**DISCRIMINATORY
VON NEUMANN-MORGENSTERN SOLUTIONS**

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Discriminatory von Neumann-Morgenstern Solutions.

J.G.C. Heijmans

The von Neumann-Morgenstern solution (vN-M solution) or stable set is arguably the most dynamic and flexible solution concept for cooperative games with side-payments. Perhaps the most striking phenomenon is that vN-M solutions often suggest intricate coalition formation processes and corresponding payoffs. Why this occurs is not well understood. On the other hand, vN-M solutions are difficult to find. This paper deals with the class of discriminatory vN-M solutions and presents results that give insights in the corresponding coalition formation process. A computationally effective procedure is presented to answer the decision problem whether or not a proposed set of imputations to a given game is a discriminatory vN-M solution.

A discriminatory set is a collection of imputations representing the scenario where certain players (the discriminated players) each receive a fixed amount, and the group of remaining players (the bargainers) can split the rest in any way they like. These discriminatory sets appear frequently as von Neumann-Morgenstern solutions or as building blocks of vN-M solutions. The best known example occurs for the monotone simple games: for every minimal winning coalition there is a corresponding discriminatory vN-M solution that assigns 0 to each player outside the minimal winning coalition.

This paper studies (for $(0,1)$ -normalised games) the discriminatory sets that are vN-M solutions. It is fairly easily seen that the set of bargainers for a discriminatory vN-M solution forms a minimal vital coalition (vital in the sense of Gillies [1959, p. 70]) and the total amount available for the bargainers is smaller than or equal to the worth of that minimal vital coalition. Minimal vital coalitions may be described as the minimal non-trivial coalitions with positive worth. An interesting new result described in this paper is that discriminatory vN-M solutions essentially appear only when the Core is empty. More precisely, if there exists a discriminatory vN-M solution that assigns a

positive amount to at least one discriminated player, then the Core of the game must be empty.

This paper gives effective characterisations of the discriminatory sets that are vN-M solutions. Besides the afore mentioned requirements regarding the set of bargaining players, the characterisations also involve a certain domination requirement on a finite set of competing discriminatory sets. Each of the competing discriminatory sets has the same set of discriminated players but now some of those players have lost their original allocations to the bargainers, so the competing discriminatory sets are more attractive to the bargainers than the original discriminatory set. The domination requirement on a competing discriminatory set is satisfied if and only if the Core of a certain reduced game (attractive for the set of bargaining players) is empty. The reduced game is similar to the well-known Davis-Maschler reduced game.

Connections with previous work on discriminatory vN-M solutions will be given at the end of section 1, and at other relevant places in this paper.

1. Terminology.

Let S be a nonempty subset of $\{1, 2, \dots, n\}$. Viewing a vector $x \in \mathbb{R}^n$ as a mapping from the set $\{1, 2, \dots, n\}$ into $\mathbb{R}$, let x^S denote the restriction of x to S . Let $\mathbb{R}^S$ denote the collection of functions $y: S \rightarrow \mathbb{R}$, and hence $x^S \in \mathbb{R}^S$. We write $x(S) = \sum_{i \in S} x_i$.

An n-person game (in characteristic function form) is a pair (n, v) where $n \geq 2$ is an integer and v is a function from $2^{\{1, 2, \dots, n\}}$ into $\mathbb{R}$ with $v(\emptyset) = 0$. The members of $N \equiv \{1, 2, \dots, n\}$ are the players and the nonempty subsets of N are the coalitions. The mapping v is the characteristic function and $v(S)$ is the worth of coalition S . (We do not assume superadditivity.) The set of imputations is

$$A \equiv \{x \in \mathbb{R}^n: x_i \geq v(\{i\}) \text{ for all } i \in N \text{ and } x(N) = v(N)\}.$$

As usual, we will only consider (0,1)-normalised games, i.e. $v(\{i\}) = 0$ for all $i \in N$ and

$v(N) = 1$. A $(0,1)$ -normalised game is symmetric if the worth $v(S)$ depends only on the size of S , and we write $v(s)$ instead of $v(S)$, where $s \equiv |S|$. A coalition S is effective for $x \in A$ if $x(S) \leq v(S)$. Let $x, y \in A$. We say that x dominates y via the coalition S ($x \text{ dom}_S y$) if S is effective for x and $x_i > y_i$ for all $i \in S$, and that x dominates y ($x \text{ dom } y$) if there exists a coalition S with $x \text{ dom}_S y$. Note that domination via N and via 1-player coalitions are empty relations. For $x \in A$ and $K \subseteq A$, we write $\text{dom}(x) \equiv \{y \in A: x \text{ dom } y\}$ and $\text{dom}(K) \equiv \bigcup_{x \in K} \text{dom}(x)$. A set $K \subseteq A$ is internally stable if the restriction of the relation dom to K is empty, and externally stable if every $y \in A - K$ is dominated by some $x \in K$. If K is both internally and externally stable then K is a von Neumann-Morgenstern solution (vN-M solution) or a stable set. The Core C is defined as

$$C = \{x \in A: x(S) \geq v(S) \text{ for all coalitions } S\},$$

and the set of all imputations that are undominated is the domcore. Clearly, $C \subseteq \text{domcore} \subseteq K$ for any vN-M solution K .

Let (n, v) be a $(0,1)$ -normalised game. Let D be a coalition different from N and suppose that $a \in \mathbb{R}^D$ satisfies

$$\begin{aligned} \text{(i)} \quad & a_i \geq 0 \text{ for all } i \in D, \text{ and} \\ \text{(ii)} \quad & \sum_{i \in D} a_i < 1. \end{aligned} \tag{1}$$

The set $K(D, a) \equiv \{x \in A: x_i = a_i \text{ for all } i \in D\}$ consisting of the imputations that allocate a to the members of D is called a discriminatory set. The players in D are said to be discriminated, the vector a is the allocation to the discriminated players, the players in $B \equiv N - D$ are the bargaining players and the quantity $b \equiv 1 - a(D) > 0$ is the bargaining amount. The center of $K(D, a)$ is denoted by $c(D, a)$, i.e. $c(D, a)_i = a_i$ if $i \in D$ and $c(D, a)_i = \frac{b}{|B|}$ if $i \in B$. For a given discriminatory set $K(D, a)$ it is useful to write $D^+ \equiv \{i \in D: a_i > 0\}$, $d = |D|$ and $d^+ = |D^+|$.

A vN-M solution that is also a discriminatory set is called a discriminatory von Neumann-Morgenstern solution. Note that the strict inequality in (1) prevents $K(D, a)$ from being a singleton, but this is not a restriction since vN-M solutions for $(0,1)$ -normalised games can not be singletons. We

now review a well-known example.

Example 1: A symmetric (0,1)-normalised 3-person game is completely determined by the worth $v(2)$ of the two-person coalitions, where w.l.o.g. $0 \leq v(2) \leq 1$. The imputation set is $A = \{x \in \mathbb{R}^3: x \geq 0, x_1 + x_2 + x_3 = 1\}$. We describe all discriminatory vN-M solutions $K(D, a)$.

There can only be one discriminated player; if there were two discriminated players, then $K(D, a)$ is a singleton, which is clearly insufficient for external stability. W.l.o.g. player 1 is discriminated. The discriminatory set $K(\{1\}, a_1)$ is a vN-M solution if and only if $1 - v(2) \leq a_1 < v(2) - \frac{1}{2}$ (see Figure 1). The condition $1 - v(2) \leq a_1$ is necessary otherwise $(1, 0, 0)$ remains undominated, and (if $a_1 \neq 0$) the condition $a_1 < v(2) - \frac{1}{2}$ is needed to dominate $(0, \frac{1}{2}, \frac{1}{2})$. It is easily seen that these conditions are also sufficient.

It follows that discriminatory vN-M solutions do exist if and only if $v(2) > \frac{3}{4}$. $\square$

Figure 1

The term discrimination is used in the literature in several different ways in connection with vN-M solutions. The most prevalent usage is in the sense of "subset of $K(D, a)$ ", i.e. vN-M solutions for which each imputation assigns a constant amount to certain players. This paper's notion of "discriminatory set" may be referred to as a purely discriminatory set. Weber's [1982, p. 53] terminology would say that $K(D, a)$ is discriminatory with respect to D and distributive with respect to B .

One traditional usage of the notion of discrimination is in attempts to inflate a vN-M solution of a game to a vN-M solution of an enlarged game by allocating a fixed amount to each of the new players (the vN-M solution of the enlarged game would be a subset of the discriminatory set $K(D, a)$, where D is the set of new players.) Gillies [1953] did so for the symmetric vN-M solutions of Bott's (n, k) -games [1953], as well as for games whose Core is a vN-M solution [1959, pg. 81]. Shapley [1967] did so by assigning zero to each of the new players. Owen [1966] obtained some general results in this direction.

Discriminatory vN-M solutions have been studied for particular classes of games. We mention here the 4-person constant-sum games (Hebert [1964] and Owen [1970]), simple games (Owen [1965] characterised all purely discriminatory vN-M solutions using the values of a large number of (incidence) matrix-games associated with minimal winning coalitions), games where only coalitions of a certain size are relevant for domination (Owen [1968], Weber [1973 a, b, c], Chang [1986], Heijmans [1986]), and absolutely stable games (Weber [1982]). Sokolina [1985 and 1986] has results on vN-M solutions whose imputations assign the same fixed amount to a particular coalition of two players and has referred to this situation as discrimination also.

2. Basic Results.

A coalition T is vital if there are imputations $x, y \in A$ with $x \text{ dom}_T y$ but x does not

dominate y via any proper subcoalition $S \subset T$ (Gillies [1959, p. 70]). It is clear that if $x \text{ dom}_U y$ for some coalition U , then there exists a vital coalition $T \subseteq U$ with $x \text{ dom}_T y$. For domination purposes one may therefore restrict attention to the vital coalitions. A minimal vital coalition is a vital coalition that has no proper vital subcoalitions. The following characterisation of minimal vital coalitions is well known.

Lemma 1: Let T be any coalition different from N . T is minimal vital if and only if $v(T) > 0$ and $v(U) \leq 0$ for all proper subcoalitions $U \subset T$.

Proof: For any coalition $S \neq N$ the binary relation dom_S is not empty if and only if $v(S) > 0$. This observation immediately proves "if". For the "only if" part it suffices to observe that T is vital implies $v(T) > 0$, and the minimality of T implies that dom_U is empty for every $U \subset T$. $\square$

For simple games it is well known (see e.g. Owen, [1982 p. 169]) that every minimal winning coalition S gives rise to a discriminatory vN -M solution $K(D, a)$, where $B = S$ and $a = 0$. In fact, Gillies [1959, p. 73] proved that if a $(0, 1)$ -normalised game has a minimal vital coalition S with $v(S) = 1$, then it has a discriminatory vN -M solution with $B = S$ and $a = 0$. We will see below using not much more than Gillies' arguments, that for any discriminatory vN -M solution $K(D, a)$ the set B of the bargaining players must be a minimal vital coalition with worth $v(B) \geq b$. For the purpose of interpretation, this indicates that B is not so much an unbreakable coalition that upon formation will then negotiate with the remaining players about how to divide $1 - v(B)$, but rather the result of a process that creates a coalition B as part of its outcome and assigns a share of B 's worth, namely $v(B) - b \geq 0$, to the discriminated players.

Lemma 2: $K(D, a)$ is internally stable if and only if $v(S) \leq 0$ for all proper subcoalitions S of B .

Proof: Domination inside $K(D, a)$ can only occur via proper subcoalitions of B . Clearly, if $v(S) \leq 0$ for all $S \subset B$, then the domination relation restricted to $K(D, a)$ is empty. On the other hand, suppose $S \subset B$ with $v(S) > 0$. Let $\delta \equiv \frac{1}{|S|} \min \{v(S), 1 - a(D)\} > 0$. Consider the imputations x and y in $K(D, a)$ defined by

$$x_i = \begin{cases} a_i & \text{if } i \in D \\ \delta & \text{if } i \in S \\ \frac{1 - a(D) - |S|\delta}{|B-S|} & \text{if } i \in B-S \end{cases} \quad y_i = \begin{cases} a_i & \text{if } i \in D \\ 0 & \text{if } i \in S \\ \frac{1 - a(D)}{|B-S|} & \text{if } i \in B-S. \end{cases}$$

Clearly $x \text{ dom}_S y$, and thus $K(D, a)$ is not internally stable. $\square$

Theorem 3: Let $K(D, a)$ be a discriminatory set for a $(0, 1)$ -normalised game (n, v) .

- (i) If $K(D, a)$ is a discriminatory vN-M solution, then B is minimal vital and $v(B) \geq b > 0$.
- (ii) If B is minimal vital and $v(B) \geq b > 0$, then $K(D, a)$ is internally stable and $\{y \in A - K(D, a) : y(B) < b\} \subseteq \text{dom}(K(D, a))$.

Proof: (i) By external stability, the imputation x defined by $x_i \equiv a_i + \frac{b}{|D|}$ if $i \in D$, $x_i \equiv 0$ if $i \in B$, is dominated by some $z \in K(D, a)$. It is clear that z must dominate x via some subcoalition of B . In light of Lemma 2 the domination must be via B itself, and thus $v(B) \geq z(B) = b > 0$. Using Lemma 2 and Lemma 1, we conclude that B is minimal vital.

(ii) From Lemma 1 and Lemma 2 we obtain that $K(D, a)$ is internally stable. Let $y \in A - K(D, a)$ with $y(B) < b$. Consider $x \in K(D, a)$ defined by $x_i = a_i$ if $i \in D$, and $x_i = y_i + \frac{b - y(B)}{|B|}$ if $i \in B$. Since $x_i > y_i$ for all $i \in B$ and $x(B) = b \leq v(B)$, it follows that $x \text{ dom}_B y$. $\square$

Gillies, Owen, Weber and Chang obtained (i) and the first conclusion of (ii) for special classes of games, and Owen [1966, p. 654] derived the second conclusion of (ii) in general.

Theorem 3 implies that in order to characterize all discriminatory vN-M solutions for $(0, 1)$ -

normalised games, it suffices to find necessary and sufficient conditions so that

$$L \equiv \{y \in A - K(D, a) : y(B) \geq b\} \subseteq \text{dom}(K(D, a)),$$

under the assumption that B is minimal vital and $0 < b \leq v(B)$. To begin, we observe that every $y \in A - K(D, a)$ with $y(D) \neq 1$ belongs to a discriminatory set with discriminated players D , namely $y \in K(D, y^D)$, where y^D is the restriction of y to D . We refer to each such $K(D, y^D)$ as a competing discriminatory set. In the following we will determine those competing discriminatory sets in L whose imputations are most difficult to dominate. The approach of searching for imputations that are hardest to dominate has been successfully employed for simple games by Owen [1965, p. 547], and Owen [1966, p. 655] also suggested it for arbitrary essential games. In our notation, the imputations in L that are most difficult to dominate by elements of $K(D, a)$ are, as suggested by Owen, those in

$$L_1 \equiv \{y \in A - K(D, a) : y_i \in \{0, a_i\} \text{ for all } i \in D, \text{ and } y_i = 0 < a_i \text{ for some } i \in D^+\}.$$

In the following lemma we show that one may indeed restrict attention to L_1 . To emphasize that L_1 is a finite union of competing discriminatory sets, we introduce the following notation. For any subset $U \subseteq D$, let $P_U: \mathbb{R}^D \mapsto \mathbb{R}^D$ denote the projection given by $P_U(\xi) = \sum_{i \in D - U} \xi_i e_i$, $\xi \in \mathbb{R}^D$, where $\{e_i\}_{i \in D}$ is the standard basis of $\mathbb{R}^D$. For example, if $D = \{1, 3, 4, 5\}$, $U = \{1, 4\}$, then $P_U(\xi_1 e_1 + \xi_3 e_3 + \xi_4 e_4 + \xi_5 e_5) = \xi_3 e_3 + \xi_5 e_5$. It is now easily seen that

$$L_1 = \bigcup_{\emptyset \neq U \subseteq D^+} K(D, P_U(a)).$$

One may view each competing discriminatory set $K(D, P_U(a))$ as being obtained from $K(D, a)$ by confiscating the allocations $a_i > 0$ from the players in $U \subseteq D^+$ and adding the total to the bargaining amount b .

Lemma 4: Let (n, v) be a $(0, 1)$ -normalised game and $K(D, a)$ a discriminatory set. Then

- (i) $b < 1$ if and only if L_1 (and L) is nonempty.
- (ii) $L_1 \subseteq \text{dom}(K(D, a))$ if and only if $L \subseteq \text{dom}(K(D, a))$.

Proof: (i) If $b = 1$, then $a_i = 0$ for all $i \in D$, and thus $K(D, a) = \{y \in A: y(B) = 1\}$ and $L = L_1 = \emptyset$. If $b < 1$, then $D^+ \neq \emptyset$, and therefore the imputation y defined by $y_i = 0$ if $i \in D$, and $y_i = \frac{1}{|B|}$ if $i \in B$ belongs to L_1 . Thus L_1 (and hence L) is nonempty.

(ii) In case $b = 1$ there is nothing to prove. Assume $b < 1$. Hence $\emptyset \neq L_1 \subseteq L$, and the 'if'-part is obvious. Assume that $L_1 \subseteq \text{dom}(K(D, a))$. Let $y \in L$. Define

$$T^+ \equiv \{i \in D: y_i < a_i\},$$

$$T^0 \equiv \{i \in D: y_i = a_i\} \text{ and}$$

$$T^- \equiv \{i \in D: y_i > a_i\}.$$

Since $y \in L$ it follows that $T^+ \neq \emptyset$. Define $z \in \mathbb{R}^n$ by

$$z_i = \begin{cases} 0 & \text{if } i \in T^+ \\ a_i & \text{if } i \in T^0 \cup T^- \\ y_i + \frac{y(T^+) + y(T^-) - a(T^-)}{|B|} & \text{if } i \in B. \end{cases}$$

Observe that $z \geq 0$ and that $z(N) = a(T^0) + a(T^-) + y(B) + y(T^+) + y(T^-) - a(T^-) = y(T^0) + y(T^+) + y(T^-) + y(B) = y(N) = 1$. Thus $z \in A$. Since $T^+ \neq \emptyset$, it follows that $z \in A - K(D, a)$ and in fact $z \in L_1$. Hence there is $x \in K(D, a)$ and a coalition S such that $x \text{ dom}_S z$. Clearly, $S \subseteq T^+ \cup B$. Since $x_i = a_i > y_i$ for all $i \in T^+$ and $x_i > z_i \geq y_i$ for all $i \in S \cap B$, we have $x \text{ dom}_S y$. Thus $L \subseteq \text{dom}(K(D, a))$. $\square$

In summary, we have the following competitive characterisation of discriminatory vN-M solutions.

Theorem 5: Let $K(D, a)$ be a discriminatory set for a $(0, 1)$ -normalised game (n, v) .

$K(D, a)$ is a discriminatory vN-M solution if and only if

- (i) B is minimal vital,
- (ii) $v(B) \geq b > 0$, and
- (iii) $K(D, P_U(a)) \subseteq \text{dom}(K(D, a))$ for all nonempty subsets $U \subseteq D^+$.

Proof: This follows immediately from Theorem 3 and Lemma 4. $\square$

The "if"-part of the following result is due to Gillies [1959, p. 73], and is best known for simple games: for each minimal winning coalition in a simple game there is a discriminatory vN-M solution where the bargainers form the minimal winning coalition and the discriminated players are totally excluded (see e.g. Owen [1982, p. 169]).

Corollary 6: Let (n, v) be a $(0, 1)$ -normalised game. Let $B \neq N$ be a coalition and let $K(N-B, 0)$ the discriminatory set where 0 is the zero-vector in $\mathbb{R}^{N-B}$. Then $K(N-B, 0)$ is a discriminatory vN-M solution if and only if $v(B) \geq 1$ and $v(S) \leq 0$ for all $S \subset B$.

Proof: For the discriminatory set $K(N-B, 0)$ we have that $b = 1$ and that $D^+ = \emptyset$. The desired result follows immediately from Theorem 5 (or Theorem 3) and Lemma 1. $\square$

Shapley [1967] determined necessary and sufficient conditions for the preservation of vN-M solutions when adding players to the game who will then be excluded (receive payoff 0). Theorem 5 supplies a straightforward direct way to obtain these results for the discriminatory vN-M solutions.

Corollary 7: Let (n, v^1) be an $(0, 1)$ -normalised game. Let $m \geq 1$ be an integer. Let $(n+m, v^2)$ be a $(0, 1)$ -normalised game such that v^1 is the restriction of v^2 to $\{1, 2, \dots, n\}$ (with the possible exception that $v^2(\{1, 2, \dots, n\}) \neq 1$). Let $K(D, a)$ be a discriminatory set of (n, v^1) . Let $(a, \theta) \in \mathbb{R}^{D \cup \{n+1, \dots, n+m\}}$, where θ is the zero-vector in $\mathbb{R}^{\{n+1, \dots, n+m\}}$. Then

$K(D, a)$ is vN-M solution for $(n, v^1) \Leftrightarrow$

$K(D \cup \{n+1, \dots, n+m\}, (a, \theta))$ is a vN-M solution for $(n+m, v^2)$.

Proof: Both discriminatory sets have the same set of bargaining players $B \subset \{1, 2, \dots, n\}$, the same

bargaining amount b and the same D^+ . Since v^1 is the restriction of v^2 to $\{1, 2, \dots, n\}$, it follows that B is minimal vital for v^1 if and only if B is minimal vital for v^2 . Since all imputations $x \in K(D \cup \{n+1, \dots, n+m\}, P_U(a, \theta))$ and in $K(D \cup \{n+1, \dots, n+m\}, (a, \theta))$ satisfy $x_{n+1} = \dots = x_{n+m} = 0$, it follows that for all $U \subseteq D^+$: $K(D, P_U(a)) \subseteq \text{dom}(K(D, a))$ if and only if $K(D \cup \{n+1, \dots, n+m\}, P_U(a, \theta)) \subseteq \text{dom}(K(D \cup \{n+1, \dots, n+m\}, (a, \theta)))$. These observations together with Theorem 5 imply the desired result. $\square$

If a game possesses a discriminatory vN-M solution, then the core C must be empty, except for the extreme case where all discriminated players are excluded. An example of the latter is the 3-person simple majority game where player 1 has vetopower. The only discriminatory vN-M solutions are $K(\{2\}, 0)$ and $K(\{3\}, 0)$, and $C = \text{domcore} = \{(1, 0, 0)\}$.

Theorem 8: If a $(0, 1)$ -normalised game (n, v) has a discriminatory vN-M solution $K(D, a)$ with $a \neq 0$, then $C = \text{domcore} = \emptyset$.

Proof. Since $C \subseteq \text{domcore} \subseteq K(D, a)$, it suffices to show that no imputation in $K(D, a)$ belongs to domcore . Let $y \in K(D, a)$ and let $j \in D^+$. Consider $z \in K(D, P_{\{j\}}(a))$ defined by $z_i \equiv y_i + \frac{a_j}{|B|}$ if $i \in B$. Since $K(D, a)$ is a vN-M solution, there exists $x \in K(D, a)$ and a coalition S with $x \text{ dom}_S z$. Since $x_i = z_i$ for all $i \in D - \{j\}$, we have that $S \subseteq B \cup \{j\}$. We have that $S \cap B \neq \emptyset$, otherwise S is a singleton. Also we have that $j \in S$, otherwise $S = B$ (B is minimal vital) and $x(B) = b < b + a_j = z(B)$. Thus

$$\min \{1, v(S)\} \geq x(S) = a_j + x(S \cap B) > a_j + z(S \cap B) > a_j + y(S \cap B) = y(S).$$

It is now easy to find an imputation $w \in A$ with $w \text{ dom}_S y$. So $y \notin \text{domcore}$. $\square$

3. Discriminatory vN-M solutions and Cores of reduced games.

When using Theorem 5 to determine whether a proposed discriminatory set $K(D, a)$ is a vN-M solution, it is clear that checking condition (iii) is hardest. We will explore this issue in this section, assuming that B is minimal vital and $v(B) \geq b > 0$.

Let $K(D, q)$ be a competing discriminatory set where $q \neq a$. We want to find necessary and sufficient conditions for $K(D, q) \subseteq \text{dom}(K(D, a))$. A partial answer was obtained in Theorem 3 (ii):

Case 1: $1 - q(D) < 1 - a(D) = b$.

In this case $K(D, q) \subseteq \text{dom}_B(K(D, a))$, since any $y \in K(D, q)$ satisfies $y(B) = 1 - q(D) < b$.

Let $U \equiv \{i \in D: q_i < a_i\}$. Thus $U \subseteq D^+$. In this case $K(D, q)$ may be dominated via subcoalitions of U . More precisely, we have:

Case 2: There exists $S \subseteq U$ with $0 < a(S) \leq v(S)$.

In this case $K(D, q) \subseteq \text{dom}_S(K(D, a))$, since any $y \in K(D, q)$ is dominated via S by every $x \in K(D, a)$.

We will now concentrate on the remaining case:

Case 3: $1 - q(D) \geq 1 - a(D) = b$ and for all $S \subseteq U$: $a(S) > v(S)$.

Lemma 9: Assume Case 3 applies to $K(D, q)$. Let $y \in K(D, q)$. Then $y \in \text{dom}(K(D, a))$ if and only if there exists $S \subseteq B \cup U$, with $S \cap B \neq B$, $S \cap B \neq \emptyset$, $S \cap U \neq \emptyset$, satisfying

$$a(S \cap U) + y(S \cap B) < v(S), \text{ and}$$

$$y(S \cap B) < b.$$

Proof. ($\Rightarrow$) Suppose $x \in K(D, a)$ dominates y via the coalition S . From the definition of U it

follows that $S \subseteq B \cup U$. Since Case 3 applies, it follows that $S \cap U \neq \emptyset$ (otherwise $S = B$ and $b = x(B) > y(B) = 1 - q(D)$) and $S \cap B \neq \emptyset$ (otherwise $v(S) \geq x(S) = a(S)$). Since $x \text{ dom}_S y$, we have that

$$\begin{aligned} v(S) &\geq x(S) = x(S \cap U) + x(S \cap B) = a(S \cap U) + x(S \cap B) > a(S \cap U) + y(S \cap B) \text{ and} \\ b = x(B) &\geq x(S \cap B) > y(S \cap B). \end{aligned}$$

Note that $S \cap B \neq B$, otherwise $1 - q(D) = y(B) = y(S \cap B) < b$, a contradiction.

($\Leftarrow$) Note that $B - S \neq \emptyset$. Define $\delta \equiv \min \{v(S) - a(S \cap U) - y(S \cap B), b - y(S \cap B)\}$. Thus $\delta > 0$. Define $x \in K(D, a)$ by

$$x_i \equiv \begin{cases} a_i & \text{if } i \in D, \\ y_i + \frac{\delta}{|S \cap B|} & \text{if } i \in S \cap B \\ \frac{b - y(S \cap B) - \delta}{|B - S|} & \text{if } i \in B - S. \end{cases}$$

Observe that $x \geq 0$ because $\delta \leq b - y(S \cap B)$, and that $x(N) = 1$, and thus indeed $x \in K(D, a)$.

Since $S \subseteq B \cup U$, it is clear that $x_i > y_i$ for all $i \in S$. Finally, because of the definition of δ ,

$$x(S) = x(S \cap U) + x(S \cap B) = a(S \cap U) + y(S \cap B) + \delta \leq v(S).$$

Hence $x \text{ dom}_S y$. □

Lemma 10: Assume Case 3 applies to $K(D, q)$. Then $K(D, q) \not\subseteq \text{dom}(K(D, a))$ if and only if there exists $y \in K(D, q)$ such that for all $\emptyset \neq T \subseteq B$,

$$y(T) \geq \min \left\{ b, \max_{\emptyset \neq R \subseteq U} v(T \cup R) - a(R) \right\}. \quad (2)$$

Proof. From Lemma 9 we obtain that $K(D, q) \not\subseteq \text{dom}(K(D, a))$ if and only if there exists $y \in K(D, q)$ such that for all $S \subseteq B \cup U$ with $S \cap B \neq B$, $S \cap B \neq \emptyset$ and $S \cap U \neq \emptyset$:

$$y(S \cap B) \geq \min \{ b, v(S) - a(S \cap U) \}.$$

Let T play the role of $S \cap B$ and R play the role of $S \cap U$. Hence $K(D, q) \not\subseteq \text{dom}(K(D, a))$ if and only if there exists $y \in K(D, q)$ such that for all $\emptyset \neq T \subseteq B$ and $\emptyset \neq R \subseteq U$:

$$y(T) \geq \min \{ b, v(T \cup R) - a(R) \}.$$

This, together with the observation that

$$\max_{\emptyset \neq R \subseteq U} \min \{ b, v(T \cup R) - a(R) \} = \min \{ b, \max_{\emptyset \neq R \subseteq U} v(T \cup R) - a(R) \},$$

proves the desired result. $\square$

Note: We may replace $\max_{\emptyset \neq R \subseteq U}$ in (2) by $\max_{R \subseteq U}$, since $y(T) \geq 0$ and $v(T) - a(\emptyset) = 0$.

In summary, let z be a proposed imputation not in $K(D, a)$. This deviation z can be blocked by B if $z(B) < b$ (case 1). The deviation can be blocked by $S \subseteq U = \{i \in D: z_i < a_i\}$ if $a(S) \leq v(S)$ (case 2). If z is not blocked in either one of these obvious ways, then z can only be blocked in the sense of Lemma 9. This cannot be done precisely when the restriction of z to B belongs to the Core of the reduced game v^* on B defined by:

$$\begin{aligned} v^*(B) &\equiv z(B), \text{ (which is } \geq b), \\ v^*(T) &\equiv \min \{ b, \max_{R \subseteq U} v(T \cup R) - a(R) \}, \quad \emptyset \neq T \subset B. \end{aligned}$$

That is, z restricted to B satisfies every subcoalition T of B in the sense that either T will do at least as well with z as with any imputation in $K(D, a)$, or that T is unable to do better by leaving B and joining forces with some players in U (who lost part of their initial allocation) by promising to restore their original share a_i . In other words, $K(D, a)$ is not externally stable if and only if there exists an imputation $z \notin K(D, a)$ that cannot be blocked by players in $U = \{i \in D: z_i < a_i\}$ alone, and whose restriction to B is a satisfactory way of splitting $z(B)$ in the sense of belonging to the Core of an attractive reduced game.

A discriminatory vN-M solution can be thought of as an assignment of payoffs b, a to the coalitions in the coalition structure $\{B, \{i\}_i \in D\}$, where B is minimal vital, where $b \leq v(B)$ (this effectiveness condition enables B to block deviations reducing its share), and where deviations that do not reduce B 's share can be blocked by a coalition of individuals not in B that are losing some of

their allocation, or fail to satisfy every subcoalition of B in the above reduced game sense.

In view of the observations made in this section, we may now reformulate Theorem 5. It is useful to introduce the following terminology. A nonempty subset $U \subseteq D^+$ is non-effective for $K(D, a)$ if $a(U) > v(U)$. A nonempty subset $U \subseteq D^+$ is totally non-effective for $K(D, a)$ if every subset $S \subseteq U$ is non-effective for $K(D, a)$.

Theorem 11: Let $K(D, a)$ be a discriminatory set for a $(0, 1)$ -normalised game (n, v) .

$K(D, a)$ is a discriminatory vN -M solution if and only if

- (i) B is minimal vital,
- (ii) $v(B) \geq b > 0$, and
- (iii) For every totally non-effective $U \subseteq D^+$, the Core of the induced game v_U is empty,

where v_U is defined by

$$v_U(B) \equiv b + a(U),$$

$$v_U(T) \equiv \min \{ b, \max_{R \subseteq U} v(T \cup R) - a(R) \}, \quad \emptyset \neq T \subset B.$$

Proof: Start with Theorem 5. If $U \subseteq D^+$ is not totally non-effective, then there exists $S \subseteq U$ such that $K(D, P_U(a)) \subseteq \text{dom}_S(K(D, a))$. In case $U \subseteq D^+$ is totally non-effective, we apply Lemma 10 to $K(D, P_U(a))$: Since $\{i \in D: P_U(a)_i < a_i\} = U$ and any $y \in K(D, P_U(a))$ satisfies $y(B) = b + a(U)$, the desired result follows immediately. $\square$

Note: The reduced games v_U that play a role here resemble the Davis-Maschler reduced game $v^{B,x}$ with respect to B and any $x \in K(D, a)$:

$$v^{B,x}(B) \equiv x(B) = b,$$

$$v^{B,x}(T) \equiv \max_{R \subseteq D} v(T \cup R) - x(R) = \max_{R \subseteq D} v(T \cup R) - a(R), \quad \emptyset \neq T \subset B.$$

The reduced games in Theorem 11 are generally not 0-normalised. Depending on the game, $K(D, a)$ may have many totally non-effective U 's. Even though every non-empty subset of a totally non-effective set is itself totally non-effective, it may be seen that one can not restrict attention to maximal totally non-effective sets in condition (iii) of Theorem 11.

Theorem 11 enables us to check whether a proposed discriminatory set $K(D, a)$ is a vN-M solution. Sometimes it also provides opportunities to determine all discriminatory vN-M solutions, as illustrated in the following example.

Example 2. Me and my Aunt (Davis and Maschler [1965]). The players in this well known game are $\{Aunt, I, P, Q, R\}$ and the characteristic function v is given by: $v(S) = 100$ if $A \in S$ and $|S| \geq 2$, $v(I, P, Q, R) = 100$ and $v(S) = 0$ otherwise (this is of course the voting game $[4; 3, 1, 1, 1, 1]$ in disguise.) The minimal vital coalitions are $\{I, P, Q, R\}$ and every 2-player coalition containing Aunt.

We determine the discriminatory vN-M solutions with $B = \{I, P, Q, R\}$ and with $B = \{A, I\}$.

Case $B = \{I, P, Q, R\}$. The case $a = 0$, (Aunt receives nothing), immediately yields a discriminatory vN-M solution since $D^+ = \emptyset$. For $a > 0$, the only totally non-effective U is $U = \{A\}$. Thus the reduced game is given by $v_{\{A\}}(I, P, Q, R) = 100$ and $v_{\{A\}}(S) = \min \{b, \max \{0, 100 - a\}\} = b$, for all other coalitions S . Since $b > 0$, the Core of this game is equal to the entire imputation set. So in order for the Core to be empty we need $b > 25$. Hence every discriminatory set that assigns less than 75 to Aunt is a vN-M solution.

Case $B = \{A, I\}$. Consider an allocation a that assigns a positive amount to at least one of the players P, Q and R . Say $a_P > 0$. Thus $\{P\}$ is totally non-effective. The reduced game $v_{\{P\}}$ satisfies $v_{\{P\}}(A, I) = b + a_P$, $v_{\{P\}}(A) = \min \{b, \max \{100 - a_P, 0\}\} = b$ and $v_{\{P\}}(I) = \min \{b, \max \{0 - a_P, 0\}\} = 0$. Thus $v_{\{P\}}$ has a nonempty Core. Hence the discriminatory set is not a vN-M solution. It is clear that the discriminatory set that completely excludes P, Q and R is a vN-M solution. $\square$

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