EXISTENCE OF PERIODIC SOLUTIONS OF NONLINEAR BOUNDARY
VALUE PROBLEMS AND THE METHOD OF UPPER AND LOWER SOLUTIONS*

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ABSTRACT

We study the existence of periodic solutions of second order nonlinear differential equations by combining the method of upper and lower solutions and the method of Alternative problems. Besides including previous results of other authors, we obtain criteria which are easily verifiable. These criteria are in a form that they can be adapted for partial differential equations of parabolic and elliptic type.

1. Introduction

Recently an attempt was made in [5] to combine two basic techniques, namely, the method of upper and lower solutions and the method of Alternative Problems to investigate the existence of periodic solutions of second order nonlinear boundary value problems. In this paper, we continue this fruitful approach to discuss the same problems from a different point of view so as to bring out the true nature of the problem. Also this modified point of view offers more realistic criteria that are verifiable. The results obtained are exhaustive in the sense that they not only contain the known results but also deal with other possible situations. The technique we follow is parallel to the classical method of upper and lower solutions. Furthermore, it is clear from our results that the method of proof is suitable to study other problems at resonance such as semilinear parabolic periodic boundary value problems with homogeneous Neumann boundary conditions in the space variable. We plan to investigate such a problem elsewhere.

2. An abstract existence Theorem

Let E be a real Hilbert space. Consider the nonlinear operator equation

$$(2.1) \quad Lu = Nu$$

where $L: \mathcal{D}(L) \subset E \rightarrow E$ is a linear operator and $N: \mathcal{D}(N) \subset E \rightarrow E$ a nonlinear operator with $\mathcal{D}(L) \cap \mathcal{D}(N) \neq \emptyset$. Let $E_0 = N(L)$ be such that $\dim E_0 < \infty$ and $E = E_0 \oplus E_1$. Suppose that $P: E \rightarrow E_0$ is the idempotent projection operator and $H: E_1 \rightarrow E_1$ the compact inverse of L on E_1 . Then it is well known [3] that the problem (2.1) is equivalent to the coupled system of operator equations

$$(2.2) \quad u_1 = H(I-P)N(u_0 + u_1),$$

$$(2.3) \quad 0 = PN(u_0 + u_1).$$

Concerning the problem (2.1), we have the following existence result [4].

Theorem 2.1: Suppose that

$$(i) \quad \|Nu\| \leq J_0, \quad u \in \mathcal{D}(N);$$

(ii) there exist $r_0, R_0 > 0$ such that

$$\langle N(u_0 + u_1), u_0 \rangle \geq 0 \quad (\text{or } \leq 0) \quad \text{whenever } \|u_0\| = R_0 \quad \text{and}$$

$$\|u_1\| \leq r_0 \quad \text{where } u_0 \in E_0 \quad \text{and } u_1 \in E_1.$$

Then the problem (2.1) admits at least one solution.

We employ this theorem as a basic tool in our discussions. In fact, this theorem can be regarded as the abstract version of Scorza-Dragoni's theorem [11] for the case of resonance.

3. Main results

Consider the periodic boundary value problem (PBVP for short)

$$(3.1) \quad -u'' = f(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $f \in C[[0, 2\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$. Let us list the following assumptions

for convenience:

$$(A_0) \quad \alpha, \beta \in C^2[[0, 2\pi], \mathbb{R}], \quad \alpha(t) \leq \beta(t), \quad t \in [0, 2\pi], \quad \alpha(0) = \alpha(2\pi),$$

$$\beta(0) = \beta(2\pi) \quad \text{and for } 0 < t < 2\pi, \quad -\alpha'' \leq f(t, \alpha, \alpha'), \quad -\beta'' \geq f(t, \beta, \beta');$$

$$(A_1) \quad f \text{ satisfies a Nagumo condition relative to } \alpha, \beta; \text{ (see [2])}$$

$$(A_2) \quad (i) \quad \alpha'(0) \geq \alpha'(2\pi) \quad \text{and} \quad -\alpha''(i) \leq f(i, \alpha(i), \alpha'(i)), \quad i = 0, 2\pi;$$

$$(ii) \quad \beta'(0) \leq \beta'(2\pi) \quad \text{and} \quad -\beta''(i) \geq f(i, \beta(i), \beta'(i)), \quad i = 0, 2\pi;$$

$$(A_3) \quad (a) \quad \text{if } \alpha'(0) < \alpha'(2\pi), \text{ then for } i = 0, 2\pi, \quad -\alpha''(i) \leq f(i, \alpha(i), \sigma) \\ - \frac{[\alpha'(2\pi) - \alpha'(0)]}{\pi} \quad \text{for all } \sigma \text{ such that } \alpha'(0) \leq \sigma \leq \alpha'(2\pi), \text{ and}$$

$$\int_0^{2\pi} f(s, \alpha(s), \alpha'(s)) ds \geq 0;$$

$$(b) \quad \text{if } \beta'(0) > \beta'(2\pi), \text{ then for } i = 0, 2\pi, \quad -\beta''(i) \geq f(i, \beta(i), \sigma) \\ + \frac{[\beta'(0) - \beta'(2\pi)]}{\pi} \quad \text{for all } \sigma \text{ such that } \beta'(2\pi) \leq \sigma \leq \beta'(0), \text{ and}$$

$$\int_0^{2\pi} f(s, \beta(s), \beta'(s)) ds \leq 0;$$

$$(A_4) \quad (A_2) \text{ (ii) and } (A_3) \text{ (a) hold;}$$

$$(A_5) \quad (A_2) \text{ (i) and } (A_3) \text{ (b) hold.}$$

Let us first consider the PBVP

$$(3.2) \quad -u'' = F(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $F(t, u, v) = f(t, p(t, u), v) + r(t, u)$,

$$p(t, u) = \max [\alpha(t), \min \{u, \beta(t)\}],$$

and

$$r(t, u) = \begin{cases} \frac{\beta(t) - u}{1 + u^2} & \text{if } u > \beta(t), \\ 0 & \text{if } \alpha(t) \leq u \leq \beta(t), \\ \frac{\alpha(t) - u}{1 + u^2} & \text{if } u < \alpha(t). \end{cases}$$

Relative to the PBVP (3.2) we prove the following lemmas.

Lemma 3.1 Suppose that (A_0) holds and that any one of (A_2) , (A_3) , (A_4) and (A_5) is satisfied. Then if u is a solution of PBVP (3.2), we have $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0, 2\pi]$.

Proof: We shall only prove that $\alpha(t) \leq u(t)$, $t \in [0, 2\pi]$ since a similar argument proves $u(t) \leq \beta(t)$, $t \in [0, 2\pi]$. Suppose that $\alpha(t) \leq u(t)$, $t \in [0, 2\pi]$ is not true. Then there exists a $t_0 \in [0, 2\pi]$ and an $\varepsilon > 0$ such that

$$(3.3) \quad \alpha(t_0) = u(t_0) + \varepsilon \quad \text{and} \quad \alpha(t) \leq u(t) + \varepsilon, \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, we have $\alpha'(t_0) = u'(t_0)$ and $\alpha''(t_0) \leq u''(t_0)$. Also, $\alpha(t_0) > u(t_0)$ and hence $\rho(t_0, u(t_0)) = \alpha(t_0)$. Using (A_0) , we get because of the definition of F ,

$$0 \leq -\alpha''(t_0) + u''(t_0) \leq f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \alpha(t_0), u'(t_0)) - \frac{(\alpha(t_0) - u(t_0))}{1 + u^2(t_0)} < 0,$$

which is a contradiction.

If $t_0 = 0$, then by (3.3) we have $\alpha(0) = u(0) + \varepsilon$ and $\alpha'(0) \leq u'(0)$.

Also, since $\alpha(0) = \alpha(2\pi)$, $u(0) = u(2\pi)$, one gets $\alpha(2\pi) = u(2\pi) + \varepsilon$ and $\alpha'(2\pi) \geq u'(2\pi)$. Consequently,

$$(3.4) \quad \alpha'(0) \leq u'(0) = u'(2\pi) \leq \alpha'(2\pi).$$

Thus, when (A_0) and (A_2) hold, it follows that $\alpha'(0) = u'(0) = u'(2\pi) = \alpha'(2\pi)$ and

$$-\alpha''(i) + u''(i) \leq f(i, \alpha(i), \alpha'(i)) - f(i, \alpha(i), u'(i)) - \frac{[\alpha(i) - u(i)]}{1 + u^2(i)}, \quad i = 0, 2\pi,$$

which is again a contradiction. If, on the other hand, (A_0) and (A_3)

are satisfied, then choosing $\sigma = u'(i)$, we get

$$-\alpha''(i) + u''(i) \leq f(i, \alpha(i), u'(i)) - \frac{\eta}{\pi} - f(i, \alpha(i), u'(i)) - \frac{[\alpha(i) - u(i)]}{1 + u^2(i)},$$

where $\eta = \alpha'(2\pi) - \alpha'(0)$. Setting $\phi(t) = \alpha(t) - u(t)$, we see that $\phi''(i) > \frac{\eta}{\pi}$, $\phi(i) = \varepsilon$, $\phi'(0) \leq 0$ and $\phi'(2\pi) \geq 0$, $i = 0, 2\pi$. It then follows that $\phi(t) \geq \phi(0) + t\phi'(0) + \frac{t^2\eta}{2\pi}$, $0 < t \leq T_0$ and $\phi(t) \geq \phi(2\pi) - (2\pi - t)\phi'(2\pi) + \frac{\eta}{2\pi}(2\pi - t)^2$, $2\pi > t \geq T_1$ for some T_0 and T_1 . Since $\phi(t) \leq \varepsilon$ for $t \in [0, 2\pi]$ and $\phi(i) = \varepsilon$, $i = 0, 2\pi$, the foregoing inequalities yield $T_0 = -\phi'(0)\frac{2\pi}{\eta}$ and $T_1 = [\eta - \phi'(2\pi)]\frac{2\pi}{\eta}$. But, $\eta - \phi'(2\pi) = \alpha'(2\pi) - \alpha'(0) - \alpha'(2\pi) + u'(2\pi) = -(\alpha'(0) - u'(0)) = -\phi'(0)$. Hence $T_0 = T_1 = T$, say. This shows that $\phi(t) \geq \frac{\eta}{\pi}$ for $0 \leq t \leq 2\pi$ since at $t = T$, we have $\phi(T) \geq \phi(0)$ and $\phi(T) \geq \phi(2\pi)$. Consequently, at $t = T \in (0, 2\pi)$, either $\phi(T) > \varepsilon$ or $\phi(T) = \varepsilon$. In either case, we are led to a contradiction. Hence, it follows that $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 2\pi]$, when (A_0) , (A_2) or (A_0) , (A_3) hold.

It is now easy to see that (A_0) , (A_4) and (A_0) , (A_5) are also sufficient to prove the lemma. Hence, the proof is complete.

Lemma 3.2 Let the assumptions (A_0) to (A_5) hold relative to the PBVP

(3.1). Then there exist α_0, β_0 such that;

$$(A_0^*) \quad \alpha_0, \beta_0 \in C[[0, 2\pi], \mathbb{R}], \quad \alpha_0(t) < \beta_0(t), \quad t \in [0, 2\pi], \quad \alpha_0(0) = \alpha_0(2\pi),$$

$$\beta_0(0) = \beta_0(2\pi) \quad \text{and for } 0 < t < 2\pi, \quad -\alpha_0'' < F(t, \alpha_0, \alpha_0'), \quad -\beta_0'' > F(t, \beta_0, \beta_0');$$

$$(A_1^*) \quad F \text{ satisfies a Nagumo condition relative to } \alpha, \beta;$$

$$(A_2^*) \quad (i) \quad \alpha_0'(0) \geq \alpha_0'(2\pi) \quad \text{and} \quad -\alpha_0''(i) < F(i, \alpha_0(i), \alpha_0'(i)), \quad i = 0, 2\pi;$$

$$(ii) \quad \beta_0'(0) \leq \beta_0'(2\pi) \quad \text{and} \quad -\beta_0''(i) > F(i, \beta_0(i), \beta_0'(i)), \quad i = 0, 2\pi;$$

$$(A_3^*) \quad (a) \quad \text{if } \alpha_0'(0) < \alpha_0'(2\pi), \quad \text{then for } i = 0, 2\pi, \quad -\alpha_0''(i) < F(i, \alpha_0(i), \sigma)$$

$$- \frac{[\alpha_0'(2\pi) - \beta_0'(0)]}{\pi} \quad \text{for all } \sigma \text{ such that } \alpha_0'(0) \leq \sigma \leq \alpha_0'(2\pi) \quad \text{and}$$

$$\int_0^{2\pi} F(s, \alpha_0(s), \alpha_0'(s)) ds \geq 0;$$

(b) if $\beta'_0(0) > \beta'_0(2\pi)$, then for $i = 0, 2\pi$, $-\beta''_0(i) > F(i, \beta_0(i), \sigma)$
 $+ \frac{[\beta'_0(0) - \beta'_0(2\pi)]}{\pi}$ for all σ such that $\beta'_0(2\pi) \leq \sigma \leq \beta'_0(0)$ and
 $\int_0^{2\pi} F(s, \beta_0(s), \beta'_0(s)) ds \leq 0$.

(A₄^{*}) A_2^* (ii) and A_3^* (a) hold;

(A₅^{*}) A_2^* (i) and A_3^* (b) hold.

Proof: Let $a < 0$, $b > 0$ and define $\alpha_0(t) = \alpha(t) + a$, $\beta_0(t) = \beta(t) + b$.

Then since $\alpha(t) \leq \beta(t)$, $t \in [0, 2\pi]$, clearly $\alpha_0(t) < \beta_0(t)$, $t \in [0, 2\pi]$.

Also, since $\alpha_0(t) < \alpha(t)$, we have by the definition of F and the
fact $\alpha'_0(t) = \alpha'(t)$, $F(t, \alpha_0, \alpha'_0) = f(t, \alpha, \alpha') + \frac{\alpha - \alpha_0}{1 + \alpha_0^2} > f(t, \alpha, \alpha') \geq -$

$\alpha'' = -\alpha''_0$ for $0 < t < 2\pi$. Similarly $-\beta''_0 < F(t, \beta_0, \beta'_0)$ for $0 < t < 2\pi$.

This proves (A₀^{*}). It is now easy to see that (A₁^{*}) and (A₂^{*}) hold. Since

$$\int_0^{2\pi} F(s, \alpha_0(s), \alpha'_0(s)) ds = \int_0^{2\pi} \left[f(s, \alpha(s), \alpha'(s)) + \frac{\alpha(s) - \alpha_0(s)}{1 + \alpha_0^2(s)} \right] ds,$$

it follows by (a) of (A₃), $\int_0^{2\pi} F(s, \alpha_0(s), \alpha'_0(s)) ds \geq 0$. Similarly one

gets $\int_0^{2\pi} F(s, \beta_0(s), \beta'_0(s)) ds \leq 0$ from (b) of (A₃). The rest of the

inequalities in (A₃^{*}) are easily seen to be true, proving (A₃^{*}). It is
therefore clear that all the conclusions (A₀^{*}) to (A₅^{*}) hold relative to
 α_0 , β_0 and F . Hence the proof is complete.

Let us next consider the following modified PBVP

$$(3.5) \quad -u'' = G(t, u), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where

$$G(t, u) = \begin{cases} F(t, \beta_0, \beta'_0) + \frac{\beta_0 - u}{1 + u^2} & \text{if } u > \beta_0, \\ \frac{u - \alpha_0}{\beta_0 - \alpha_0} \left[F(t, \beta_0, \beta'_0) - F(t, \alpha_0, \alpha'_0) \right] + F(t, \alpha_0, \alpha'_0) & \text{if } \alpha_0 \leq u \leq \beta_0, \\ F(t, \alpha_0, \alpha'_0) + \frac{\alpha_0 - u}{1 + u^2} & \text{if } u < \alpha_0. \end{cases}$$

Since $\alpha_0(t) < \beta_0(t)$, $t \in [0, 2\pi]$, $G(t, u)$ is well defined. Relative to the PBVP (3.5), we have the following result.

Lemma 3.3 Let the assumptions (A_0) , (A_1) hold. Then any one of the conditions (A_2) to (A_5) implies that there exists a unique solution u for the PBVP (3.5) such that $\alpha_0(t) \leq u(t) \leq \beta_0(t)$, $t \in [0, 2\pi]$.

Proof: We now apply Theorem 2.1 to prove Lemma 3.3. We first observe that $G(t, u)$ is continuous on $[0, 2\pi] \times \mathbb{R}$ and is bounded everywhere. Thus, we have $|G(t, u)| \leq J$ on $[0, 2\pi] \times \mathbb{R}$ where J depends only on α_0, β_0 . We let $E = L_2[0, 2\pi]$, $Lu = -u''$, $\mathcal{D}(L) = \{u \in E: u, u' \text{ are absolutely continuous, } u'' \in E \text{ and } u(0) = u(2\pi), u'(0) = u'(2\pi)\}$ and N be the Nemytskii operator generated by G . Now the PBVP (3.5) can be translated as the operator equation (2.1). It is easy to see that $E_0 = N(L)$ consists of constant functions and hence E_1 , where $E = E_0 \oplus E_1$, is the class of all functions whose average is zero. Then the operators P and H can be defined so as to satisfy the conditions required in Theorem 2.1. Because of the fact G is bounded, we can find positive constants A and B which depend only on α_0, β_0 such that any solution u_1 of (2.2) satisfies the estimates $|u_1(t)| \leq A$ and $|u'_1(t)| \leq B$, $t \in [0, 2\pi]$. Hence by Theorem 2.1 it is enough to find an $R_0 > 0$ such that

$$(3.6) \quad \langle N(u_0 + u_1), u_0 \rangle \geq 0 \quad (\text{or } \leq 0)$$

for all $|u_0| = R_0$ and $|u_1| \leq A$, $|u_1'| \leq B$ on $[0, 2\pi]$. Since $E_0 = \langle 1 \rangle$ and $u_0 \in E_0$, the relation (3.6) can be written as

$$(3.7) \quad \int_0^{2\pi} G(s, R_0 + u_1(s)) ds \leq 0 \quad \text{and} \quad \int_0^{2\pi} G(s, -R_0 + u_1(s)) ds \geq 0.$$

We now choose $R_0 > 0$ large enough so that we have

$$R_0 + u_1(t) > \max_{[0, 2\pi]} \beta_0(t) \quad \text{and} \quad -R_0 + u_1(t) < \min_{[0, 2\pi]} \alpha_0(t).$$

This choice, in view of the definition of G , reduces (3.7) to

$$(3.8) \quad \int_0^{2\pi} G(s, \beta_0(s)) ds \leq 0 \quad \text{and} \quad \int_0^{2\pi} G(s, \alpha_0(s)) ds \geq 0.$$

However, by (A_3^*) which is a consequence of Lemma 3.2, we have

$$\begin{aligned} \int_0^{2\pi} G(s, \beta_0(s)) ds &= \int_0^{2\pi} F(s, \beta_0(s), \beta_0'(s)) ds \leq 0 \quad \text{and} \\ \int_0^{2\pi} G(s, \alpha_0(s)) ds &= \int_0^{2\pi} F(s, \alpha_0(s), \alpha_0'(s)) ds \geq 0. \end{aligned}$$

Hence by Theorem 2.1 there exists a solution u for the PBVP (3.5).

It is now easy to show that $\alpha_0(t) \leq u(t) \leq \beta_0(t)$, $t \in [0, 2\pi]$ following the proof of Lemma 3.1, the definition of G and the fact α_0, β_0 satisfy (A_0^*) to (A_5^*) . It therefore follows that u is a solution of

$$(3.9) \quad -u'' = \frac{(u - \alpha_0)}{\beta_0 - \alpha_0} \left[F(t, \beta_0, \beta_0') - F(t, \alpha_0, \alpha_0') \right] + F(t, \alpha_0, \alpha_0'),$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

Let us show that this solution u is unique for the PBVP (3.9). If not,

let v be another solution of (3.9) such that $\alpha_0(t) \leq v(t) \leq \beta_0(t)$, $t \in [0, 2\pi]$.

Then setting $m(t) = u(t) - v(t)$, we obtain

$$(3.10) \quad -m'' = c(t)m, \quad m(0) = m(2\pi), \quad m'(0) = m'(2\pi),$$

where $c(t)[\beta_0 - \alpha_0] = F(t, \beta_0, \beta_0') - F(t, \alpha_0, \alpha_0')$. Also, letting $z = \beta_0 - \alpha_0$,

we see by (A_0^*) , (A_2^*) that $z(t) > 0$ on $[0, 2\pi]$ and for $0 < t < 2\pi$,

$$(3.11) \quad (a) \quad -z'' = -\beta_0'' + \alpha_0'' > F(t, \beta_0, \beta_0') - F(t, \alpha_0, \alpha_0') = c(t)z, \quad z(0) = z(2\pi),$$

$$(b) \quad -z''(i) > c(i)z(i), \quad i = 0, 2\pi, \quad z'(0) \leq z'(2\pi).$$

Furthermore, in case of (A_3^*) , we have, if $z'(0) > z'(2\pi)$,

choosing $\sigma = \beta'_0(i)$ and $\alpha'_0(i)$ respectively,

$$(3.12) \quad -z''(i) > c(i)z(i) + \frac{z'(0) - z'(2\pi)}{\pi}, \quad i = 0, 2\pi.$$

It is now easy to show that $m(t) \leq 0$ on $[0, 2\pi]$. If not, there exists a minimal $\lambda > 0$ and a $t_0 \in [0, 2\pi]$ such that

$$(3.13) \quad m(t) \leq \lambda z(t), \quad t \in [0, 2\pi] \quad \text{and} \quad m(t_0) = \lambda z(t_0).$$

If $t_0 \in (0, 2\pi)$, we have $-m'' + \lambda z'' < c(t_0)m - \lambda c(t_0)z = c(t_0)[m - \lambda z] = 0$, which is impossible since the function $m + \lambda z$ attains maximum at t_0 .

If $t_0 = i$, $i = 0, 2\pi$, we get using (3.10), (3.11) and (3.13),

$$m(0) = \lambda z(0), \quad m(2\pi) = \lambda z(2\pi), \quad m'(0) \leq \lambda z'(0) \quad \text{and} \quad m'(2\pi) \geq \lambda z'(2\pi).$$

Hence $\lambda z'(0) \geq m'(0) = m'(2\pi) \geq \lambda z'(2\pi)$ and by (3.11) (b), it follows that $z'(0) = z'(2\pi)$. Thus, by (3.11), for $i = 0, 2\pi$,

$$-m''(i) + \lambda z''(i) < c(i)[m(i) - \lambda z(i)] = 0,$$

a contradiction, as before.

Now let $\phi(t) = m(t) - \lambda z(t)$ so that $\phi(0) = 0$, $\phi(2\pi) = 0$, $\phi'(0) \leq 0$ and $\phi'(2\pi) \geq 0$. It is easy to obtain, using (3.12), $\phi''(i) > \frac{\rho}{\pi}$ where $\rho = z'(0) - z'(2\pi) > 0$, $i = 0, 2\pi$. Proceeding now as in the last part of the proof of Lemma 3.1, it is easy to prove that $m(t) \leq 0$ on $[0, 2\pi]$. Defining $n(t) = v(t) - u(t)$, and arguing in a similar manner, we can show that $n(t) \leq 0$ on $[0, 2\pi]$. This then shows that $u(t) \equiv v(t)$ on $[0, 2\pi]$ and the proof is complete.

We are now in a position to prove our main result.

Theorem 3.1 Consider the PBVP (3.1). Let the conditions (A_0) and (A_1) be satisfied. Then any one of the conditions (A_2) to (A_5) implies that there exists a solution u of (3.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ and $|u'(t)| \leq N$ on $[0, 2\pi]$, where N depends only on α, β and the Nagumo function.

Proof: Let us consider the PBVP

$$(3.14) \quad -u'' = H(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $H(t, u, u') = \lambda F(t, u, u') + (1 - \lambda)G(t, u)$, $\lambda \in [0, 1]$. It is easy to verify because of Lemma 3.2 that α_0, β_0 satisfy (A_0^*) to (A_5^*) with respect to H also. Consequently, for any $\lambda \in (0, 1)$, if u_λ is a solution of (3.13), we can show, by using the arguments as in Lemma 3.1, that $\alpha_0(t) \leq u_\lambda(t) \leq \beta_0(t)$, $t \in [0, 2\pi]$. Since H satisfies a Nagumo condition, it also follows that $|u'_\lambda(x)| \leq c$, $t \in [0, 2\pi]$, where c is independent of λ . Thus it is clear in view of Lemma 3.3 that for

$\lambda \in [0, 1)$ all possible solutions of (3.13) are such that $\alpha_0(t) \leq u_\lambda(t) \leq \beta_0(t)$ and $|u'_\lambda(t)| \leq c$, $t \in [0, 2\pi]$. Furthermore for $\lambda = 0$, the problem (3.13) has a unique solution. We can therefore choose a bounded, closed, convex subset B in the (u, u') space $H^1[0, 2\pi]$ such that the problem (3.13) has no solution on the boundary of B for $\lambda \in (0, 1)$ and has a unique solution in the interior of B for $\lambda = 0$. Hence by Leray-Schauder theory, the PBVP (3.13) has a solution u for $\lambda = 1$. This implies by Lemma 3.1 that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0, 2\pi]$ and as a result it follows that u is also a solution of PBVP (3.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0, 2\pi]$. (A_1) then implies that $|u'(t)| \leq N$, $t \in [0, 2\pi]$ where N is the Nagumo constant. The proof is therefore complete.

Remark 3.1. Clearly conditions (A_3) , (A_4) and (A_5) are weaker than (A_2) .

For (A_0) and (A_2) imply

$$\int_0^{2\pi} f(s, \beta(s), \beta'(s)) ds \leq \int_0^{2\pi} -\beta''(s) ds = \beta'(0) - \beta'(2\pi) \leq 0,$$

and similarly $\int_0^{2\pi} f(s, \beta(s), \beta'(0)) ds \leq 0$. The rest of the conditions in

(A_3) hold automatically since in this case it follows that $\alpha'(0) = \alpha'(2\pi)$

and $\beta'(0) = \beta'(2\pi)$. See (3.4) in the proof of Lemma 3.1. In the literature

condition (A_2) is employed to prove Theorem 3.1 by different techniques [1,6,7,8,10]. We remark also that π occurring in (A_3) is one half of the length of the interval $[0, 2\pi]$. Hence, when the interval is changed, π has to change suitably.

We offer below two examples to demonstrate that when (A_2) is not satisfied there is a solution to PBVP (3.1) because (A_5) holds.

Example 3.1 Consider the PBVP

$$(3.15) \quad u'' = (u')^{4/3} + p(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

Let us take $\alpha(t) = 1$ and $\beta(t) = 1 + \epsilon t(2\pi - t) + w \cos t + k$

where ϵ, w and $k > 0$ are to be chosen suitably, later. Clearly

$\alpha(0) = \alpha(2\pi)$, $\beta(0) = \beta(2\pi)$ and $\alpha'(0) = \alpha'(2\pi)$. However, $\beta'(0) = 2\pi\epsilon >$

$\beta'(2\pi) = -2\pi\epsilon$. Further $\alpha'' = 0$, $\alpha' = 0$, $\beta'' = -2\epsilon - w \cos t$ and $\beta'(t) =$

$\epsilon(2\pi - 2t) - w \sin t$. Hence, in order to verify $-\alpha'' \leq -(\alpha')^{4/3} - p(t)$ for

$t \in [0, 2\pi]$, $-\beta'' \leq -(\beta')^{4/3} - p(t)$ for $t \in (0, \pi)$ and condition $A_3(b)$ we need to have

$$(3.16) \quad \begin{aligned} -2\epsilon - w \cos t &\leq [\epsilon(2\pi - 2t) - w \sin t]^{4/3} + p(t), \\ 2\epsilon - w &\leq (\epsilon 2\pi)^{4/3} + p(0), \\ p(t) &\geq -[\epsilon(2\pi - 2t) - w \sin t]^{4/3} \end{aligned}$$

and $p(t) \leq 0$.

By choosing $w < 2\pi$, it can be shown that w, ϵ and $k > 0$ can be obtained so that there exist $p(t)$ which is 2π -periodic in t satisfying

$$(3.17) \quad p(t) \leq 0, \quad p(t) \geq -(2\epsilon + w \cos t), \quad p(t) \geq -[\epsilon(2\pi - 2t) - w \sin t]^{4/3}$$

and $p(0) \geq 2\epsilon - w - [2\pi\epsilon]^{4/3}$. Thus we see that in this example (A_2) does not hold while (A_5) does.

The next example does not depend on u' .

Example 3.2 Consider the PBVP

$$(3.18) \quad u'' = u^2 + p(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

where $p(t)$ is a 2π -periodic function to be determined later. Take $\alpha(t) = 1$ and $\beta(t) = 1 + \epsilon t(2\pi - t) + \epsilon \cos t + 2\epsilon$ where $\epsilon > 0$ is to be chosen suitably later. Clearly $\alpha(0) = \alpha(2\pi)$, $\alpha'(0) = \alpha'(2\pi)$, $\beta(0) = \beta(2\pi)$. But $\beta'(0) = 2\pi\epsilon > \beta'(2\pi) = -2\pi\epsilon$. To verify $-\alpha'' \leq -\alpha^2 - p(t)$, $-\beta'' \geq -\beta^2 - p(t)$ and $A_3(b)$ we need to have

$$(3.19) \quad p(t) \geq -\beta^2, \quad p(t) \geq -2\epsilon - \epsilon \cos t - \beta^2 \quad \text{and} \quad p(0) \geq -(1+3\epsilon)^2 + \epsilon.$$

But $2\epsilon + \epsilon \cos t \geq 0$ for $t \in [0, 2\pi]$. Thus, it suffices to have

$$(3.20) \quad -1 \geq p(t) \geq -\beta^2 \quad \text{and} \quad p(0) \geq -(1+3\epsilon)^2 + \epsilon.$$

It is easy to see that one can find a $p(t)$ satisfying these two inequalities.

A simple example is the problem

$$u'' = u^2 - \epsilon \cos t - (1 + \epsilon + \epsilon \cos t)^2$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

Clearly $u(t) = 1 + \epsilon + \epsilon \cos t$ is a solution to this problem satisfying $1 \leq u(t) \leq 1 + \epsilon t(2\pi - t) + \epsilon \cos t + 2\epsilon$.

Remark 3.2 In the case f in (3.1) is independent of u' the proof of Theorem 3.1 follows exactly the classical modified approach and is almost similar to the existence proof of first order PBVP (see [9] for details). That method of proof yields, in case f depends on u' , the following inequalities

$$(3.21) \quad \int_0^{2\pi} \left[f(s, \beta(s), q(u'_1(s))) + \frac{\beta(s) - u(s)}{1 + u^2(s)} \right] ds \leq 0 \quad \text{and}$$

$$\int_0^{2\pi} \left[f(s, \alpha(s), q(u'_1(s))) + \frac{\alpha(s) - u(s)}{1 + u^2(s)} \right] ds \geq 0.$$

Here $q(u') = \max \left[-c, \min\{u', c\} \right]$ with $c > \max \left[N, \max |\alpha'|, \max_t |\beta'| \right]$

where N is the Nagumo constant resulting from (A_1) , $u = (\pm)R_0 + u_1$ and u_1 is any solution of (2.2) with $u_0 = \pm R_0$. Verifying the inequalities (3.19) from the assumed inequalities, namely

$$(3.22) \quad \int_0^{2\pi} f(s, \beta(s), \beta'(s)) ds \leq 0 \quad \text{and} \quad \int_0^{2\pi} f(s, \alpha(s), \alpha'(s)) ds \geq 0,$$

remains an open question. If (3.20) implies (3.19) then, at least, the PBVP (3.1) can be proved in a direct way as in [9]. Even then we have to use the continuation argument to show that $|u'| < N$, (see [2, p. 32]). To extend this technique to other resonance cases, this continuation argument is not suitable. Employing Leray-Schauder theory as we have utilized in the proof of Theorem 3.1 appears to be natural and this requires that the problem for $\lambda = 0$ should have a unique solution (or the degree of the map corresponding to G is not equal to zero). This requirement is not necessarily true when one employs the classical modified approach and consequently a suitable change of this classical technique is essential. Our proof incorporates precisely this needed change which in turn solves all the difficulties that are encountered in discussing any resonance problem.

Remark 3.3 The condition (A_2) that is usually assumed does not indicate the interplay between the linear and nonlinear parts of the given problem whereas condition (A_3) does. For example, suppose that the BVP (3.1) is of nonresonance type, namely

$$-u'' = f(t, u, u'), \quad B_\mu u(\mu) = \alpha_\mu u(\mu) + (-1)^{\mu+1} \beta_\mu u'(\mu) = b_\mu, \quad u = 0, 2\pi,$$

where $\alpha_\mu, \beta_\mu, b_\mu \in \mathbb{R}$ and $\alpha_0, \alpha_1 \geq 0, \beta_0, \beta_1 > 0$. In this case (A_0) takes the form

($\bar{A}_0$)

$$\alpha, \beta \in C^2[0, 2\pi], \mathbb{R}, \alpha(t) \leq \beta(t), t \in [0, 2\pi] \text{ and for } 0 \leq t \leq 2\pi,$$

$$-\alpha'' \leq f(t, \alpha, \alpha'), B_\mu \alpha(\mu) \leq b_\mu, -\beta'' \geq f(t, \beta, \beta'), B_\mu \beta(\mu) \geq b_\mu.$$

Then since $N(L) = \{0\}$, we will have $Pu = \int_0^{2\pi} u(s) ds = 0$. Consequently the integral conditions in (A_3) are automatically satisfied. The "if" requirement demanded in (A_3) relative to the end point conditions does not exist because the boundary conditions of the problem will suffice for the proof. It is thus easy to see that our results cover the non-resonance case trivially. Furthermore, our method of proof offers another alternative which works for both resonance and nonresonance cases whereas the classical technique is good only for the nonresonance case as was explained in Remark 3.2.

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