SAMPLE SOLUTIONS OF STOCHASTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT

We prove existence theorems for nonlinear stochastic Sturm-Liouville problems which improve results from [4]. In the simplest case this is done by means of a known result about measurable selections of multivalued maps and a new fixed point theorem for stochastic nonlinear operators which is more realistic than existing ones.

1. INTRODUCTION

We consider the boundary value problem

- (1) $x'' = f(t, x, x', \omega)$ in $J = [0, 1]$
- (2) $B_i(\omega)x(\cdot, \omega) = \alpha_i(\omega)x(i, \omega) + (-1)^{i+1}\beta_i(\omega)x'(i, \omega) = b_i(\omega)$
 for $i = 0, 1$.

We shall always assume that we have a probability measure space $(\Omega, \mathcal{A}, P)$, the functions $\alpha_i, \beta_i, b_i : \Omega \rightarrow \mathbb{R}$ are measurable, Green's function $G(t, s, \omega)$ for

- (3) $x'' = 0$ and $B_i(\omega)x(\cdot, \omega) = 0$ for $i = 0, 1$

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exists for every $\omega \in \Omega$ and the function $f : J \times R^2 \times \Omega \rightarrow R$ satisfies at least condition

(C₁) $f(\cdot, p, q, \cdot)$ is $(\mathcal{L}_1 \otimes \mathcal{A}, \mathcal{B}_1)$ -measurable for all (p, q) and $f(t, \cdot, \cdot, \omega)$ is continuous for all (t, ω) ; here $\mathcal{L}_1$ are the Lebesgue-measurable subsets of J and $\mathcal{B}_1$ is the Borel σ -algebra of R .

This assumption is reasonable since in typical situations one gets f from a deterministic g by replacing some coefficients etc. by stochastic processes z , i.e. $f(t, p, q, \omega) = g(t, p, q, z(t, \omega))$, where some interesting z do not have continuous but measurable samples $z(\cdot, \omega)$ and z is measurable in (t, ω) .

By a sample solution of (1), (2) we understand a stochastic process $x : J \times \Omega \rightarrow R$, satisfying the boundary condition (2) on Ω , such that $x'(\cdot, \omega)$ is a.c. (absolutely continuous) and $x(\cdot, \omega)$ satisfies (1) μ -a.e. (μ the one-dimensional Lebesgue measure), for all $\omega \in \Omega$. Since we shall also assume some integrability conditions on f , the existence of a sample solution to (1), (2) is therefore equivalent to

$$(4) \quad x(t, \omega) = \psi(t, \omega) + \int_0^1 G(t, s, \omega) f(s, x(s, \omega), x'(s, \omega), \omega) ds,$$

i.e. to the existence of an $x : J \times \Omega \rightarrow R$ such that $x(\cdot, \omega)$ is continuously differentiable, $x(t, \cdot)$ is measurable and (4) is satisfied on $J \times \Omega$; here $\psi(\cdot, \omega) \in C^2(J)$ is the unique solution of $\psi'' = 0$ and $B_i(\omega)\psi(\cdot, \omega) = b_i(\omega)$ for $i = 0, 1$. By the definition of ψ and G it is easy to see that these functions are measurable in ω . We start with the basic case where $f(t, \cdot, \cdot, \omega)$ is bounded. Then we try to extend these results along traditional lines (upper- and lower solutions, Nagumo condition, etc.) to right hand sides f which are unbounded in (p, q) .

2. THE CASE WHERE f IS BOUNDED IN (p,q) .

In addition to the assumptions mentioned in the first section we now also assume that f satisfies

$$(C_2) \quad \left\{ \begin{array}{l} |f(t,p,q,\omega)|_1 \leq M(t,\omega) \text{ on } J \times \mathbb{R}^2 \times \Omega, \text{ with a measurable } M \\ \text{such that } \int_0 M(t,\omega) dt < \infty \text{ on } \Omega. \end{array} \right.$$

Theorem 1. Let the functions in (2) be measurable and such that Green's function for (3) exists on Ω . Suppose also that f satisfies (C_1) and (C_2) . Then problem (1), (2) has a sample solution.

For the proof of this result we need

Lemma 1. Let $(\Omega, \mathcal{O})$ be a measurable space, (X, d) a separable metric space and $G : \Omega \rightarrow 2^X \setminus \{\emptyset\}$ a multivalued map such that $G(\omega)$ is complete for all $\omega \in \Omega$ and $\rho(x, G(\cdot))$ is measurable for every $x \in X$ ($\rho(x, A)$ is the distance from x to A). Then G admits an $(\mathcal{O}, \mathcal{B})$ -measurable selection, i.e. an $g : \Omega \rightarrow X$ such that $g^{-1}(\mathcal{B}) \subset \mathcal{O}$ and $g(\omega) \in G(\omega)$ for all $\omega \in \Omega$ ($\mathcal{B}$ is the Borel σ -algebra of (X, d)).

This is Theorem III.6 in [3] or Theorem 24.3 in [7] since the measurability of $\rho(x, G(\cdot))$ for every x is equivalent to $G^{-1}(A) \in \mathcal{O}$ for every open $A \subset X$. We also need the following fixed point theorem.

Lemma 2. Let X be a separable Banach space, $(\Omega, \mathcal{O}, P)$ a probability measure space and $r : \Omega \rightarrow \mathbb{R}$ measurable. Let $F : \Omega \times X \rightarrow X$ be such that $F(\cdot, x)$ is measurable for every $x \in X$, $F(\omega, \cdot)$ is continuous for every $\omega \in \Omega$ and $F(\omega, \cdot) : \bar{B}_{r(\omega)}(0) = \{x \in X : |x| \leq r(\omega)\} \rightarrow \bar{B}_{r(\omega)}(0)$ is compact (i.e. $F(\omega, \bar{B}_{r(\omega)}(0))$ compact) for every $\omega \in \Omega$. Then there exists a measurable $x : \Omega \rightarrow X$ such that $x(\omega) \in \bar{B}_{r(\omega)}(0)$ and $F(\omega, x(\omega)) = x(\omega)$ on Ω .

Proof. Let $B_\omega = \bar{B}_{r(\omega)}(0)$ and $R_\omega : X \rightarrow B_\omega$ the retraction of X onto B_ω , defined by $R_\omega x = x$ on B_ω and $R_\omega x = r(\omega)x/|x|$ for $|x| > r(\omega)$. Define $\tilde{F} : \Omega \times X \rightarrow X$ by $\tilde{F}(\omega, x) = F(\omega, R_\omega x)$ and let $S(\omega) = \{x \in X : \tilde{F}(\omega, x) = x\}$. Since $\tilde{F}(\omega, \cdot)$ is continuous and $\tilde{F}(\omega, X) = F(\omega, B_\omega)$ is relatively compact, $S(\omega)$ is nonempty and compact, by Schauder's fixed point theorem, $S(\omega) = \{x \in B_\omega : F(\omega, x) = x\}$ by definition of $\tilde{F}$. Furthermore, $\tilde{F}(\cdot, x)$ is measurable since, for $B \in \mathcal{B}$, we have

$$\{\omega : \tilde{F}(\omega, x) \in B\} = (F(\cdot, x)^{-1}(B) \cap r^{-1}([|x|, \infty))) \cup$$

$$\cup (\{\omega : F(\omega, r(\omega)x/|x|) \in B\} \cap r^{-1}((0, |x|))),$$

where the first intersection is obviously in $\mathcal{A}$ while the second one is in $\mathcal{A}$ since $r(\cdot)$ is the pointwise limit of step functions.

We want to apply Lemma 1 to $S(\cdot)$ and know already that $S(\omega)$ is complete. Given $x_0 \in X$, to show measurability of $\rho(x_0, S(\cdot))$ means to show $\{\omega : \rho(x_0, S(\omega)) \leq \alpha\} \in \mathcal{A}$ for all $\alpha \geq 0$. For $\alpha = 0$ we have $\{\omega : x_0 \in S(\omega)\} = \tilde{F}(\cdot, x_0)^{-1}(\{x_0\}) \in \mathcal{A}$. For $\alpha > 0$, $\rho(x_0, S(\omega)) \leq \alpha$ means $S(\omega) \cap \bar{B}_\alpha(x_0) \neq \emptyset$. Since X is separable we have a countable dense subset $\{x_j : j \in \mathbb{N}\}$ of $\bar{B}_\alpha(x_0)$ and therefore

$$\{\omega : \rho(x_0, S(\omega)) \leq \alpha\} = \bigcap_{n \geq 1} \bigcup_{j \geq 1} \{\omega : |x_j - \tilde{F}(\omega, x_j)| \leq 1/n\} \in \mathcal{A},$$

since $\tilde{F}(\omega, X)$ is relatively compact and $\tilde{F}(\cdot, x_j)^{-1}(B_{1/n}(x_j)) \in \mathcal{A}$. Hence Lemma 1 applies.

q.e.d.

Now the proof to Theorem 1 is a direct consequence of Lemma 2: Choose $X = C^1(J)$, define $F(\omega, x)(t) = \psi(t, \omega) + \int_0^t G(t, s, \omega) f(s, x(s), x'(s), \omega) ds$ and

$$r(\omega) = \max \left\{ \left| \psi(\cdot, \omega) + \int_0^1 |G(\cdot, s, \omega)| M(s, \omega) ds \right|_0, \left| \psi'(\cdot, \omega) + \int_0^1 |G_t(\cdot, s, \omega)| M(s, \omega) ds \right|_0 \right\},$$

where $|y|_0 = \max_J |y(t)|$ for $y \in C(J)$.

Remarks 1. Theorem 1 remains true for $\mathbb{R}^n$ -valued functions, i.e. finite systems (1), (2), and it improves Theorem 2.1 in [4] which has the restrictive hypothesis that f be continuous in t and $\int_0^1 M(t, \omega) dt \leq C < \infty$ (P-a.e.).

2. Lemma 2 is much more realistic than the procedure chosen in a flood of papers on stochastic fixed point theorems, where the set in which one looks for fixed points is not allowed to vary with ω ; see e.g. the survey [2]. The result remains true if $F(\omega, \cdot)$:

$B_\omega = \bar{B}_{r(\omega)}(0) \rightarrow B_\omega$ is only γ -condensing, i.e. $\gamma(F(\omega, A)) < \gamma(A)$ for all $A \subset B_\omega$ with $\gamma(A) > 0$, where γ is either Kuratowski's or the ball-measure of noncompactness (see e.g. [9] or § 7 in [7]), since $\gamma(R_\omega(A)) \leq \gamma(A)$ for all bounded $A \subset X$.

3. Theorem 1 remains true in any Banach space X (instead of $\mathbb{R}$ or $\mathbb{R}^n$; the functions α_i, β_i in (2) still realvalued) provided we add a condition on f guaranteeing the relative compactness of $\{F(\omega, x)(t) : x \in B_\omega\}$. The natural formulation of such extra conditions is also in terms of measures of noncompactness, i.e. estimates of type

$$\gamma(f(t, A, B, \omega)) \leq \rho(t, \gamma(A) \wedge \gamma(B), \omega) \quad \text{for bounded } A, B \subset X$$

which imply $\max\{\gamma(F(\omega, B_\omega)(t)), \gamma(F(\omega, B_\omega)'(t))\} = 0$ for all $(t, \omega) \in J \times \Omega$, since " $\gamma(A) = 0$ iff A is relatively compact" is one of the properties of such measures of noncompactness; a basic paper for deterministic problems (1), (2) in any Banach space is [10].

3. CONSEQUENCES OF THEOREM 1.

We first consider the case where f has at most linear growth in p and q .

Theorem 2. Let the functions in (2) be as in Theorem 1. Suppose that f satisfies (C_1) and

(5) $|f(t, p, q, \omega)| \leq k_1(t, \omega)|p| + k_2(t, \omega)|q| + k_3(t, \omega)$ on $J \times \mathbb{R}^2 \times \Omega$,
 with measurable $k_i \geq 0$ such that (for every $\omega \in \Omega$) $\int_0^1 k_1(t, \omega) dt < \infty$
 and the linear operator $H(\omega) : C(J)^2 \rightarrow C(J)^2$, defined by

$$H(\omega)(x, y)(t) = \left(\int_0^1 G(t, s, \omega) |p(s, x(s), y(s), \omega)| ds, \int_0^1 G_t(t, s, \omega) |p(\dots)| ds \right)$$

 has spectral radius < 1 ; here $p(t, p, q, \omega) = k_1(t, \omega)|p| + k_2(t, \omega)|q|$.

Then (1), (2) has a sample solution.

Proof. The reduction to the situation of Theorem 1 goes as follows.
 For a sample solution x of (1), (2) we have the a-priori estimate
 (consider (4))

$$(|x(\cdot, \omega)|, |x'(\cdot, \omega)|) \leq H(\omega)(|x(\cdot, \omega)|, |x'(\cdot, \omega)|) + (M_0(\cdot, \omega), M_1(\cdot, \omega))$$

with

$$M_0(t, \omega) = |\psi(t, \omega)| + \int_0^1 |G(t, s, \omega)| k_3(s, \omega) ds,$$

$$M_1(t, \omega) = |\psi'(t, \omega)| + \int_0^1 |G_t(t, s, \omega)| k_3(s, \omega) ds,$$

hence $(|x(t, \omega)|, |x'(t, \omega)|) \leq (I - H(\omega))^{-1} (M_0(\cdot, \omega), M_1(\cdot, \omega))$, and there-
 fore $\max_j |x(t, \omega)| \leq M_2(\omega)$ and $\max |x'(t, \omega)| \leq M_2(\omega)$ with a measurable
 $M_2(\cdot)$ since $(I - H(\omega))^{-1} = \sum_{m=0}^{\infty} H(\omega)^m$. Now, consider $\pi : \mathbb{R} \times \Omega \rightarrow [0, 1]$
 continuous in τ and measurable in ω such that $\pi(\tau, \omega) = 1$ for
 $|\tau| \leq M_2(\omega)$ and $\pi(\tau, \omega) = 0$ for $|\tau| \geq M_2(\omega) + 1$. Then $\tilde{f} : J \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R}$,
 defined by

$$\tilde{f}(t, p, q, \omega) = \pi(p, \omega) \pi(q, \omega) f(t, p, q, \omega),$$

satisfies (C_1) and (C_2) . Consequently, Theorem 1 yields a sample
 solution x of (1), (2) with $\tilde{f}$ instead of f . Since $\tilde{f}$ also satisfies
 (5) we get $\pi(x(t, \omega), \omega) = \pi(x'(t, \omega), \omega) = 1$ and therefore x is a
 sample solution of (1), (2).

q.e.d.

As another extension of Theorem 1 we consider the situation in which
 the behaviour of f with respect to x is controlled by upper and
 lower solutions and its behaviour in x' is governed by Nagumo's
 condition which means essentially at most quadratic growth in x' .

If f is only measurable in t , a reasonable definition of (sample) upper and lower solutions goes as follows.

A function $\alpha: J \times \Omega \rightarrow \mathbb{R}$ is a lower (sample) solution of (1), (2) if $\alpha(t, \cdot)$ is measurable for all t , $\alpha'(\cdot, \omega)$ is a.c. for every ω , $B_1(\omega)\alpha(\cdot, \omega) \leq b_1(\omega)$ on Ω and for every $\omega \in \Omega$ we have $\alpha''(t, \omega) \geq f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega)$ μ -a.e. on J . The definition of an upper (sample) solution β is obtained by replacing α by β and reversing the inequalities.

If α is a lower and β is an upper solution for (1), (2) such that $\alpha \leq \beta$ then we say that f satisfies a Nagumo condition w.r. to (α, β) if

(6) $|f(t, p, q, \omega)| \leq h(|q|, \omega)$ for $(t, \omega) \in J \times \Omega, \alpha(t, \omega) \leq p \leq \beta(t, \omega), q \in \mathbb{R}$ with $h: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+ \setminus \{0\}$ such that $h(\cdot, \omega)$ is continuous, $h(\tau, \cdot)$ is measurable and

$$(7) \quad \int_{\lambda(\omega)}^{N(\omega)} \frac{\tau d\tau}{h(\tau, \omega)} > \max_J \beta(t, \omega) - \min_J \alpha(t, \omega) \text{ on } \Omega$$

for some measurable $N(\cdot)$ and

$$\lambda(\omega) = \max(|\beta(0, \omega) - \alpha(1, \omega)|, |\beta(1, \omega) - \alpha(0, \omega)|).$$

Such a function $N(\cdot)$ exists if $\int_{\lambda(\omega)}^{\infty} \frac{\tau d\tau}{h(\tau, \omega)} = \infty$ on Ω . To see this notice that $H(s, \omega) := \int_{\lambda(\omega)}^s \frac{\tau d\tau}{h(\tau, \omega)}$ is strictly increasing in s , denote the right hand side of (7) by $\psi(\omega)$ and define $N(\omega)$ by $H(N(\omega), \omega) = \psi(\omega) + 1$; then $\{\omega: N(\omega) > r\} = \{\omega: \psi(\omega) + 1 < H(r, \omega)\} \in \mathcal{A}$ for every $r \in \mathbb{R}$ and therefore $N(\cdot)$ is measurable. Now we have

Theorem 3. Let the functions in (2) be as in Theorem 1 and such that $\alpha_i \geq 0$, $\beta_i \geq 0$ and $\alpha_i(\omega) + \beta_i(\omega) > 0$ on Ω for $i = 0, 1$. Let $f: J \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R}$ satisfy condition (C_1) . Suppose also that there exist a lower sample solution α and an upper sample solution β such that $\alpha(t, \omega) \leq \beta(t, \omega)$ on $J \times \Omega$ and f satisfies a Nagumo condition w.r. to (α, β) . Then (1), (2) has a sample solution.

Proof. 1. Let $c(\omega) = 1 + \max\{N(\omega), \max_J |\alpha'(t, \omega)|, \max_J |\beta'(t, \omega)|\}$ with $N(\cdot)$ from (7). Define $F : J \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R}$ by

$$F(t, p, q, \omega) = f(t, \rho_1(t, p, \omega), \rho_2(q, \omega), \omega) + r(t, p, \omega)$$

where

$$\rho_1(t, p, \omega) = \max\{\alpha(t, \omega), \min\{p, \beta(t, \omega)\}\}$$

$$\rho_2(q, \omega) = \max\{-c(\omega), \min\{q, c(\omega)\}\}$$

$$r(t, p, \omega) = \begin{cases} \frac{p - \beta(t, \omega)}{1 + p^2} & \text{if } p > \beta(t, \omega) \\ 0 & \text{if } \alpha(t, \omega) \leq p \leq \beta(t, \omega) \\ \frac{p - \alpha(t, \omega)}{1 + p^2} & \text{if } p < \alpha(t, \omega) \end{cases}$$

It is easy to see that F satisfies (C_1) and (C_2) , and therefore (1), (2) with F instead of f has a sample solution and if we can show that there is a sample solution x satisfying $\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)$ and $|x'(t, \omega)| \leq N(\omega)$ on $J \times \Omega$ then we are done.

2. Since f satisfies only Caratheodory's condition, the comparison $\alpha \leq x \leq \beta$ is not directly obtainable. Therefore, we first approximate F by functions F_n which are measurable in (t, ω) and locally Lipschitz in (p, q) such that (C_2) holds with the same M for all F_n and $F_n(t, p, q, \omega) \rightarrow F(t, p, q, \omega)$ as $n \rightarrow \infty$, uniformly in (p, q) for every (t, ω) . Then we let (omitting ω), $\alpha_n(t) = \alpha(t) - 1/n$ and $\beta_n(t) = \beta(t) + 1/n$,

$$g_n(t, p, q) = F_n(t, p, q) + \frac{p - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)} [f(t, \beta, \beta') - F_n(t, \beta_n, \beta_n')] + \frac{\beta_n(t) - p}{\beta_n(t) - \alpha_n(t)} [f(t, \alpha, \alpha') - F_n(t, \alpha_n, \alpha_n')],$$

$$G_n(t, p, q) = g_n(t, \rho_n(t, p), q) + r_n(t, p),$$

where ρ_n and r_n are ρ_1 and r with (α_n, β_n) instead of (α, β) . By these definitions and Theorem 1 it is obvious that (1), (2) with G_n instead of f has a sample solution x_n and α_n, β_n are lower and upper sample solutions, respectively, of this new problem.

If we are able to show that $\alpha_n \leq x_n \leq \beta_n$ then we can use the idea of the proof to Theorem 1 in [6] to obtain a sample solution x of (1), (2) with F instead of f satisfying $\alpha \leq x \leq \beta$ on $J = \Omega$.

3. We only show $\alpha_n \leq x_n$ since the inequality $x_n \leq \beta_n$ then follows similarly. Suppose on the contrary that $\alpha_n \leq x_n$ is wrong. Then there exists an ω such that $\alpha_n(\cdot, \omega) - x_n(\cdot, \omega)$ has a positive maximum, say at $t_0 \in J$. In the sequel we shall omit ω .

Suppose that $t_0 \in (0, 1)$. Then we have $\alpha_n(t) \geq x_n(t) + \epsilon$ in $J_\delta = [t_0, t_0 + \delta)$ for some positive ϵ and δ , $\alpha'_n(t_0) = x'_n(t_0)$ and $|x'_n(t)| < c(\omega)$ in J_δ , by the continuity of x'_n and the definition of $c(\omega)$. Therefore

$$(8) \quad \begin{cases} \gamma'(t) \geq g_n(t, \alpha_n(t), \gamma(t)), \\ z'(t) \leq g_n(t, \gamma_n(t), z(t)) - \epsilon_1 \end{cases} \quad (\text{u-a.e. on } J_\delta), \gamma(t_0) = z(t_0)$$

for $\gamma(t) = \alpha'_n(t)$ and $z(t) = x'_n(t)$. Hence $\gamma(t) \geq r_*(t)$ and $z(t) \leq r^*(t)$, where r_* and r^* are the minimal and the maximal solution of

$u' = g_n(t, \alpha_n(t), u)$, $u(t_0) = \gamma(t_0)$, respectively (see e.g. Theorem 1.10.1 in [8]). But $r_* = r^*$ since g_n is locally Lipschitz in q , hence integration yields $\alpha(t) - x(t) \geq \alpha(t_0) - x(t_0)$, which means $\alpha(t) = x(t) + c$ in J_δ for some $c > 0$ and therefore $\gamma(t) = z(t)$ in J_δ , a contradiction to (8) since integration of (8) yields $\gamma(t) \geq z(t) + \epsilon_1(t - t_0)$ in J_δ with $\epsilon_1 > 0$. Hence $\alpha - x$ cannot attain its maximum at a $t_0 \in (0, 1)$.

Suppose next that $t_0 = 0$. Then $0 < \alpha_n(0) - x_n(0) \geq \alpha_n(t) - x_n(t)$ for all small $t > 0$ implies $\alpha'_n(0) \leq x'_n(0)$. By the first boundary condition we also have

$$\alpha_0(x_n(0) - \alpha_n(0)) - \beta_0(x'_n(0) - \alpha'_n(0)) \geq 0$$

and therefore $\alpha'_n(0) = x'_n(0)$, since $\alpha_0 \geq 0$, $\beta_0 \geq 0$ and $\alpha_0 + \beta_0 > 0$. Thus we may proceed as in the case $t_0 \in (0, 1)$ and likewise one sees that the maximum can also not be at $t_0 = 1$. Hence $\alpha_n(t, \omega) \leq x_n(t, \omega)$ on $J = \Omega$.

3. We now have a sample solution x satisfying $\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)$, and therefore

$$x''(t, \omega) = f(t, x(t, \omega), \phi_2(x'(t, \omega), \omega), \omega) \quad \mu\text{-a.e. in } J \text{ for every } \omega \in \Omega.$$

Since $c(\omega) > N(\omega)$, we may now repeat (for every $\omega \in \Omega$) the argument given in the proof to Theorem 1.4.1. in [1] to see that

$$|x'(t, \omega)| \leq N(\omega) \text{ on } J \times \Omega.$$

q.e.d.

Remarks. Theorem 3 improves Theorem 2.2 in [4] considerably. Notice that the Nagumo condition implies boundedness of f in t . The question is whether this can be relaxed keeping the quadratic growth in x' . For an interesting application of stochastic BVPs (1), (2) see [5].

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