right)$$

$$\times \Gamma\left(\frac{k+c+1}{2}\right)\Gamma^{-1}\left(\frac{k+n+c}{2}\right)\left[1-\mathrm{B}_\lambda\left(\frac{k+c+1}{2},\frac{n-1}{2}\right)\right]$$

From Eq. (3.15) we have

$$E[S_u^{2a}S_v^{2b}r_{uv}^c|\, r_{uv}^2 \ge \lambda] = E[S_u^{2a}S_v^{2b}r_{uv}^c] - E[S_u^{2a}S_v^{2b}r_{uv}^c|\, r_{uv}^2 < \lambda]$$

Choosing different values of *a, b* and *c* we obtain

$$E[S_uS_vr_{uv} \mid r_{uv}^2 \ge \lambda] = \frac{1}{P(r_{uv}^2\ge\lambda)}\frac{2(1-\rho^2)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}\sigma_u\sigma_v d$$

$$\times \sum_{k=0}^{\infty}\frac{1}{2}[(-1)^k+1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n+1}{2}\right)\Gamma\left(\frac{k+2}{2}\right)\left[1-\mathrm{B}_\lambda\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right]$$

$$E[r_{uv}^2S_u^2S_v^2 \mid r_{uv}^2 \ge \lambda] = \frac{1}{P(r_{uv}^2\ge\lambda)}\frac{4(1-\rho^2)^{2+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^2}\sigma_u^2\sigma_v^2$$

$$\times \sum_{k=0}^{\infty}\frac{1}{2}[(-1)^k+1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{k+3}{2}\right)\left[1-\mathrm{B}_\lambda\left(\frac{k+3}{2},\frac{n-1}{2}\right)\right]$$

$$E[r_{uv}S_u^3S_v \mid r_{uv}^2 \ge \lambda] = \frac{1}{P(r_{uv}^2\ge\lambda)}\frac{4(1-\rho^2)^{2+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^2}\sigma_u^3\sigma_v d$$

$$\times \sum_{k=0}^{\infty}\frac{1}{2}[(-1)^k+1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+\frac{3}{2}\right)\Gamma\left(\frac{k+2}{2}\right)\left[1-\mathrm{B}_\lambda\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right]$$

$$E[r_{uv}S_uS_v^3 \mid r_{uv}^2 \ge \lambda] = \frac{1}{P(r_{uv}^2\ge\lambda)}\frac{4(1-\rho^2)^{2+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n^2}\sigma_u\sigma_v^3 d$$

$$\times \sum_{k=0}^{\infty}\frac{1}{2}[(-1)^k+1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+\frac{3}{2}\right)\Gamma\left(\frac{k+2}{2}\right)\left[1-\mathrm{B}_\lambda\left(\frac{k+2}{2},\frac{m-1}{2}\right)\right]$$

Let $A_1 = \frac{2\left(1-\rho^2\right)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}$; $\quad A_2 = \frac{1}{2}[(-1)^k + 1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{k+1}{2}\right)$ and

$A_3 = \frac{1}{2}[(-1)^k + 1]\frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n+1}{2}\right)\Gamma\left(\frac{k+2}{2}\right)$

Note the following two results (see Joarder, 2006)

$$Var[S_u^4] = \frac{n+2}{n}\sigma_u^4$$

$$E[S_u^2 S_v^2] = \frac{n+2\rho^2}{n}\sigma_u^2\sigma_v^2$$

With the previous equations we get

$$E[\hat{\sigma}_x^2] = \frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}d\sigma_u\sigma_v A_1 \sum_{k=0}^{\infty} A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right] \tag{3.17}$$

$$\begin{aligned} Var[\hat{\sigma}_x^2] &= \frac{1}{16}\left(E[S_u^4] + E[S_v^4] + 2\,E[S_u^2 S_v^2]\right) \\ &+ \frac{1}{4}\left(E[r_{uv}^2 S_u^2 S_v^2 | r_{uv}^2 \geq \lambda] + E[r_{uv} S_u^3 S_v | r_{uv}^2 \geq \lambda] + E[r_{uv} S_u S_v^3 | r_{uv}^2 \geq \lambda]\right) \\ &- \left\{\frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}E[r_{uv} S_u S_v | r_{uv}^2 \geq \lambda]\right\}^2 \\ &= \frac{n+2}{16n}(\sigma_u^4 + \sigma_v^4) + \frac{n+2\rho^2}{8n}\sigma_u^2\sigma_v^2 \\ &+ \frac{1}{4}\sigma_u^2\sigma_v^2\left\{\frac{2(1-\rho^2)}{n}A_1\sum_{k=0}^{\infty} A_2\left(\frac{k+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+3}{2},\frac{n-1}{2}\right)\right]\right\} \\ &+ \frac{1}{4}d(\sigma_u^3\sigma_v + \sigma_u\sigma_v^3)\left\{\frac{2(1-\rho^2)}{n}A_1\sum_{k=0}^{\infty} A_3\left(\frac{k+n+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right]\right\} \\ &- \left\{\frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}d\sigma_u\sigma_v A_1\sum_{k=0}^{\infty} A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right]\right\}^2 \end{aligned} \tag{3.18}$$

So the bias of $\hat{\sigma}_x^2$ is

$$Bias[\hat{\sigma}_x^2] = E[\hat{\sigma}_x^2] - \sigma_x^2 \tag{3.19}$$

The MSE of $\hat{\sigma}_x^2$ is

$$MSE[\hat{\sigma}_x^2] = Var[\hat{\sigma}_x^2] + (Bias[\hat{\sigma}_x^2])^2 \tag{3.20}$$

3.3 Estimator of σ_x^2 After Preliminary One-tailed Test

If it is known that $\sigma_y^2 \leq \sigma_x^2$ then we can concentrate on the one-tailed test and the hypothesis for the preliminary test become

$H_0: \sigma_x^2 = \sigma_y^2$ VS $H_1: \sigma_y^2 \leq \sigma_x^2$

Under this new scenario, $r_{uv} \geq 0$. In the case of independence of the two samples, one estimator of σ_x^2 after preliminary test was developed by Bancroft (1944), he proposed the following estimator, when the sample sizes are equal

$$\hat{\sigma}_x^{2**} = \begin{cases} \frac{1}{2}\left(S_x^2 + S_y^2\right) & if\ \frac{S_x^2}{S_y^2} < f_\alpha \\ S_x^2 & if\ \frac{S_x^2}{S_y^2} \geq f_\alpha \end{cases} \tag{3.21}$$

Where f_α is the value on the F-distribution with (n-1, n-1) degrees of freedom corresponding to some assigned significance level α.

When the samples are equal, the Bancroft equations for the expectation, variance, bias and MSE become

$$E[\hat{\sigma}_x^{2**}] = \left\{1 + \frac{1}{2}\left[\mathrm{B}_{x_0}\left(\frac{n}{2}, \frac{n+2}{2}\right)\frac{\sigma_y^2}{\sigma_x^2} - \mathrm{B}_{x_0}\left(\frac{n+2}{2}, \frac{n}{2}\right)\right]\right\}\sigma_x^2$$

$$Var[\hat{\sigma}_x^{2**}] = \left(\frac{1}{4} + \frac{1}{2n}\right)\left[\mathrm{B}_{x_0}\left(\frac{n+4}{2}, \frac{n}{2}\right)\sigma_x^4 + \mathrm{B}_{x_0}\left(\frac{n}{2}, \frac{n+4}{2}\right)\sigma_y^4\right]$$

$$+ \frac{1}{2}\mathrm{B}_{x_0}\left(\frac{n+2}{2}, \frac{n+2}{2}\right)\sigma_x^2\sigma_y^2 + \left(\frac{n+2}{n}\right)\left[1 - \mathrm{B}_{x_0}\left(\frac{n+4}{2}, \frac{n}{2}\right)\right]\sigma_x^4$$

$$- \left[1 + \frac{1}{2}\mathrm{B}_{x_0}\left(\frac{n}{2}, \frac{n+2}{2}\right)\frac{\sigma_y^2}{\sigma_x^2} - \frac{1}{2}\mathrm{B}_{x_0}\left(\frac{n+2}{2}, \frac{n}{2}\right)\right]^2 \sigma_x^4$$

$Bias[\hat{\sigma}_x^{2**}] = \frac{1}{2}\left[\mathrm{B}_{x_0}\left(\frac{n}{2},\frac{n+2}{2}\right)\frac{\sigma_y^2}{\sigma_x^2} - \mathrm{B}_{x_0}\left(\frac{n+2}{2},\frac{n}{2}\right)\right]\sigma_x^2$

$MSE[\hat{\sigma}_x^{2**}] = Var[\hat{\sigma}_x^{2**}] + [E[\hat{\sigma}_x^{2**}] - \sigma_x^2]^2$

where $x_0 = \frac{f_\alpha}{\frac{\sigma_x^2}{\sigma_y^2}+f_\alpha}$

We define now an estimator of σ_x^2 after one-tailed preliminary test, for dependence samples, as

$$\hat{\sigma}_x^{2*} = \begin{cases} \frac{1}{2}\left(S_x^2 + S_y^2\right) & if\;\; r_{uv} < (\lambda^*)^{1/2} \quad (r_{uv}^2 < \lambda^*) \\ S_x^2 & if\;\; r_{uv} \geq (\lambda^*)^{1/2} \quad (r_{uv}^2 < \lambda^*) \end{cases} \tag{3.22}$$

where λ^* is the $100(1-2\alpha)$ percentage point of the beta distribution

$$\lambda^* = B_{1-2\alpha}\left(\frac{1}{2},\frac{1}{2}n-1\right)$$

Note that in a two-tailed test we reject $H_0\colon \sigma_x^2 = \sigma_y^2$ if, using the distribution of r_{uv}, we have $P\left(r_{uv} > \lambda^{1/2}\right) = \frac{\alpha}{2}$ or $P\left(r_{uv} < -\lambda^{1/2}\right) = \frac{\alpha}{2}$. This is equivalent to $P(r_{uv}^2 < \lambda) = \alpha$ where $r_{uv}^2 \sim B\left(\frac{1}{2},\frac{1}{2}n-1\right)$. In the one-tailed case, we reject H_0 if $P\left(r_{uv} > \lambda^{1/2}\right) = \frac{\alpha}{2}$, which is equivalent to $P(r_{uv}^2 < \lambda) = 2\alpha$.

The expectation and variance for $\hat{\sigma}_x^{2*}$ are obtained in a similar way as the two-tailed test, in the procedure we only consider the rejection region, $r_{uv} \geq \lambda^{1/2}$.

$$E[\hat{\sigma}_x^{2*}] = \frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}\sigma_u\sigma_v A_1 \sum_{k=0}^{\infty} A_3^* \left[1 - \mathrm{B}_{\lambda^*}\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right] \tag{3.23}$$

$$Var[\hat{\sigma}_x^{2*}] = \frac{n+2}{16n}(\sigma_u^4 + \sigma_v^4) + \frac{n+2\rho^2}{8n}\sigma_u^2\sigma_v^2 \tag{3.24}$$

$$+\frac{1}{4}\sigma_u^2\sigma_v^2\left\{\frac{2(1-\rho^2)}{n}A_1\sum_{k=0}^{\infty}A_2^*\left(\frac{k+1}{2}\right)\mathrm{B}_{\lambda^*}\left(\frac{k+3}{2},\frac{n-1}{2}\right)\right\}$$

$$+\frac{1}{4}(\sigma_u^3\sigma_v + \sigma_u\sigma_v^3)\left\{\frac{2(1-\rho^2)}{n}A_1\sum_{k=0}^{\infty}A_3^*\left(\frac{k+n+1}{2}\right)\left[1-\mathrm{B}_{\lambda^*}\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right]\right\}$$

$$-\left\{\frac{1}{4}(\sigma_u^2+\sigma_v^2) + \frac{1}{2}\sigma_u\sigma_v A_1\sum_{k=0}^{\infty}A_3^*\left[1-\mathrm{B}_{\lambda^*}\left(\frac{k+2}{2},\frac{n-1}{2}\right)\right]\right\}^2$$

$$MSE[\hat{\sigma}_x^{2*}] = Var[\hat{\sigma}_x^{2*}] + [E[\hat{\sigma}_x^{2*}] - \sigma_x^2]^2$$

where $A_2^* = \frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{k+1}{2}\right)$ and $A_3^* = \frac{(2\rho)^k}{k!}\Gamma\left(\frac{k+n+1}{2}\right)\Gamma\left(\frac{k+2}{2}\right)$

The preliminary test estimator given by Bancroft (1944) is based on the F distribution, meanwhile the preliminary test that we proposed is based on a Beta distribution, in section 3.4, Table 3.3, we will see that when the samples are independent, $\rho = 0$, the values that we get from both methods are similar

3.4 Discussion and Numerical Example

The value of λ is a function of α and n. In Figure 3.1, we can see that when n goes to infinity the value of λ goes to zero for any value of α, that is we never pool.

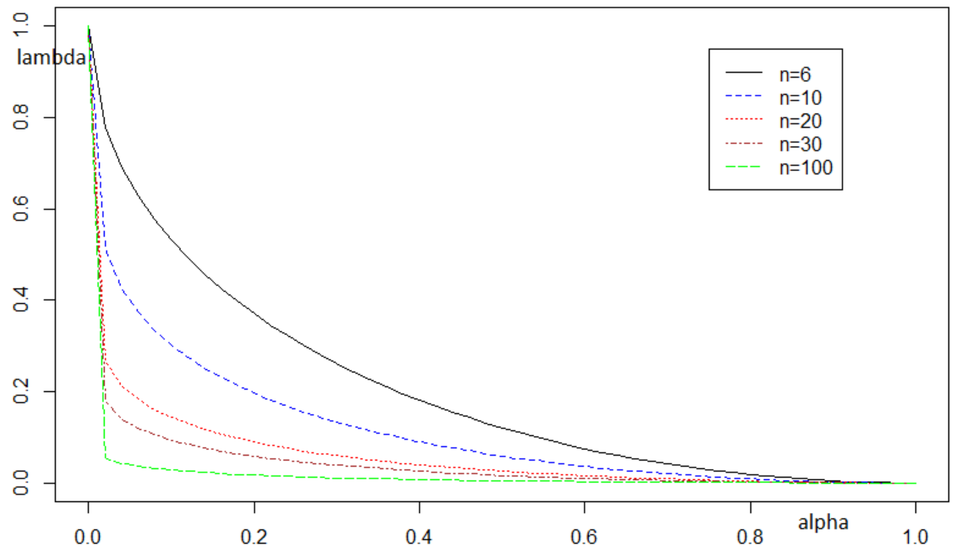


Figure 3.1 Graph of λ *vs* α for different values of *n*.

If we always pool then $\lambda = 0$, expressions for $E[\hat{\sigma}_x^2]$ and $Var[\hat{\sigma}_x^2]$ become

$$E[\hat{\sigma}_x^2] = \frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}\sigma_u\sigma_v A_1 \sum_{k=0}^{\infty} A_3 = E[S_x^2]$$

$$Var[\hat{\sigma}_x^2] = \frac{n+2}{16n}(\sigma_u^4 + \sigma_v^4) + \frac{n+2\rho^2}{8n}\sigma_u^2\sigma_v^2 + \frac{1}{4}\sigma_u^2\sigma_v^2 \left\{\frac{2(1-\rho^2)}{n} A_1 \sum_{k=0}^{\infty} A_2 \left(\frac{k+1}{2}\right)\right\}$$

$$+ \frac{1}{4}(\sigma_u^3\sigma_v + \sigma_u\sigma_v^3)\left\{\frac{2(1-\rho^2)}{n} A_1 \sum_{k=0}^{\infty} A_3 \left(\frac{k+n+1}{2}\right)\right\}$$

$$- \left\{\frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}\sigma_u\sigma_v A_1 \sum_{k=0}^{\infty} A_3\right\}^2$$

$$= \frac{n+2}{16n}(\sigma_u^4 + \sigma_v^4) + \frac{n+2\rho^2}{8n}\sigma_u^2\sigma_v^2$$

$$+ \frac{1}{4}\left\{\sigma_u^2\sigma_v^2 \frac{2(1-\rho^2)}{n} A_1 \sum_{k=0}^{\infty} A_2 \left(\frac{k+1}{2}\right) - \left(\sigma_u\sigma_v A_1 \sum_{k=0}^{\infty} A_3\right)^2\right\}$$

$$+ \frac{1}{4}\left\{(\sigma_u^2 + \sigma_v^2)\sigma_u\sigma_v \frac{2(1-\rho^2)}{n} A_1 \sum_{k=0}^{\infty} A_3 \left(\frac{k+n+1}{2}\right)\right\}$$

$$-\frac{1}{4}\left((\sigma_u^2+\sigma_v^2)\sigma_u\sigma_v A_1 \sum_{k=0}^{\infty} A_3\right)^2$$

$$= Var\left[\frac{1}{4}(S_u^2+S_v^2)\right] + Var\left[\frac{1}{2}r_{uv}S_uS_v\right] + 2Cov\left[\frac{1}{4}(S_u^2+S_v^2), \frac{1}{2}r_{uv}S_uS_v\right]$$

$$= Var[S_x^2]$$

On the other hand, if we always pool then $\lambda = 1$

$$E[\hat{\sigma}_x^2] = \frac{1}{4}(\sigma_u^2+\sigma_v^2) = E\left[\frac{1}{2}\left(S_x^2+S_y^2\right)\right]$$

$$Var[\hat{\sigma}_x^2] = \frac{n+2}{16n}(\sigma_u^4+\sigma_v^4) + \frac{n+2\rho^2}{8n}\sigma_u^2\sigma_v^2 - \left(\frac{1}{4}(\sigma_u^2+\sigma_v^2)\right)^2$$

$$= \frac{1}{16}\left\{\left(\frac{n+2}{n}\sigma_u^4 - \sigma_u^4\right) + \left(\frac{n+2}{n}\sigma_v^4 - \sigma_v^4\right) + 2\left(\frac{n+2\rho^2}{n}\sigma_u^2\sigma_v^2 - \sigma_u^2\sigma_v^2\right)\right\}$$

$$= \frac{1}{16}\{Var(S_u^4) + Var(S_v^4) + 2Cov(S_u^2S_v^2)\}$$

$$= Var\left[\frac{1}{2}\left(S_x^2+S_y^2\right)\right]$$

In both cases the results are consistent. If we let $\lambda = 1$, the expectation and variance of $\hat{\sigma}_x^2$ become the expectation and variance of the pooled estimator. If we let $\lambda = 0$, the expectation and variance of $\hat{\sigma}_x^2$ become the expectation and variance of the never pooled estimator.

In the expressions for $E[\hat{\sigma}_x^2]$ and $Var[\hat{\sigma}_x^2]$ we have an infinite sum, the terms in the infinite sum approach to zero quickly as *k* increase. In Table 3.1, values of $E[\hat{\sigma}_x^2]$ and $Var[\hat{\sigma}_x^2]$ for different values of *k* are given, in each case the infinite series is truncated at *k*. When the difference between σ_x^2 and σ_y^2 are small then the infinite series stabilized quickly, but if the difference is big then the infinite series do not stabilized and we have a computational limitation. We realized that in general when $0.3 < \frac{\sigma_y^2}{\sigma_x^2} < 3$, $|\rho_{xy}| < 0.85$ and n < 50, the infinite series stabilized quickly and we have good approximations if we truncate the infinite series at k=150

Table 3.1Values of $E[\hat{\sigma}_x^2]$ and $MSE[\hat{\sigma}_x^2]$ for different values of k in the infinite serie

	Case 1		Case 2		Case 3		Case 4	
k	$E[\hat{\sigma}_x^2]$	$MSE[\hat{\sigma}_x^2]$	$E[\hat{\sigma}_x^2]$	$MSE[\hat{\sigma}_x^2]$	$E[\hat{\sigma}_x^2]$	$MSE[\hat{\sigma}_x^2]$	$E[\hat{\sigma}_x^2]$	$MSE[\hat{\sigma}_x^2]$
0	3.90904	2.59311	4.16742	0.84041	12.50000	61.94500	10.00000	34.86250
1	3.90904	2.59311	4.16742	0.84041	12.50000	61.94500	10.00000	34.86250
2	3.92984	2.67101	4.17213	0.85329	12.50000	61.94500	10.00000	34.86250
4	3.93082	2.67653	4.17220	0.85353	12.50000	61.94500	10.00000	34.86250
6	3.93085	2.67673	4.17220	0.85353	12.50000	61.94500	10.00000	34.86250
8	3.93085	2.67673	4.17220	0.85353	12.50000	61.94500	10.00000	34.86250
10	3.93085	2.67673	4.17220	0.85353	12.50000	61.94500	10.00000	34.86250
20	3.93085	2.67673	4.17220	0.85353	12.49999	61.94501	10.00000	34.86239
25	3.93085	2.67673	4.17220	0.85353	12.49991	61.94502	10.00001	34.86213
50	3.93085	2.67673	4.17220	0.85353	12.46170	61.87192	10.00225	34.81283
100	3.93085	2.67673	4.17220	0.85353	9.19725	42.72293	10.17059	31.98959
150	3.93085	2.67673	4.17220	0.85353	5.36552	6.17148	11.12305	21.20019
154	3.93085	2.67673	4.17220	0.85353	5.27790	5.03658	11.23821	20.22346

Case 1: $\sigma_x^2 = 4$, $\sigma_y^2 = 3.6$, $\sigma_{xy} = 3$, n=10, $\alpha = 0.1$
Case 2: $\sigma_x^2 = 4.2$, $\sigma_y^2 = 4$, $\sigma_{xy} = 2$, n=30, $\alpha = 0.05$
Case 3: $\sigma_x^2 = 5$, $\sigma_y^2 = 20$, $\sigma_{xy} = 8.5$, n=50, $\alpha = 0.05$
Case 4: $\sigma_x^2 = 15$, $\sigma_y^2 = 5$, $\sigma_{xy} = 8.5$, n=20, $\alpha = 0.05$

A simulation was performed using Monte Carlo method. A random sample of size n from the bivariate random distribution was generated for the specified parameter, then and estimation of r_{uv}^2 was gotten, if the estimation for

r_{uv}^2 was less than λ we pool otherwise we use the estimation for the variance of the sample for x, we did the process 10,000 times and we got the mean square error form the 10,000 values.

Values of $MSE[\hat{\sigma}_x^2]$ for different values of n and α in three scenarios are given in Table 3.2. In the approximations we truncate the infinite series at $k=150$. The computations for the tables were performed using programs written in R version 2.15.1. The data of the simulation are in line with those of the equations that we got.

In the case of the one-tailed preliminary test we compute values for $MSE[\hat{\sigma}_x^{2*}]$ and $MSE[\hat{\sigma}_x^{2**}]$, that are given in Table 3.3. Again when the samples are independent, $\rho = 0$, the values that we get are similar to those of Bancroft (1944). We can say that, with a particular case of the preliminary test given in 3.20, samples sizes equal and $\rho = 0$, we can get similar values of those gotten with Bancroft equations

Table 3.2 Values of $MSE[\hat{\sigma}_x^2]$ for different values of n and α, for the equation that we got and for a simulation

		case 1		case 2		case 3	
n	alpha	Equation	Simulation	Equation	Simulation	Equation	Simulation
10	0.05	2.5533144	2.911120	2.394272	2.7064590	3.1821721	3.509223
10	0.1	2.6767311	3.0143865	2.642882	2.8811301	3.6559338	3.925595
10	0.2	2.8571791	3.2379324	3.031702	3.2622289	4.4057203	4.396387
10	0.3	2.9820949	3.4044709	3.318479	3.5227358	4.9653973	4.949612
10	0.5	3.1316896	3.5886548	3.685264	3.7855203	5.6915728	5.534832
20	0.05	1.3180163	1.4391192	1.226799	1.3029027	1.6407944	1.688116
20	0.1	1.3744933	1.4501049	1.34478	1.4357759	1.8732247	1.911863
20	0.2	1.4444464	1.5060883	1.511789	1.6157959	2.2109313	2.179347
20	0.3	1.4858448	1.6126757	1.625205	1.7218582	2.4466554	2.372776
20	0.5	1.5270103	1.6449276	1.759088	1.8211920	2.7349696	2.508156
30	0.05	0.8989703	0.9419439	0.824650	0.8705788	1.1011387	1.131195
30	0.1	0.9336375	0.9782433	0.899396	0.9479946	1.2519441	1.243935
30	0.2	0.9723144	1.0404832	1.001002	1.0431716	1.4650806	1.427387
30	0.3	0.9925231	1.0552206	1.067436	1.1006743	1.6104255	1.545268
30	0.5	1.0088253	1.1046119	1.142460	1.1695195	1.7841129	1.661979
50	0.05	0.5609575	0.5844762	0.499923	0.5187593	0.6626018	0.678982
50	0.1	0.5788562	0.5973554	0.541664	0.5654707	0.7498874	0.728851
50	0.2	0.5954322	0.6206048	0.595768	0.6139027	0.8703457	0.868053
50	0.3	0.6018261	0.6294167	0.629439	0.6882561	0.9507437	0.925180
50	0.5	0.6034024	0.6450123	0.665016	0.6919349	1.0445668	0.970123
100	0.05	0.3040210	0.3120257	0.255583	0.2579829	0.3310965	0.326093
100	0.1	0.3095092	0.3238052	0.274290	0.2854258	0.3727815	0.386592
100	0.2	0.3116961	0.3174539	0.296758	0.3150369	0.4290758	0.433176
100	0.3	0.3103489	0.3267571	0.309536	0.3316886	0.4658524	0.440745
100	0.5	0.3054034	0.3207741	0.321188	0.3381699	0.5076396	0.490568

Case 1: $\sigma_x^2 = 4$, $\sigma_y^2 = 3.6$, $\sigma_{xy} = 3$.
Case 2: $\sigma_x^2 = 4.2$, $\sigma_y^2 = 4$, $\sigma_{xy} = 2$.
Case 3: $\sigma_x^2 = 5.03$, $\sigma_y^2 = 5$, $\sigma_{xy} = 1$

Table 3.3 Computation of $MSE[\hat{\sigma}_x^{2*}]$ and $MSE[\hat{\sigma}_x^{2**}]$ for different values of n and α

Case	n	alpha	$MSE[\hat{\sigma}_x^{2*}]$	$MSE[\hat{\sigma}_x^{2**}]$
	10	0.05	1.925520	2.002319
	10	0.1	2.199892	2.265530
	10	0.2	2.517807	2.549530
	20	0.05	1.020834	1.036401
	20	0.1	1.141644	1.153503
1	20	0.2	1.261998	1.266878
	50	0.05	0.446506	0.448529
	50	0.1	0.485764	0.487057
	50	0.2	0.516336	0.516701
	10	0.05	2.118077	2.196695
	10	0.1	2.406708	2.477641
	10	0.2	2.763990	2.801367
	20	0.05	1.092560	1.108616
2	20	0.1	1.222677	1.235996
	20	0.2	1.366805	1.373137
	50	0.05	0.451022	0.453216
	50	0.1	0.497209	0.498873
	50	0.2	0.542688	0.543374
	10	0.05	3.069182	3.175031
	10	0.1	3.465581	3.564854
	10	0.2	3.979782	4.035333
	20	0.05	1.561619	1.583126
3	20	0.1	1.741288	1.760203
	20	0.2	1.954906	1.964766
	50	0.05	0.627204	0.630133
	50	0.1	0.692082	0.694541
	50	0.2	0.764635	0.765841

Case 1: $\sigma_x^2 = 4$, $\sigma_y^2 = 3.6$, $\sigma_{xy} = 0$.
Case 2: $\sigma_x^2 = 4.2$, $\sigma_y^2 = 4$, $\sigma_{xy} = 0$.
Case 3: $\sigma_x^2 = 5.03$, $\sigma_y^2 = 5$, $\sigma_{xy} = 0$

Chapter 4

Relative Efficiency

4.1 Relative Efficiency for Preliminary Test on Pooling Variances

The relative efficiency (RE) of $\hat{\sigma}_x^2$ to S_x^2 is defined as (Bancroft and Han, 1983)

$$RE = \frac{1/MSE(\hat{\sigma}_x^2)}{1/MSE(S_x^2)} = \frac{2\sigma_x^4}{nMSE(\hat{\sigma}_x^2)} \tag{4.1}$$

The MSE of $\hat{\sigma}_x^2$ is given by using equations (3.17) and (3.18)

$$MSE[\hat{\sigma}_x^2] = Var[\hat{\sigma}_x^2] + [E[\hat{\sigma}_x^2] - \sigma_x^2]^2 \tag{4.2}$$

$$= \frac{n+2}{16n}(\sigma_u^4 + \sigma_v^4) + \frac{n+2\rho_{uv}^2}{8n}\sigma_u^2\sigma_v^2$$

$$+ \frac{1}{4}\sigma_u^2\sigma_v^2\left\{\frac{2(1-\rho_{uv}^2)}{n}A_1\sum_{k=0}^{\infty}A_2\left(\frac{k+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+3}{2}, \frac{n-1}{2}\right)\right]\right\}$$

$$+ \frac{1}{4}d(\sigma_u^3\sigma_v + \sigma_u\sigma_v^3)\left\{\frac{2(1-\rho_{uv}^2)}{n}A_1\sum_{k=0}^{\infty}A_3\left(\frac{k+n+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right]\right\}$$

$$-\left\{\frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}d\sigma_u\sigma_v A_1\sum_{k=0}^{\infty}A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right]\right\}^2$$

$$+\left\{\frac{1}{4}(\sigma_u^2 + \sigma_v^2) + \frac{1}{2}d\sigma_u\sigma_v A_1\sum_{k=0}^{\infty}A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right] - \sigma_x^2\right\}^2$$

$$= \frac{n+2}{16n}(\sigma_u^4 + \sigma_v^4) + \frac{n+2\rho_{uv}^2}{8n}\sigma_u^2\sigma_v^2 - \frac{1}{16}(\sigma_u^2 + \sigma_v^2)^2 + \left\{\frac{1}{4}(\sigma_u^2 + \sigma_v^2) - \sigma_x^2\right\}^2$$

$$+ \frac{1}{4}\sigma_u^2\sigma_v^2\left\{\frac{2(1-\rho_{uv}^2)}{n}A_1\sum_{k=0}^{\infty}A_2\left(\frac{k+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+3}{2}, \frac{n-1}{2}\right)\right]\right\}$$

$$+ \frac{1}{4}d(\sigma_u^3\sigma_v + \sigma_u\sigma_v^3)\left\{\frac{2(1-\rho_{uv}^2)}{n}A_1\sum_{k=0}^{\infty}A_3\left(\frac{k+n+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right]\right\}$$

$$-d\sigma_x^2\sigma_u\sigma_v A_1\sum_{k=0}^{\infty}A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right]$$

where λ is the $100(1-\alpha)\%$ point of the beta distribution corresponding to an α-level of significance for the preliminary test

$$A_1 = \frac{2\left(1-\rho_{uv}^2\right)^{1+\frac{n}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n}; \quad A_2 = \frac{1}{2}[(-1)^k + 1]\frac{(2\rho_{uv})^k}{k!}\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{k+1}{2}\right) \text{ and}$$

$$A_3 = \frac{1}{2}[(-1)^k + 1]\frac{(2\rho_{uv})^k}{k!}\Gamma\left(\frac{k+n+1}{2}\right)\Gamma\left(\frac{k+2}{2}\right)$$

$$d = \begin{cases} (-1)^c & if\ r_{uv} < 0 \\ 1 & if\ r_{uv} \geq 0 \end{cases}$$

Using the following

$$\sigma_u^2 = \sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}; \qquad \sigma_v^2 = \sigma_x^2 + \sigma_y^2 - 2\sigma_{xy};$$

$$\sigma_{uv} = \sigma_x^2 - \sigma_y^2; \qquad \frac{1}{4}(\sigma_u^2 + \sigma_v^2) = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2\right)$$

we have

$$\sigma_u^2\sigma_v^2 = \sigma_x^4 + \sigma_y^4 + 2\sigma_x^2\sigma_y^2 - 4\sigma_{xy}^2 = \sigma_x^4\left[1 + \frac{\sigma_y^4}{\sigma_x^4} + \frac{2\sigma_y^2}{\sigma_x^2} - \frac{4\sigma_{xy}^2}{\sigma_x^4}\right]$$

$$(\sigma_u^2 + \sigma_v^2)^2 = 4\sigma_x^4\left[1 + \frac{\sigma_y^4}{\sigma_x^4} + \frac{2\sigma_y^2}{\sigma_x^2}\right]$$

$$(\sigma_u^4 + \sigma_v^4) = (\sigma_u^2 + \sigma_v^2)^2 - 2\sigma_u^2\sigma_v^2 = 2\sigma_x^4\left[1 + \frac{\sigma_y^4}{\sigma_x^4} + \frac{2\sigma_y^2}{\sigma_x^2} + \frac{4\sigma_{xy}^2}{\sigma_x^4}\right]$$

$$\sigma_u^3\sigma_v + \sigma_u\sigma_v^3 = \sigma_u\sigma_v(\sigma_u^2 + \sigma_v^2) = 2\left(\sigma_x^2 + \sigma_y^2\right)\sigma_x^2\left[1 + \frac{\sigma_y^4}{\sigma_x^4} + \frac{2\sigma_y^2}{\sigma_x^2} - \frac{4\sigma_{xy}^2}{\sigma_x^4}\right]^{1/2}$$

$$= 2\sigma_x^4\left(1 + \frac{\sigma_y^2}{\sigma_x^2}\right)\left[1 + \frac{\sigma_y^4}{\sigma_x^4} + \frac{2\sigma_y^2}{\sigma_x^2} - \frac{4\sigma_{xy}^2}{\sigma_x^4}\right]^{1/2}$$

$$\left[\frac{1}{4}(\sigma_u^2 + \sigma_v^2) - \sigma_x^2\right]^2 = \left[\frac{1}{2}(\sigma_y^2 - \sigma_x^2)\right]^2 = \frac{\sigma_x^4}{4}\left[1 + \frac{\sigma_y^4}{\sigma_x^4} - \frac{2\sigma_y^2}{\sigma_x^2}\right]$$

$$\frac{1}{2}(\sigma_y^2 - \sigma_x^2)\sigma_u\sigma_v = \frac{\sigma_x^4}{2}\left(\frac{\sigma_y^2}{\sigma_x^2} - 1\right)\left[1 + \frac{\sigma_y^4}{\sigma_x^4} + \frac{2\sigma_y^2}{\sigma_x^2} - \frac{4\sigma_{xy}^2}{\sigma_x^4}\right]^{1/2}$$

Let $\phi = \frac{\sigma_y^2}{\sigma_x^2}$ then $\frac{\sigma_{xy}^2}{\sigma_x^4} = \frac{\rho_{xy}^2\sigma_x^2\sigma_y^2}{\sigma_x^4} = \rho_{xy}^2\phi$ and

$$d = \begin{cases} (-1)^c & if\ \phi \geq 1 \\ 1 & if\ \phi < 1 \end{cases}$$

We can express ρ_{uv}^2 in terms of ϕ and ρ_{xy}^2 as

$$\rho_{uv} = \frac{\sigma_x^2 - \sigma_y^2}{\left[(\sigma_x^2 + \sigma_y^2)^2 - 4\rho_{xy}^2\sigma_x^2\sigma_y^2\right]^{1/2}}$$

$$\rho_{uv}^2 = \frac{\sigma_x^4 + \sigma_y^4 - 2\sigma_{xy}^2}{\sigma_x^4 + \sigma_y^4 + 2\sigma_{xy}^2 - 4\rho_{xy}^2\sigma_x^2\sigma_y^2} = \frac{\sigma_x^4\left(1 + \frac{\sigma_y^4}{\sigma_x^4} - 2\frac{\sigma_y^2}{\sigma_x^2}\right)}{\sigma_x^4\left(1 + \frac{\sigma_y^4}{\sigma_x^4} + 2\frac{\sigma_y^2}{\sigma_x^2} - 4\frac{\sigma_{xy}^2}{\sigma_x^4}\right)} = \frac{1 + \phi^2 - 2\phi}{1 + \phi^2 + 2\phi - 4\phi\rho_{xy}^2} \tag{4.3}$$

Our final expression for RE^{-1} is

$$RE^{-1} = \frac{nMSE(\hat{\sigma}_x^2)}{2\sigma_x^4} \tag{4.4}$$

$$= \frac{n+2}{16}\left[1 + \phi^2 + 2\phi + 4\rho_{xy}^2\phi\right] + \frac{n+2\rho_{uv}^2}{16}\left[1 + \phi^2 + 2\phi - 4\rho_{xy}^2\phi\right] - \frac{n}{2}\phi$$

$$+ \frac{n}{8}\left[1 + \phi^2 + 2\phi - 4\rho_{xy}^2\phi\right]\left\{\frac{2(1-\rho_{uv}^2)}{n}A_1\sum_{k=0}^{\infty}A_2\left(\frac{k+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+3}{2}, \frac{n-1}{2}\right)\right]\right\}$$

$$+ \frac{n}{4}d(1 + \phi)\left[1 + \phi^2 + 2\phi - 4\rho_{xy}^2\phi\right]^{\frac{1}{2}}$$

$$\times \left\{\frac{2(1-\rho_{uv}^2)}{n}A_1\sum_{k=0}^{\infty}A_3\left(\frac{k+n+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right]\right\}$$

$$- \frac{n}{2}d\left[1 + \phi^2 + 2\phi - 4\rho_{xy}^2\phi\right]^{1/2}\left\{A_1\sum_{k=0}^{\infty}A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right]\right\}$$

For given *n*, *RE* is a function of α, ρ_{xy}^2, and ϕ. Some special cases are given as follows

i) If $\phi = 1$ and $\rho_{xy}^2 = 1$ then *RE=1.*

ii) If *n* goes to infinite then λ goes to cero and $Var[\hat{\sigma}_x^2] = Var[S_x^2]$ so *RE=1.*

iii) If $\phi = 1$, then $\rho_{uv} = 0$ and the infinite series take values only when k=0

$$A_1 = \frac{2}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)n};$$

$$\sum_{k=0}^{\infty} A_2\left(\frac{k+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{k+3}{2}, \frac{n-1}{2}\right)\right] = \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{3}{2}, \frac{n-1}{2}\right)\right]$$

$$= \Gamma\left(\frac{n}{2}\right)\left(\frac{n\sqrt{\pi}}{4}\right)\left[1 - \mathrm{B}_\lambda\left(\frac{3}{2}, \frac{n-1}{2}\right)\right]$$

$$\sum_{k=0}^{\infty} A_3\left[1 - \mathrm{B}_\lambda\left(\frac{k+2}{2}, \frac{n-1}{2}\right)\right] = \Gamma\left(\frac{n+1}{2}\right)\left[1 - \mathrm{B}_\lambda\left(1, \frac{n-1}{2}\right)\right]$$

Hence

$$RE^{-1} = \frac{1}{4}\left(n + n\rho_{xy}^2 + 2 + 2\rho_{xy}^2\right) + \frac{n}{4}\left(1 - \rho_{xy}^2\right) - \frac{n}{2}$$

$$+\frac{1}{2}\left(1 - \rho_{xy}^2\right)\left[1 - \mathrm{B}_\lambda\left(\frac{3}{2}, \frac{n-1}{2}\right)\right]$$

$$+\left(1 - \rho_{xy}^2\right)^{\frac{1}{2}}\left(\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}\right)\left[1 - \mathrm{B}_\lambda\left(1, \frac{n-1}{2}\right)\right]\left\{\frac{(n+1)}{n} - 1\right\}$$

$$= 1 - \frac{1}{2}\left(1 - \rho_{xy}^2\right)\mathrm{B}_\lambda\left(\frac{3}{2}, \frac{n-1}{2}\right) \tag{4.5}$$

$$+\left(1 - \rho_{xy}^2\right)^{\frac{1}{2}}\left(\frac{2}{B\left(\frac{1}{2}, \frac{n}{2}\right)n}\right)\left[1 - \mathrm{B}_\lambda\left(1, \frac{n-1}{2}\right)\right]$$

In the last equation *RE* is function of n, α and ρ_{xy}^2 . If furthermore $\rho_{xy} = 0$ then

$$RE^{-1} = 1 - \frac{1}{2}\mathrm{B}_\lambda\left(\frac{3}{2}, \frac{n-1}{2}\right) + \left(\frac{2}{B\left(\frac{n}{2}, \frac{1}{2}\right)n}\right)\left[1 - \mathrm{B}_\lambda\left(1, \frac{n-1}{2}\right)\right] \tag{4.6}$$

The difference of RE^{-1} when $\rho_{xy} = 0$ minus RE^{-1} when $\rho_{xy} \neq 0$ is

$$Dif = \frac{1}{2}\mathrm{B}_\lambda\left(\frac{3}{2}, \frac{n-1}{2}\right)\rho_{xy}^2 - \frac{1}{2}\left(\frac{4\left[1 - \mathrm{B}_\lambda\left(1, \frac{n-1}{2}\right)\right]}{nB\left(\frac{1}{2}, \frac{n}{2}\right)}\right)\left(1 - \left(1 - \rho_{xy}^2\right)^{\frac{1}{2}}\right) \tag{4.7}$$

Note that the difference is positive when n is big or α is not big (λ not small). From the graph in figure 4.1 we found that the difference is always positive when $\alpha < 0.33$. On the scenario mentioned before, for $\phi = 1$, RE at $\rho_{xy} = 0$ is greater than RE at $\rho_{xy} \neq 0$, so there is a maximum of RE at $\rho_{xy} = 0$, $\phi = 1$.

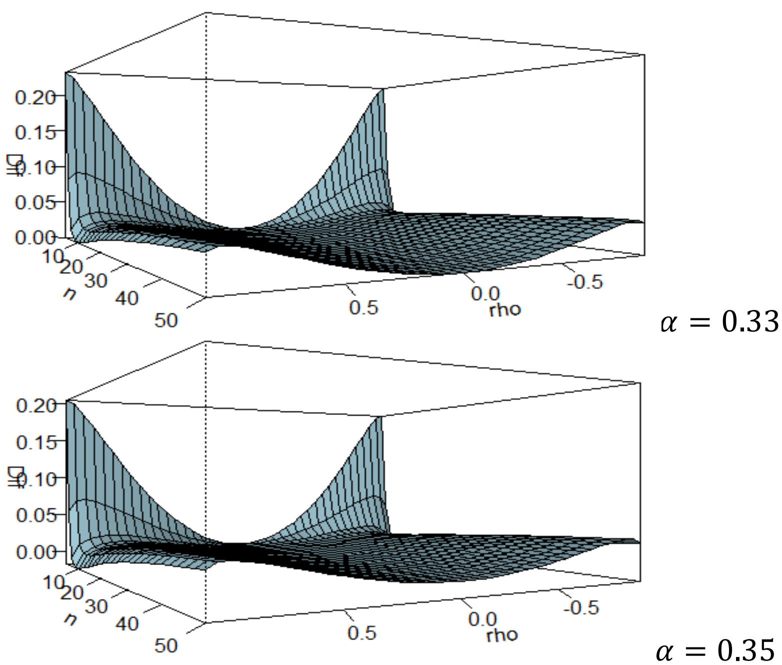


Figure 4.1 Graph of *Dif*, Eq 4.7 for two values of α

4.2 Computations of RE and Recommendations of Significant Level

The quantity ϕ is a parameter that is beyond the control of the experimenter. As mentioned by Bancroft and Han (1983), we need to find an appropriate λ level such that the relative efficiency of $\hat{\sigma}_x^2$ to S_x^2 should be high. We want to choose an estimator for estimating σ_x^2 and we would like an estimator with the smallest MSE since the bias is part of MSE, so it is reasonable consider only the MSE or the relative efficiency.

There are computational limitations when we evaluate the infinite series of equation (4.4) for some values, particularly when $\phi < 0.25 \; or \; \phi > 3$ and $|\rho_{xy}| > 0.85$. For the other cases if we truncate the infinite series at k=150 we got good approximations.

The mathematical way to get the maximum of RE with respect to ϕ and ρ_{xy} is not easy, instead we use software (R. ver. 2.15.1) to get the maximum for different parameters values.

The experimenter would choose the estimator with largest relative efficiency. In order to have a criterion for selecting the estimator, or equivalently the α level of the preliminary test we use the criterion proposed by Han and Bancroft (1968). If the experimenter does not know the size of ϕ and is willing to accept and estimator which has a relative efficiency of no less than RE_0, then among the set of estimators with $\alpha \in A$, where $A = \{\alpha | RE(\alpha, \phi, \rho_{xy}) \geq RE_0,$

$for\ all\ \phi, \rho_{xy}\}$, the estimator is chosen to maximize $RE(\alpha, \phi, \rho_{xy})$ over all α, ϕ and ρ_{xy}. Since Maximum of RE occurs when $\rho_{xy} = 0$ he selects the $\alpha \in A$ (say α^*), which maximizes $RE(\alpha, \phi, 0)$ (say RE^*). This criterion will guarantee that the relative efficiency of the chosen estimator is at least RE_0 and it may become as large as RE^*. This criterion is called the max-min criterion.

In Table 4.1 we can find values of $RE_0(Min\ RE)$, the corresponding α to use and the maximum relative efficiency $RE^*(Max\ RE)$. For given *n*, one may select from Tale 4.1, the $Min\ RE$ which has the smallest relative efficiency he wishes to accept. For example if *n*=10 and the experimenter wants to have an estimator which has a relative efficiency no less than 0.8, then he would use $\alpha = 0.4$ and the maximum relative efficiency he can obtain is 1.3297

The behavior of the maximum *RE* is as follows. As *n* and α increase the maximum *RE* decrease. The maximum *RE* is reached when ϕ was between 0.75 and 1.1 and ρ_{xy} is zero. The interval for ϕ, such that *RE* > 1, is shorter as n increase. Table 4.1, show an interval for ϕ such that the value of *RE* is greater than 1.

In Figure 4.1 and 4.2, we can see the 3D graphs of *RE* as a function of ϕ (phi) and ρ_{xy} (rho) for different values of α and *n* (for graphs of other values see Appendix B). In the graphs and Table 4.1, we can see that the maximum take place when $\rho_{xy} = 0$ for all values of ϕ and α.

Table 4.1 Maximum and minimum relative efficiency for different values of α and n on the interval $0.25 \leq \phi \leq 3$

		Minimum			Maximum			Interval for ϕ where RE >1	
n	α	ϕ	ρ_{xy}	$Min\ RE$	ϕ	ρ_{xy}	$Max\ RE$	"A"	"B"
6	0.01	3	0	0.182586	0.75	0	2.216875	0.25	1.45
6	0.05	3	0	0.237209	0.8	0	1.944299	0.3	1.45
6	0.1	3	0	0.298054	1.0001	0	1.731745	0.35	1.45
6	0.2	3	0	0.432787	1.0001	0	1.567841	0.4	1.5
6	0.3	3	0	0.580347	1.0001	0	1.476133	0.4	1.6
6	0.4	3	0	0.730338	1.0001	0	1.428314	0.35	1.75
10	0.01	3	0	0.163757	0.85	0	2.067694	0.4	1.4
10	0.05	3	0	0.245429	0.85	0	1.766577	0.45	1.4
10	0.1	3	0	0.335943	1.0001	0	1.617651	0.5	1.4
10	0.2	3	0	0.50864	1.0001	0	1.455161	0.5	1.45
10	0.3	3	0	0.666917	1.0001	0	1.371893	0.5	1.5
10	0.4	3	0	0.801385	1.02	0	1.329681	0.45	1.65
15	0.01	3	0	0.154986	0.9	0	1.991582	0.5	1.35
15	0.05	3	0	0.270731	1.0001	0	1.708694	0.55	1.35
15	0.1	3	0	0.389173	1.0001	0	1.558836	0.6	1.35
15	0.2	2.7	0	0.582202	1.0001	0	1.396958	0.6	1.35
15	0.3	2.55	0	0.730312	1.02	0	1.316258	0.55	1.45
15	0.4	2.55	0	0.845438	1.05	0	1.282605	0.55	1.6
20	0.01	3	0	0.160292	0.9	0	1.949785	0.55	1.3
20	0.05	2.85	0	0.31017	1.0001	0	1.681045	0.6	1.3
20	0.1	2.55	0	0.438315	1.0001	0	1.527942	0.65	1.3
20	0.2	2.35	0	0.624007	1.0001	0	1.365753	0.65	1.35
20	0.3	2.25	0	0.761452	1.03	0	1.289551	0.25	1.4
20	0.4	2.25	0	0.865911	1.05	0	1.259216	0.25	1.5
30	0.01	2.75	0	0.195337	0.95	0	1.916409	0.25	1.25
30	0.05	2.3	0	0.365674	1.0001	0	1.651568	0.25	1.25
30	0.1	2.15	0	0.492274	1.0001	0	1.494913	0.25	1.25
30	0.2	2	0	0.667448	1.01	0	1.333481	0.25	1.25
30	0.3	1.95	0	0.792776	1.04	0	1.263734	0.25	1.35
30	0.4	1.95	0	0.885851	1.1	0	1.236624	0.25	1.4

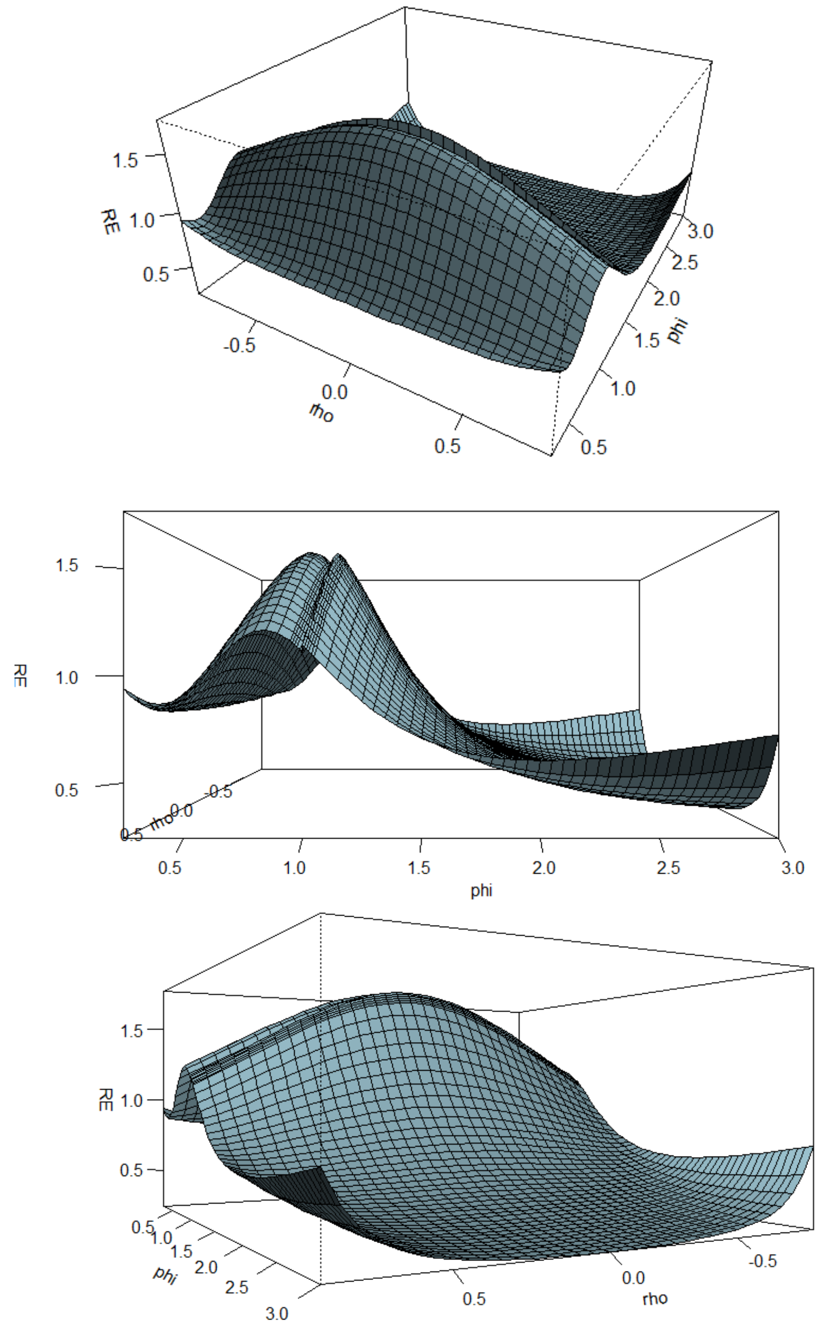


Figure 4.2 Graphs for RE as a function of ϕ (phi) and ρ_{xy} (rho) for

$\alpha = 0.05$ and n=*10*

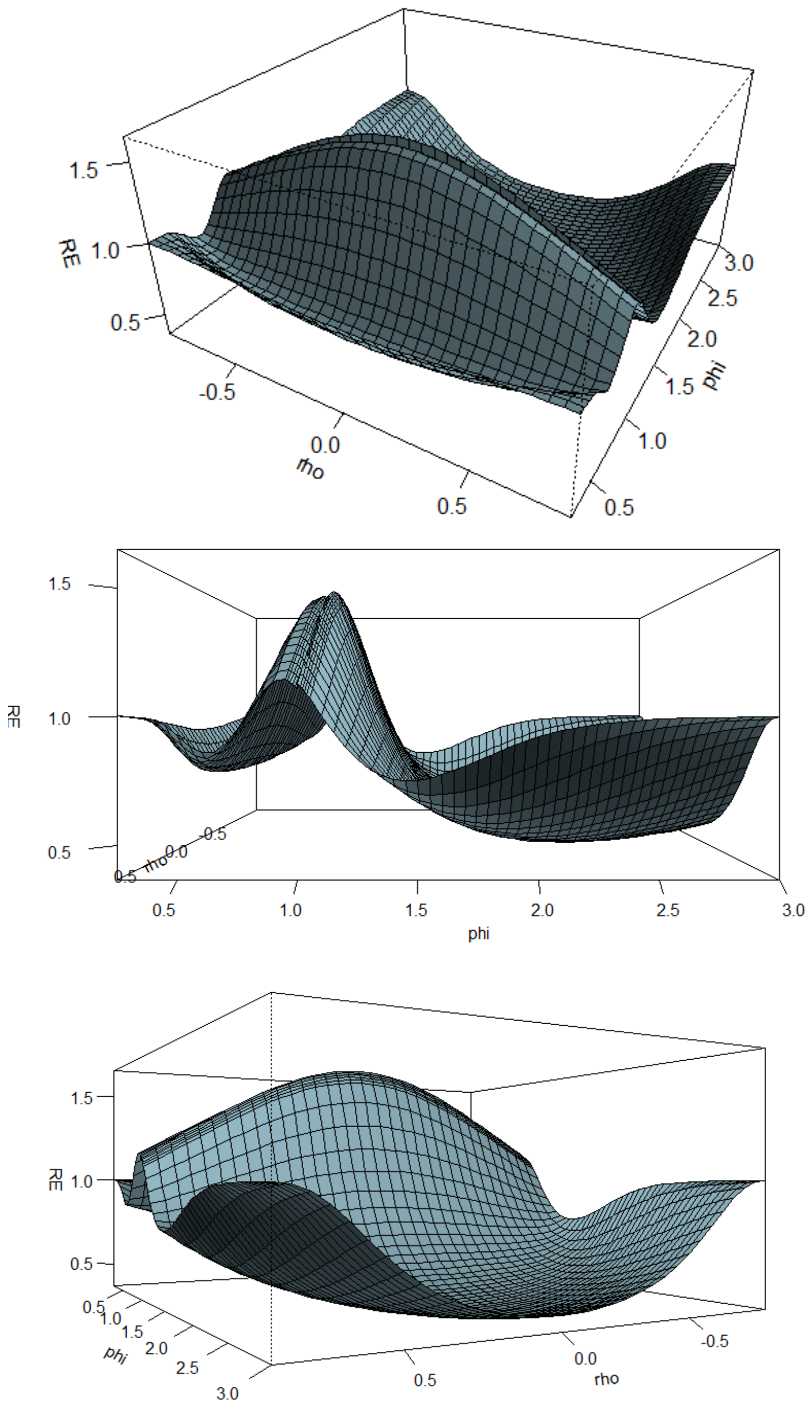


Figure 4.3 Graphs for RE as a function of ϕ (phi) and ρ_{xy} (rho) for $\alpha = 0.05$ and

n=30

Chapter 5

Conclusions

By making the necessary adjustment of a theorem derived by Joarder (2006), in Chapter 2, we obtained a general result to get any product moments of the variances and correlation coefficient in a bivariate normal sample. The final expression is given in terms of the hypergeometric function which is a well known function and there exists computational routines to be evaluated.

In Chapter 3, we proposed two estimations for the variance after preliminary test of homogeneity of variance; one for two-tailed preliminary test and another one for one-tailed preliminary test. The mean, variance and MSE of the estimators were derived. Partial checks for the cases of never and always pool estimators are made.

Finally in Chapter 4, we have discussed the relative efficiency for estimator of variance after preliminary test, we got and expression for RE in terms of the ratio of two variances (ϕ) and the correlation (ρ_{xy}). Computations of RE for different parameters were given and 3D graphs were analyzed. It was found that for a subset of parameter space ϕ the value of RE is greater than one, for each value of ρ_{xy}. The maximum RE is obtained that occurred when $\rho_{xy} = 0$ and ϕ is around 1. We found that when n and α increase the maximum RE decrease. As a recommendation one may select the α level for the preliminary test by using the max-min criterion, ie, by specifying a minimum RE and select the α level to achieve the maximum RE. The behavior of RE of pooling two variances is similar when the samples are correlated or not, as mentioned by Bancroft and Han (1983), if the significance level of the preliminary test is carefully selected, the preliminary test estimator can be used when the experimenter is uncertain whether the variances are equal and the sample size is not big.

Appendix A

R Program Code

```
############### Program #####################################
############ Product Moments of Wishart Distribution ############
library(hypergeo)
# Some variable definitions
n<-10
a<-1
b<-1/2
c<-2
var1<-16
var2<-1
cov12<-1
sd1<-sqrt(var1)
sd2<-sqrt(var2)
r<-cov12/(sd1*sd2)
r2<-r^2
U1<-(n/2)+a
U2<-(n/2)+b
U3<-(c+1)/2
L1<-(n+c)/2
L2<-1/2
F<-genhypergeo(U=c(U1,U2,U3), L=c(L1,L2),z=r2)
A<-(2^(a+b))*(1-r2)^((2+n)/2)/(pi^(1/2)*gamma(n/2)*n^(a+b))
G<-
((sd1^(2*a))*(sd1^(2*b))*gamma(U1)*gamma(U2)*gamma(U3))/gamma(L1)
Eabc<-A*G*F
Eabc
```

```
############### Program ##########################################
### Function to get the expectation and MSE, Eq 3.17 and Eq. 3.20 #########
 EqMSE<-function(n, alpha, varx, vary, covxy){
        varu<-varx+vary+(2*covxy)
        varv<-varx+vary-(2*covxy)
        covuv<-varx-vary
        sdu<-varu^(1/2)
        sdv<-varv^(1/2)
        ruv<-covuv/(sdu*sdv)
        ruv2<-ruv^2
        #Expectation of the variance
        lambda<-qbeta(1-(alpha),0.5,(0.5*n)-1)
        A1<-((1-ruv2)^((2+n)/2))/(pi^(1/2)*gamma(n/2)*n)
        cA1<-(2*(1-ruv2))/n
        sv1<-0
        sv2<-0
        sv3<-0
        for (k in 0:150) {
                B0<-(n-1)/2
                B1<-(k+1)/2
                B2<-(k+2)/2
                B3<-(k+3)/2
                v1<-
((2*ruv)^k*gamma((k+n+1)/2)*gamma((k+2)/2)/factorial(k))*(1-
pbeta(lambda,B2,B0))*((k+n+1)/2)*A1*cA1*(((-1)^k)+1)
                v2<-
((2*ruv)^k*gamma((k+n+2)/2)*gamma((k+1)/2)/factorial(k))*(1-
pbeta(lambda,B3,B0))*((k+1)/2)*A1*cA1*(((-1)^k)+1)
```

```
                v3<-
((2*ruv)^k*gamma((k+n+1)/2)*gamma((k+2)/2)/factorial(k))*(1-
pbeta(lambda,B2,B0))*A1*(((-1)^k)+1)
                sv1<-sv1+v1
                sv2<-sv2+v2
                sv3<-sv3+v3
         }
        var0<-(sdu^4+sdv^4)*(n+2)/(16*n)+(sdu^2*sdv^2)*(2*ruv2+n)/(8*n)
        var1<-((sdu^3*sdv+sdu*sdv^3)/4)*sv1
        var2<-((sdu^2*sdv^2)/4)*sv2
        E<-((sdu^2+sdv^2)/4)+(1/2)*sdu*sdv*sv3
        Var<-var0+var1+var2-E^2
        MSE<-Var+(E-varx)^2
        RE<-(2*varx^2)/(n*MSE)
        return(c(varx,vary,covxy,n,alpha,E,Var,MSE,RE))
 }
### Computations of Mean and MSE for different values of n and alpha ###
 varx<-5.03
 vary<-5
 covxy<-1
 vn<-c(10,20,30,50,100)                    ##Different values of n
 valpha<-c(0.05,0.1,0.2,0.3,0.5)           ##Different values of alpha
 MSE<-matrix(1,nrow=1,ncol=9)              ##Matrix to add the computations
 for (j in 1:5) {                          ##Number of different values of n
  n<-vn[j]
        for (k in 1:5) {
         alpha<-valpha[k]
         MSE1<-EqMSE(n, alpha, varx, vary, covxy)
```

```
        MSE<-rbind(MSE,MSE1)              ##Add a matrix to a previous matrix,
to keep the information
        }
 }
 MSEnames<-c("varx","vary","covxy","n","alpha","E","Var","MSE","RE")
 MSEnames
 MSE

############### Program ####################################
##########   Bancroft Equations for Mean and MSE ##############
#Some variable definitions
n<-50
varx<-5.03
vary<-5
covxy<-0
ratio<-vary/varx
alpha<-0.05
lambda<-qf(1-alpha,n,n)
x0<-lambda/((1/ratio)+lambda)
B0<-n/2
B2<-(n+2)/2
B4<-(n+4)/2
Ex05<-varx*(1+(1/2)*(pbeta(x0,B0,B2)*ratio-pbeta(x0,B2,B0)))
v105<-(1/4+1/(2*n))*(pbeta(x0,B4,B0)*varx^2+pbeta(x0,B0,B4)*vary^2)
v205<-(1/2)*(pbeta(x0,B2,B2)*varx*vary)+((n+2)/n)*(1-
pbeta(x0,B4,B0))*varx^2
v305<-varx^2*(1+(1/2)*pbeta(x0,B0,B2)*ratio-(1/2)*pbeta(x0,B2,B0))^2
v05<-v105+v205-v305
```

```
MSE05<-v05+(Ex05-varx)^2
x0
Ex05
MSE05

######## Program ###################################
###### Function to get the Mean and MSE, One Tail Test ####
 EqMSE<-function(n, alpha, varx, vary, covxy){
	varu<-varx+vary+(2*covxy)
	varv<-varx+vary-(2*covxy)
	covuv<-varx-vary
	sdu<-varu^(1/2)
	sdv<-varv^(1/2)
	ruv<-covuv/(sdu*sdv)
	ruv2<-ruv^2
	#Expectation of the variance
	lambda<-qbeta(1-(2*alpha),0.5,(0.5*n)-1)
	A1<-((1-ruv2)^((2+n)/2))/(pi^(1/2)*gamma(n/2)*n)
	cA1<-(2*(1-ruv2))/n
	sv1<-0
	sv2<-0
	sv3<-0
	for (k in 0:150) {
		B0<-(n-1)/2
		B1<-(k+1)/2
		B2<-(k+2)/2
		B3<-(k+3)/2
```

```
                    v1<-
((2*ruv)^k*gamma((k+n+1)/2)*gamma((k+2)/2)/factorial(k))*(1-
pbeta(lambda,B2,B0))*((k+n+1)/2)*A1*cA1
                    v2<-
((2*ruv)^k*gamma((k+n+2)/2)*gamma((k+1)/2)/factorial(k))*(1-
pbeta(lambda,B3,B0))*((k+1)/2)*A1*cA1
                    v3<-
((2*ruv)^k*gamma((k+n+1)/2)*gamma((k+2)/2)/factorial(k))*(1-
pbeta(lambda,B2,B0))*A1
                    sv1<-sv1+v1
                    sv2<-sv2+v2
                    sv3<-sv3+v3
            }
          var0<-(sdu^4+sdv^4)*(n+2)/(16*n)+(sdu^2*sdv^2)*(2*ruv2+n)/(8*n)
          var1<-((sdu^3*sdv+sdu*sdv^3)/4)*sv1
          var2<-((sdu^2*sdv^2)/4)*sv2
          E<-((sdu^2+sdv^2)/4)+(1/2)*sdu*sdv*sv3
          Var<-var0+var1+var2-E^2
          MSE<-Var+(E-varx)^2
          RE<-(2*varx^2)/(n*MSE)
          return(c(varx,vary,covxy,n,alpha,E,Var,MSE,RE))
 }

###### Computations of Mean and MSE, One tail test, for different values of n
and alpha #################
 varx<-4.2
 vary<-4
 covxy<-0
```

```
 vn<-c(10,20,50)                          ##Diferent values of n
 valpha<-c(0.05,0.1,0.2)                  ##Diferent values of alpha
 MSE<-matrix(1,nrow=1,ncol=9)             ##Matrix to add the computations
 for (j in 1:3) {                         ##Number of diferent values of n
  n<-vn[j]
          for (k in 1:3) {
           alpha<-valpha[k]
           MSEb<-EqMSE(n, alpha, varx, vary, covxy)
           MSE<-rbind(MSE,MSEb)           ##Add a matrix to a previous matrix,
to keep the information
          }
 }
 MSEnames<-c("varx","vary","covxy","n","alpha","E","Var","MSE","RE")
 MSEnames
 MSE

############### Program ####################################
############### Simulation  ##################################
library(mvtnorm)
## Simulation function
simulation<-function(n, alpha, sigx, sigy, covxy){
          lambda<-qbeta(1-(alpha),0.5,0.5*n-1)
          ns<-10000
          mu<-c(0,0)                      #mean vector
          sig<-matrix(c(sigx,covxy,covxy,sigy),ncol=2)          #variance matrix
          vec<-seq(1,ns,by=1)
          vect<-seq(1,ns,by=1)
          vecSx<-seq(1,ns,by=1)
```

```
for (k in 1:ns) {
        data<-rmvnorm(n,mean=mu,sigma=sig)        #generate a sample
        x<-data[,1]
        y<-data[,2]
        u<-x+y
        v<-x-y
        Sx<-var(x)
        vecSx[k]<-Sx
        Sxy<-(var(x)+var(y))/2
        r2<-cov(u,v)^2/(var(u)*var(v))
        if (r2<lambda) vx<-Sxy else vx<-Sx
        vec[k]<-vx
        ### Using t distribution
        t<-((r2*(n-2))/(1-r2))^1/2
        lambdat<-qt(1-alpha/2,n-2)
        if (t<lambdat) tvx<-Sxy else tvx<-Sx
        vect[k]<-tvx
 }
Exy<-mean(vec)
Varxy<-var(vec)
MSExy<-Varxy+(Exy-sigx)^2
Ex<-mean(vecSx)
Varx<-var(vecSx)
MSEx<-Varx+(Ex-sigx)^2
RE<-MSEx/MSExy
Etxy<-mean(vect)
Vartxy<-var(vect)
MSEtxy<-Vartxy+(Etxy-sigx)^2
```

```
        return(c(sigx,sigy,covxy,n,alpha,Exy,MSExy,MSEx,RE,MSEtxy))
}

# Some variable definitions
# sigx<-5; # sigy<-5; # covxy<-2.5; # alpha<-0.3; # n<-20
# simula1<-simulation(n, alpha, sigx, sigy, covxy)
# simulanames<-
c("sigx","sigy","covxy","n","alpha","Exy","MSExy","MSEx","RE","MSEtxy")
# simulanames
# simula1

 ###### Simulations for different values of n and alpha ##################
 sigx<-5.03
 sigy<-5
 covxy<-1
 vn<-c(10,20,30,50,100)                    ##Different values of n
 valpha<-c(0.05,0.1,0.2,0.3,0.5)           ##Different values of alpha
 Simula<-matrix(1,nrow=1,ncol=10)          ##Matrix to add the computations
 for (j in 1:5) {                          ##Number of different values of n
  n<-vn[j]
        for (k in 1:5) {
         alpha<-valpha[k]
         Simula1<-simulation(n, alpha, sigx, sigy, covxy)
         Simula<-rbind(Simula,Simula1)            ##Add a matrix to a previous
matrix, to keep the information
        }
 }
```

```
simulanames<-
c("sigx","sigy","covxy","n","alpha","Exy","MSExy","MSEx","RE","MSEtxy")
simulanames
Simula

############### Program #####################################
########## Relative Efficiency Function ########################
re<- function(n, alpha, a, rxy) {
        rxy2<-rxy^2
        ruv2<-(1+a^2-2*a)/(1+a^2+2*a-4*a*rxy2)
        ruv<-sqrt(ruv2)
        if (a>1) {
                ruv<--ruv
                d<--1}
        else d<-1
        lambda<-qbeta(1-(alpha),0.5,(0.5*n)-1)
        A1<-((1-ruv2)^((2+n)/2))/(pi^(1/2)*gamma(n/2)*n)
        cA1<-(2*(1-ruv2))/n
        sv1<-0
        sv2<-0
        sv3<-0
        for (k in 0:150) {
                B0<-(n-1)/2
                B1<-(k+1)/2
                B2<-(k+2)/2
                B3<-(k+3)/2
```

```
                v1<-
((2*ruv)^k*gamma((k+n+2)/2)*gamma((k+1)/2)/factorial(k))*(1-
pbeta(lambda,B3,B0))*((k+1)/2)*A1*cA1*(((-1)^k)+1)
                v2<-
((2*ruv)^k*gamma((k+n+1)/2)*gamma((k+2)/2)/factorial(k))*(1-
pbeta(lambda,B2,B0))*((k+n+1)/2)*A1*cA1*(((-1)^k)+1)
                v3<-
((2*ruv)^k*gamma((k+n+1)/2)*gamma((k+2)/2)/factorial(k))*(1-
pbeta(lambda,B2,B0))*A1*(((-1)^k)+1)
                sv1<-sv1+v1
                sv2<-sv2+v2
                sv3<-sv3+v3
          }
 var0<-((n+2)/16)*(1+a^2+2*a+4*rxy2*a)+((n+2*ruv2)/16)*(1+a^2+2*a-
4*rxy2*a)-(n/2)*a
 var1<-(n/8)*(1+a^2+2*a-4*rxy2*a)*sv1
 var2<-(n/4)*d*(1+a)*(1+a^2+2*a-4*rxy2*a)^(1/2)*sv2
 var3<-(n/2)*d*((1+a^2+2*a-4*rxy2*a)^(1/2))*sv3
 RE1<-var0+var1+var2-var3
 RE<-1/RE1
 return(c(n,alpha,a,rxy,RE))
 }
 #########Some variable definitions
# n<-10
# a<-2
# rxy<-0.5
# alpha<-0.3
# re1<-re(n, alpha, a, rxy)                    ## Run the function to get the RE
```

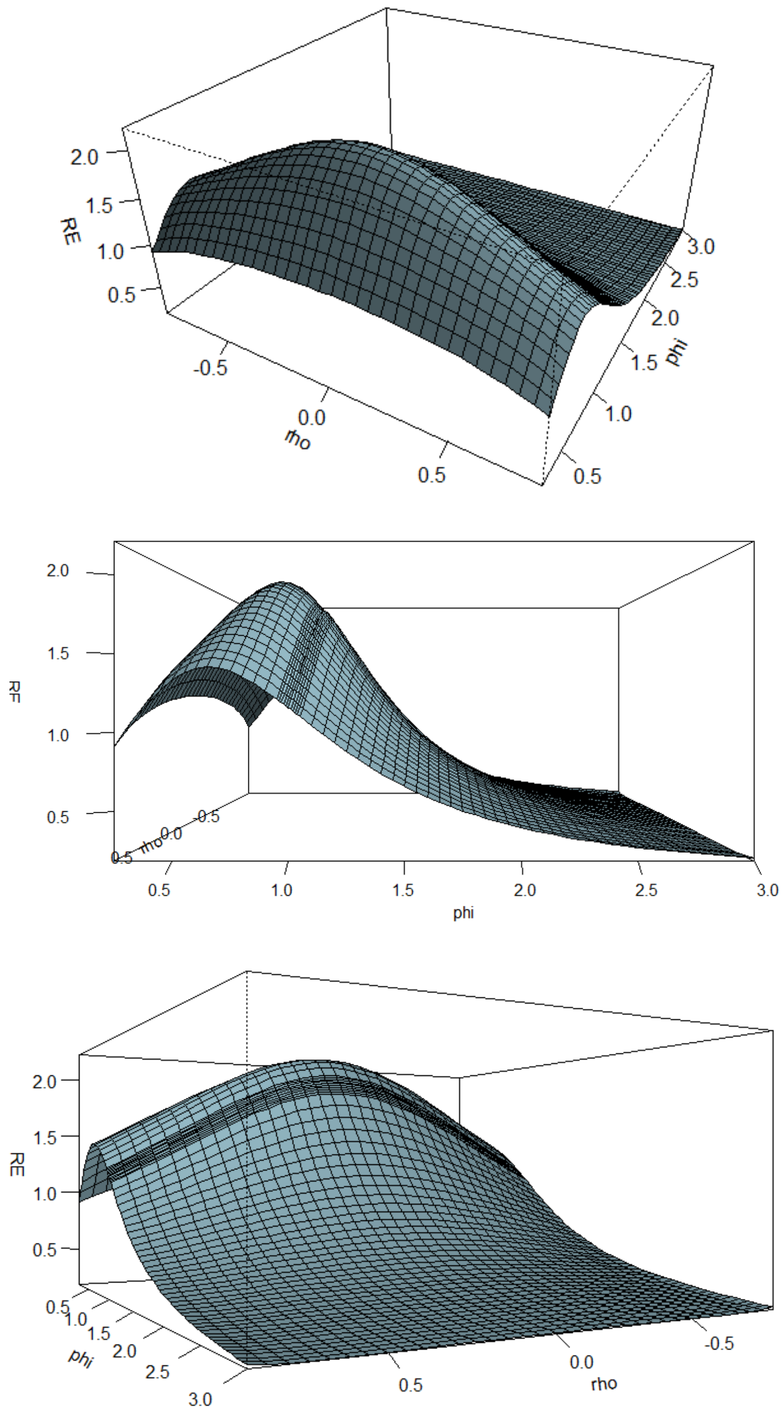
```
# re1names<-c("n","alpha","phi","rho","RE")
# re1

############### Program ####################################
########### Computations of Efficiency #######################
 n<-20
 alpha<-0.5
 vphi<-c(seq(0.1,0.90,by=0.05),seq(0.95,1.05,by=0.01),seq(1.1,2.95,by=0.05))
 vrho<-c(seq(-0.9,0.9,by=0.05))
 RE<-matrix(1,nrow=1,ncol=5)
 for (j in 1:66) {
  a<-vphi[j]
           for (k in 1:37) {
            r<-vrho[k]
            re1<-re(n, alpha, a, r)
            RE<-rbind(RE,re1)              ##Add a matrix to a previous matrix, to keep
the information
           }
 }
 a<-66*37+1                          #to drop the first raw
 RE1<-RE[2:a,]
 vphi
 vrho
 RE1
 ###### Graphs for the Efficiency
 require(grDevices) # for trans3d
 RE2<-RE1[,5]
 RE3<-matrix(RE2, 37, 66)
```

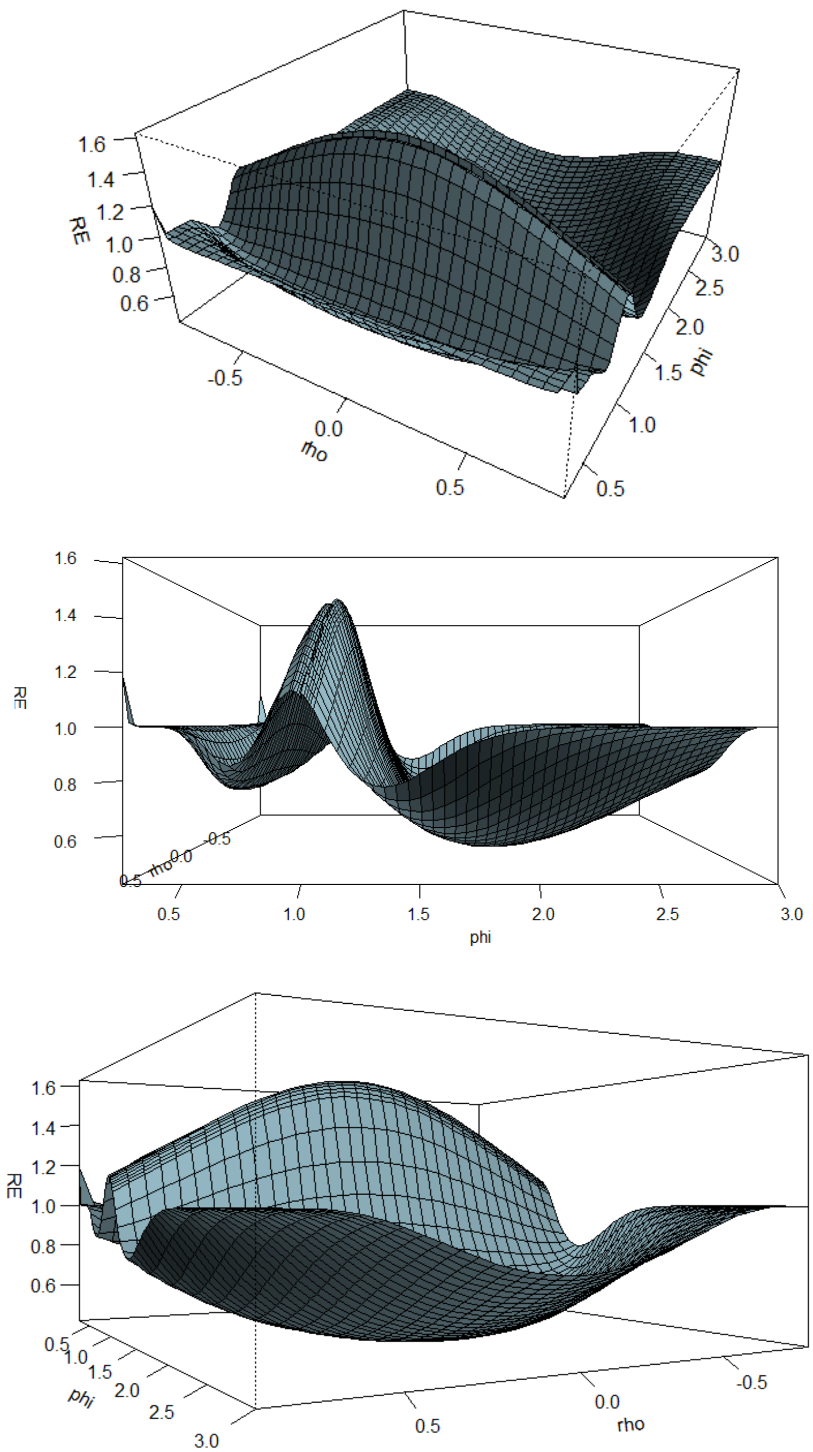
```
op <- par(bg = "white")
persp(vrho, vphi, RE3, theta = 30, phi = 30, col = "lightblue", expand = 0.5,
    ltheta = 120, shade = 0.75, ticktype = "detailed",
    xlab = "rho", ylab = "phi", zlab = "RE" )
###### see the graph for a different perspective
persp(vrho, vphi, RE3, theta = 90, phi = 0, col = "lightblue", expand = 0.5,
    ltheta = 120, shade = 0.75, ticktype = "detailed",
    xlab = "rho", ylab = "phi", zlab = "RE" )
#see the graph for a different perspective
persp(vrho, vphi, RE3, theta = 150, phi = 0, col = "lightblue", expand = 0.5,
    ltheta = 120, shade = 0.75, ticktype = "detailed",
    xlab = "rho", ylab = "phi", zlab = "RE" )
```

Appendix B

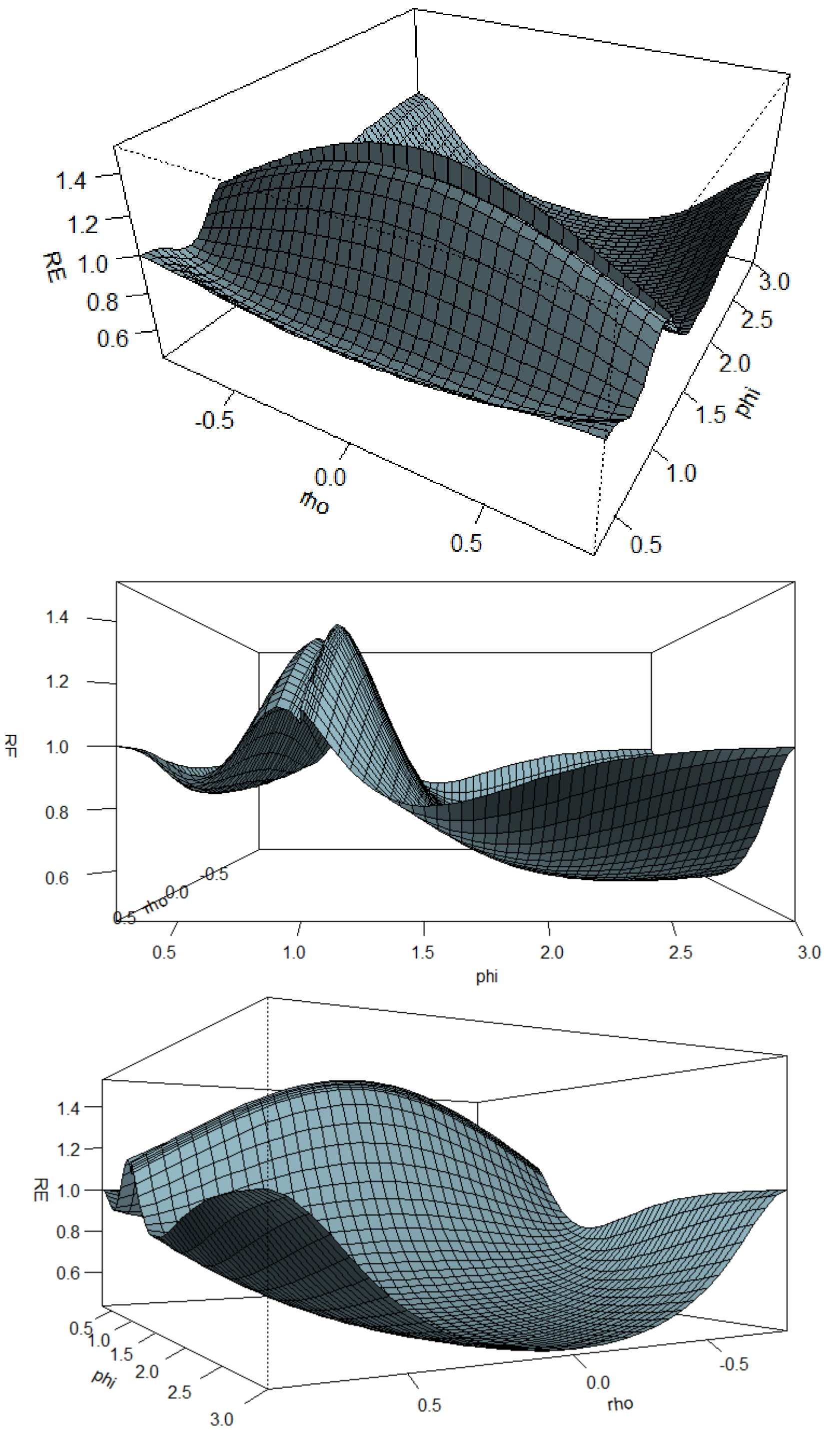
Relative Efficiency Graphs



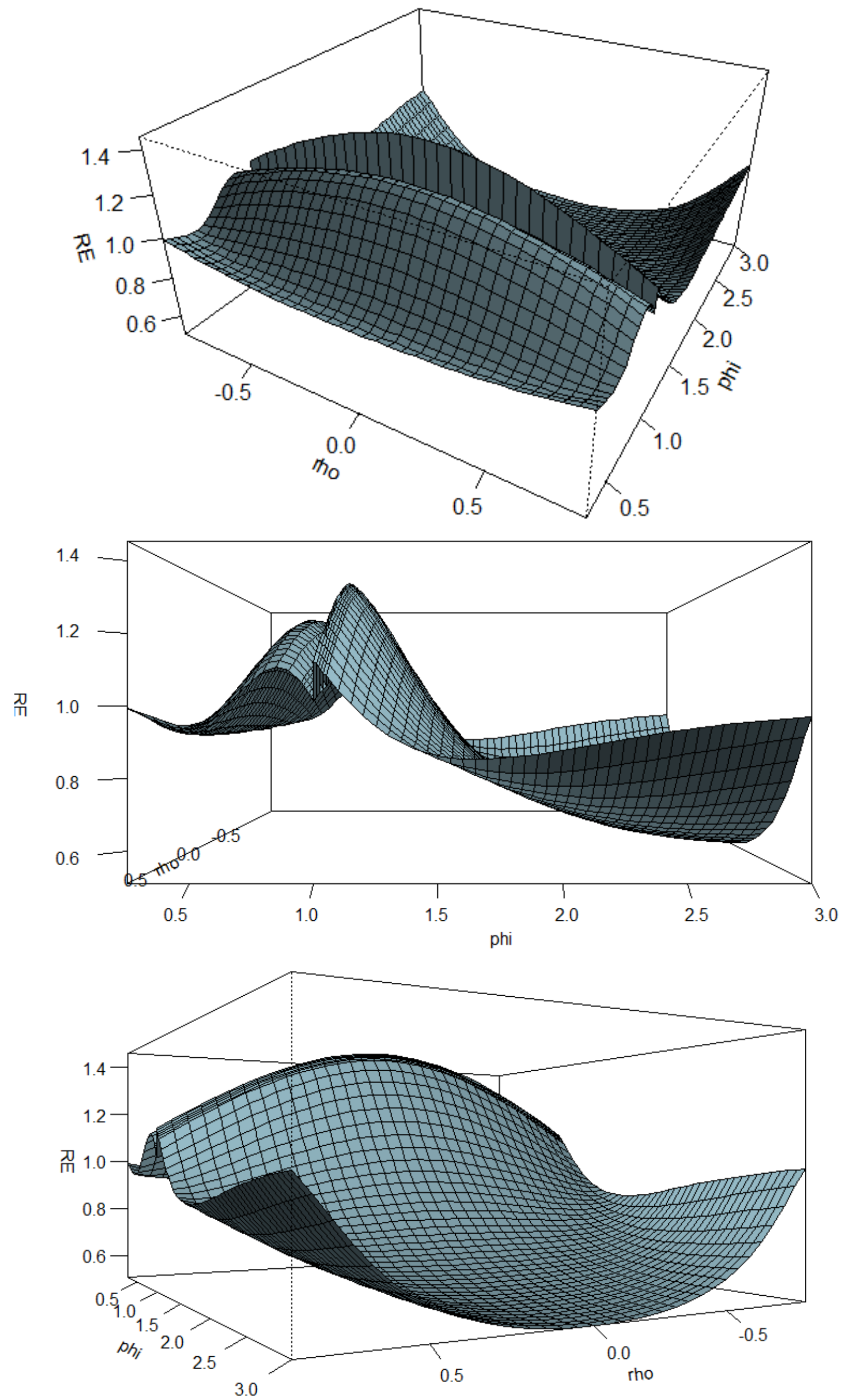
$\alpha = 0.01$ and *n=6*



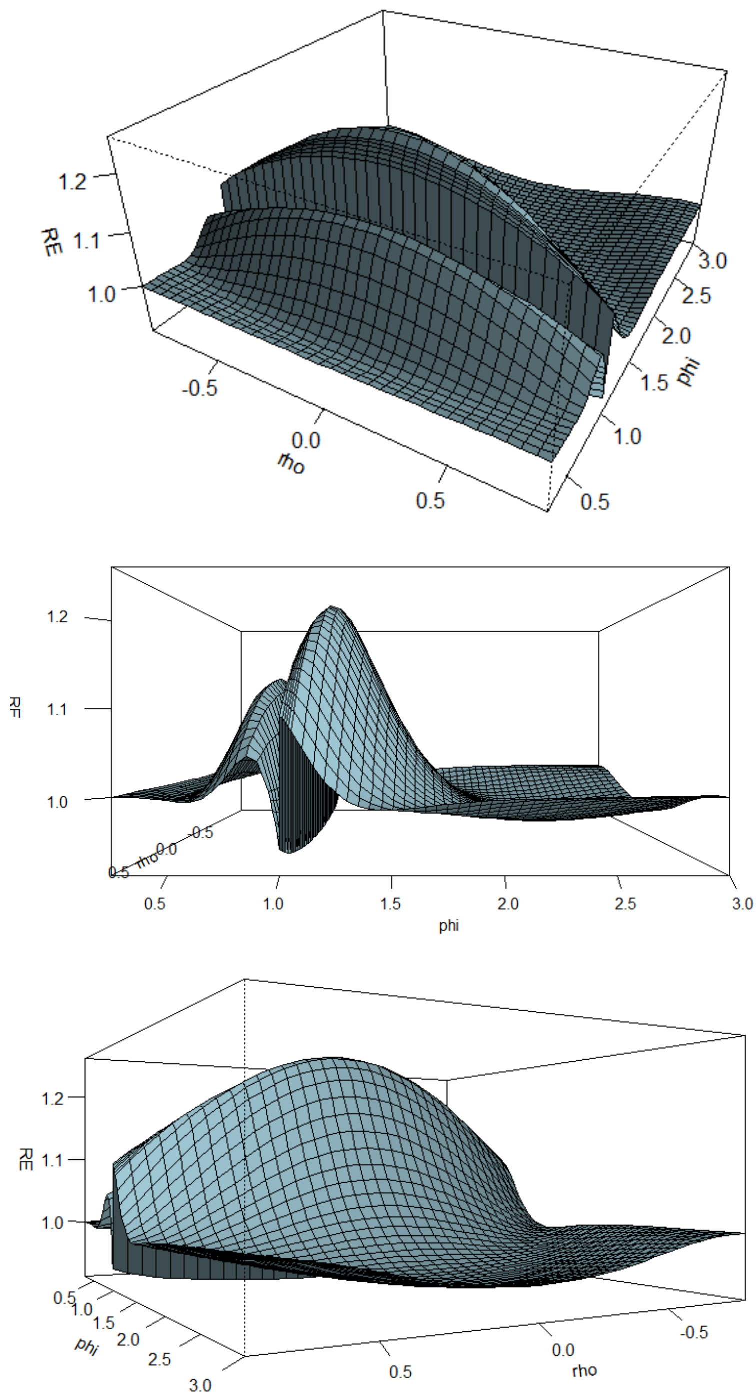
$\alpha = 0.05$ and *n=50*



$\alpha = 0.1$ and *n=20*



$\alpha = 0.2$ and *n=10*



$\alpha = 0.5$ and *n=20*

References

Akcakaya, H. R. (2002). Estimating the variance of survival rates and fecundities. *Animal conservation*, 5, 333-336.

Anderson, T.W. (2003). An introduction to multivariate statistical analysis. Wiley, London.

Bailey, W.N. (1964). Generalized hypergeometric series. Stechert-Hafner Service Agency. New York and London 1964. Chapter 2, 8-12.

Bancroft, T. A. (1944). On biases in estimation due the use of preliminary test of significance, *Annals of Mathematical Statistics*, 15, 190-204.

Bancroft, T. A. and Han, C.P. (1983). A note on pooling variances. *Journal of the American Statistical Association*, 78, 981-983.

Bennet, B. M. (1952). Estimations of means on the basis of preliminary test of significance. *Annals Institute of Statistical mathematics*. 4, 31-43.

Chiou, P and Han C.P. (1994). Conditional interval estimation of the exponential scale parameter following rejection of a preliminary test. *Journal of Statistical Research*, 28, 65-72.

Chiou, P and Han C.P. (1995). Conditional interval estimation of the exponential location parameter following rejection of a pre-test. *Communications in statistics: Theory and Methods*, 24, 1481-1492.

Chiou, P and Han C.P. (1998). Conditional interval estimation of the ratio of variance components following a rejection of a pre-test. *J. Statist. Comput.*

Simul. 63, 105-119.

Cohen, A. (1986). Comparing variances of correlated variables. *Psychometrika.* 51(3), 379-391.

Engle, Robert F. (1983). Estimates of variance of U. S. inflation based upon the ARCH model. *Journal of Money, Credit and Banking,* 15(3), 286-301.

Falconer, D. S and Mackey T. F. C. (1996).Introduction to quantitative genetic Ed 4, Longmans Green, Harlow, Essex, UK.

Han, C.P. (1968). Testing the homogeneity of a set of correlated variances. *Biometrika*. 55(2), 317-326.

Han, C.P. (1978). Nonnegative and Preliminary Test Estimators of Variance Components. *Journal of the American Statistical Association*. 73, 855-858.

Han, C. P. and Bancroft, T. A. (1968). On pooling means when variance is unknown. *Journal of the American Statistical Association*, 63, 1333-1342.

Harris, P. (1985). Testing the homogeneity of correlated variances. *Biometrika*, 72(1), 103-107.

Henderson, C. R. (1953). Estimation of Variance and Covariance Components, *Biometrics*, 9(2), 226-252.

Joarder, A.H. (2006). Product moments of a bivariate Wishart distribution. *Journal of Probability and Statistical Science*, 4(2), 2006, 233-244.

Joarder A.H. (2009) Moments of the product and ratio of two correlated chi-square variables. *Stat Papers*, 50, 2009, 581-592.

Joarder, A.H. (2013). Exact distribution of the sum of two correlated chi-square variables and its applications. *Kuwait Journal of Science*, 40(2), 61-81.

Johnson, J. P., Bancroft, T. A. and Han C. P. (1977). A pooling Methodology for regression in Prediction. *Biometrics*, 33, 57-67.

Mahdi, S (2000). Conditional estimation in exponential models. *Matematicki Vesnik*, 52, 27-45.

Mahdi, S (2003). Optimal conditional confidence interval for the shape parameter of a Weibull distribution. *Brazilian Journal of Probability and Statistics*, 17, 57-74.

Mahdi, S (2004). Conditional confidence interval for the scale parameter of a Weibull distribution. *Serdica Math. J.* 30, 55-70.

Meeks, S. L. and D'Angostino, R. B. (1983). A note on the use of confidence limits following rejection of a null hypothesis. *The American Statistician*, 37, 134-136.

Morgan, W. A. (1939). A test for the significance of the difference between two variances in a sample from a normal bivariate population. *Biometrika*, 31, 13-9.

Mosteller, F. (1948). On pooling data. *Journal of the American Statistical Association*. 43, 231-242.

Pitman, E. J. G. (1939). A note on normal correlation. *Biometrika* 31, 9-12.

Rash, D. and Masata, O. (2006). Methods of variance component estimation. *Czech J. Anim. Sci*.,51(6), 227-235.

Saleh, A. K. (2006) Theory of preliminary test and Stein-type estimation with applications, John Wiley & Sons.

Sisodia, B. V. S. and Rai, S. N. (1991). On estimation procedures of variance subsequent to preliminary test of significance. *Microelectron Rehab.* 31(6), 1101-1104.

Srivastava, S.R. (1966). Estimation of variance using a preliminary test of significance. *Trabajos de Estadistica*, 17(2), 101-108.

Toyota, T. and Wallace, T.D. (1975). Estimation of variance after a preliminary test of homogeneity and optimal levels of significance for the pre-test. *Journal of Econometrics*, 3, 395-404

Biographical Information

Juan Manuel Romero Padilla received his bachelor degree in statistics from Chapingo Autonomous University, Mexico and M. Sc. in statistics from Postgraduate Collage, Mexico. He started his Ph.D. under the supervision of Dr. Chien-Pai Han at the University of Texas at Arlington in 2010. After earned his M. Sc. he was employed as chief of quality and statistics by Kantar World Panel Mexico for five years. His current research interest is in mathematical statistics, sample survey and statistical genetics