STOCHASTIC ANALYSIS OF COMPRESSIBLE GAS LUBRICATION
SLIDER BEARING PROBLEM *

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ABSTRACT

By employing the theory of stochastic differential inequalities and a comparison result for the stochastic boundary value problem, the effects of roughness for nonlinear gas lubricated problem is analyzed. In particular, by summarizing analytic techniques, estimates on the absolute mean and root-mean-square deviations of a normalized load carrying capacity from the smooth case is obtained for a general nonlinear one-dimensional problem.

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1. INTRODUCTION:

In this paper, we investigate the effect of roughness in (compressible) lubricated slider bearing. Specifically, we present some analytical methods to estimate the deviation of the mean pressure from that developed when the surfaces are assumed to be smooth. The calculations are based on the investigation of a nonlinear stochastic boundary value problem associated with 1-dimensional Reynolds equation. Our basic tools are derived from the theory of stochastic differential inequalities [6] and a comparison theorem for such stochastic boundary value problems (SBVP for short). In this paper, we, however, do not dwell upon rigorous mathematical analysis but merely summarize and interpret relevant mathematical results details of which can be found in [2]. The main purpose of this paper is, therefore, to indicate a systematic procedure and provide some precise quantitative insight into the effect of roughness for nonlinear gas lubricated slider bearing problem. Results of similar nature have been reported in [3,4,5,7] for the case of linear incompressible case. Very recently, by linearising the transverse flow term and using simple geometry bearing, exact solutions and effects of surface roughness of a model of compressible lubrication equation for very thin film are investigated [8]. The deterministic nonlinear problem was treated rigorously in [1].

Consider a gas lubricated slider bearing governed by the nonlinear stochastic Reynolds equation in one-dimension

$$(H_1^3(x,\omega)p(x,\omega)p'(x,\omega))' = \Lambda(H_2(x,\omega)p(x,\omega))' \quad (1)$$

subject to the boundary conditions

$$p(0, \omega) = p(1, \omega) = 1 \quad (2)$$

with probability one (henceforth denoted by w.p. 1). Here $p(x, \omega)$ and $H_1(x, \omega)$ are real-valued random functions denoting, respectively, the unknown pressure and the known profile of the film thickness processes, ω is an element of sample space Ω and $x \in I = [0, 1]$. We shall assume that

$$H_1(x, \omega) = h(x) + \delta_1(x, \omega) + \delta_2(x, \omega) \quad (3)$$

and

$$H_2(x, \omega) = h(x) + v(\delta_1(x, \omega) - \delta_2(x, \omega)) \quad (4)$$

where $h(x) = \langle H_1(x, \omega) \rangle$ is the mean film thickness; δ_1 and δ_2 are the roughness profiles of the lower and upper surfaces; $v = (u_2 - u_1)/(u_1 + u_2)$, u_1, u_2 being the rolling or sliding speeds of the lower and upper surfaces; $\Lambda = 6\mu(u_1 + u_2)$ and μ is the viscosity of the lubricant. Furthermore, we assume that δ_1 and δ_2 are sample continuous stochastic processes satisfying

$$|\delta_1(x, \omega)| + |\delta_2(x, \omega)| < h(x) \text{ w.p. 1 on } I. \quad (5)$$

By setting $p^2 = 2u$, the SBVP (1) - (2) is transformed into the quasi-linear problem given by

$$(H_1^3(x, \omega)u'(x, \omega))' = \Lambda(H_2(x, \omega)\sqrt{2u(x, \omega)})' \quad (6)$$

with

$$u(0, \omega) = u(1, \omega) = \frac{1}{2}. \quad (7)$$

2. STOCHASTIC BOUNDARY VALUE PROBLEM:

Suppose that $\alpha(x, \omega)$ is a sample continuously differentiable stochastic process on I satisfying the random differential inequality

$$(H_1^3(x, \omega) \alpha'(x, \omega))' \geq \Lambda(H_2(x, \omega) \sqrt{2\alpha(x, \omega)})', \text{ w.p. } 1, \quad (8)$$

and

$$\alpha(0, \omega) \leq \frac{1}{2}, \alpha(1, \omega) \leq \frac{1}{2}, \text{ w.p. } 1. \quad (9)$$

Then $\alpha(x, \omega)$ is referred to as a lower sample solution process for SBVP (6) - (7). One obvious choice of α is $\alpha(x, \omega) \equiv \frac{1}{2}$ if $H_2'(x, \omega) \leq 0$, w.p. 1. One can define an upper sample solution process $\beta(x, \omega)$ in an analogous manner with inequalities reversed in (8) and (9). A choice of β is given by

$$\beta(x, \omega) = \frac{1}{2} \left[1 + \Lambda \int_0^t H_1^{-1}(s, \omega) H_2(s, \omega) ds \right]^2.$$

In [2], for a more general class of nonlinear SBVP (which certainly includes our present problem of interest), it was demonstrated that the unique sample solution process of SBVP (6) - (7) can be constructed by two iterative sequences; one starting from a lower sample solution α converging uniformly and monotonically upward w.p. 1 to the sample solution, and the other starting from an upper sample solution β converging uniformly and monotonically downwards w.p. 1 to the sample solution provided $\alpha \leq \beta$. Thus with our choice of α and β , there exist two sequences $\{\alpha_n(x, \omega)\}$, $\{\beta_n(x, \omega)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ which converge uniformly and monotonically w.p. 1 to the sample solution process $u(x, \omega)$ of SBVP (6) - (7), and

$$\alpha(x, \omega) \leq \alpha_1(x, \omega) \leq \dots \leq \alpha_n(x, \omega) \leq u(x, \omega) \leq \beta_n(x, \omega) \leq \dots \leq \beta_1(x, \omega) \leq \beta(x, \omega) \quad (10)$$

on I w.p. 1. The iterates α_n and β_n are solutions of certain auxiliary linear stochastic boundary value problems (see, Appendix-I).

3. ESTIMATES ON DEVIATION:

In this section, we obtain an upper estimate on deviation

$$\Delta u(x, \omega) = u(x, \omega) - m(x), \quad (11)$$

where $u(x, \omega)$ is a sample solution process of (6) - (7) and $m(x)$ is the solution of corresponding deterministic boundary value problem in which the surfaces are assumed smooth, that is, $m(x)$ is the solution of

$$(h^3(x)m')' = \Lambda(h(x)\sqrt{2m})' \quad (12)$$

such that

$$m(0) = m(1) = \frac{1}{2}, \quad (13)$$

where h and Λ are as defined in (6).

Setting

$$V(x, u-m) = (u-m)^n, \text{ for any integer } n \geq 1, \quad (14)$$

we obtain the following random differential inequality (see the Appendix-II)

$$(H(x, \omega)V'(x, u-m))' \geq q(x, \omega)V(x, u-m) + \lambda(x, \omega). \quad (15)$$

From (7), (13) and (14), we have

$$V(0, u(0, \omega) - m(0)) = V(1, u(1, \omega) - m(1)) = 0. \quad (16)$$

By the application of comparison theorem 4.2 in [2] (see, Remark 4.1) we get

$$[u(x, \omega) - m(x)]^n \leq r(x, \omega), \quad (17)$$

where $u(x, \omega)$ and $m(x)$ are as defined in (11); $r(x, \omega)$ is the sample solution process of comparison SBVP

$$(H(x, \omega)r')' = q(x, \omega)r + \lambda(x, \omega) \quad (18)$$

with

$$r(0, \omega) = r(1, \omega) = 0. \quad (19)$$

We note that $q(x, \omega)$ and $H(x, \omega) > 0$ w.p. 1, and hence

$$r(x, \omega) = \int_0^1 G(x, s, \omega) \lambda(s, \omega) ds \quad (20)$$

where $G(x, s, \omega)$ is the stochastic Green's function associated with SBVP (18 - (19)). From (17) and (20), we have

$$[u(x, 1) - m(x)]^n \leq \int_0^1 G(x, s, \omega) \lambda(s, \omega) ds. \quad (21)$$

We further note that $G(x, s, \omega)$ depends continuously (in the sense of sample) on u and m where $\alpha(x, \omega) \leq u \leq \beta(x, \omega)$ and $\bar{\alpha}(x) \leq m \leq \bar{\beta}(x)$, respectively (see Appendix-II). Therefore, from (21), we obtain

$$[u(x, \omega) - m(x)]^n \leq \int_0^1 G^*(x, s, \omega) \lambda^*(s, \omega) ds \quad \text{w.p. 1,} \quad (22)$$

where

$$G^*(x, s, \omega) = \sup_{\substack{\alpha \leq u \leq \beta \\ \bar{\alpha} \leq m \leq \bar{\beta}}} \{G(x, s, \omega)\} \quad \text{and} \quad \lambda^*(s, \omega) = \sup_{\bar{H}} \{\lambda(s, \omega)\}.$$

We note that $G^*(x, s, \omega)$ and $\lambda^*(x, s, \omega)$ are independent of parameters u and m .

We remark that for $n = 2$, (22) gives an estimate on the quadratic deviation. Other sample estimates on the various types of deviations can be obtained, similarly.

Sample estimates on various kinds of deviations can be obtained, directly, from problems (6) - (7) and (12) - (13). In order to obtain such estimates, we set

$$V(x, \omega) = u(x, \omega) - m(x), \quad (23)$$

where $u(x, \omega)$ and $m(x)$ are as defined in (11). From (23), we observe that

$$V(0, \omega) = V(1, \omega) = 0. \quad (24)$$

Now, after some computations (see Appendix-II), we obtain

$$(H_1^3(x, \omega) V'(x, \omega))' = (q_1(x, \omega) V(x, \omega))' + f'(x, \omega). \quad (25)$$

After solving the SBVP (24) - (25) and further estimating the sample solution process, we arrive at (see, Appendix-III)

$$|u(x, \omega) - m(x)| \leq \int_0^1 |f(s, \omega)| H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds. \quad (26)$$

Moreover, from the definition of $f(x, \omega)$ (see, Appendix-II) and (26),

we have

$$|u(x, \omega) - m(x)| \leq \Lambda \int_0^1 \left[|H_2(s, \omega) - h(s)| \sqrt{2\bar{\beta}(s)} + |H_1^3(s, \omega) - h^3(s)| h^{-2}(s) v(h(s), \bar{\beta}(s), \bar{N}) \right] H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \quad \text{w.p. 1}, \quad (27)$$

where

$$v(h(s), \bar{\beta}(s), \bar{N}) = \left| \frac{h(s) \sqrt{2\bar{\beta}(s)} - h(0) + \Lambda^{-1} h^3(0) \bar{N}}{h(s)} \right| \quad (\text{see, Appendix-III}).$$

4. ESTIMATES ON MEAN AND MEAN-SQUARE DEVIATIONS:

Sample estimates of Section 3 can be utilized to obtain estimates on mean and mean-square deviation. By taking mean both sides of random inequality (22), we obtain

$$\langle [u(s, \omega) - m(x)]^n \rangle \leq \int_0^1 \langle G^*(x, s, \omega) \lambda(s, \omega) \rangle ds.$$

This, together with the application of Schwartz's inequality, yields

$$\langle [u(x, \omega) - m(x)]^n \rangle \leq \left[\int_0^1 \langle (G^*(x, s, \omega))^2 \rangle ds \int_0^1 \langle (\lambda^*(s, \omega))^2 \rangle ds \right]^{\frac{1}{2}}. \quad (28)$$

For $n = 2$, (28) gives estimates on absolute mean-square deviation of $u(x, \omega)$ from the smooth case.

From the above analysis, we remark that the estimate on the mean of solution process of comparison SBVP (18) - (19) is the estimate on absolute mean-square deviation. From (28), we observe that the estimate on mean-square deviation are not, directly, related with the roughness profile processes, however, it is related through the solution process of (18) - (19) whose coefficients are stochastic processes of roughness profile processes (see Appendix-II).

On the other hand, if we employ the sample estimate on deviation in (27), one can obtain the estimates on the absolute mean and mean-square deviation in terms of the roughness correlation of $\delta_1(x, \omega) + \delta_2(x, \omega)$ and $\delta_2(x, \omega) - \delta_1(x, \omega)$.

To obtain the absolute mean-deviation, we take mean and then apply Schwartz's inequality to the expression in (27) (see, Appendix-III). Thus, we have

$$\langle |u(x, \omega) - m(x)| \rangle \leq \sqrt{2} \ A \left[2v^2 \int_0^1 R_\eta(s) \bar{\beta}(s) ds + 49 \int_0^1 R_\xi(s) v^2(h(s), \bar{\beta}(s), \bar{N}) ds \right]. \quad (29)$$

where $R_\eta(s) = \langle (\delta_1(s, \omega) - \delta_2(s, \omega))^2 \rangle$, $R_\xi(s) = \langle (\delta_1(s, \omega) + \delta_2(s, \omega))^2 \rangle$. For

further detailed deviation of (29), see Appendix-III. Similarly, squaring the expression (27), a bound for the absolute mean-square deviation can be obtained, analogously.

Let $|\Delta W|$ denote the mean deviation of the load carrying capacity defined as

$$|\Delta W| = \int_0^1 \langle |u(s, \omega) - m(s)| \rangle ds. \quad (30)$$

Then, noting (29), the unit mean-deviation is given by

$$\rho = \left| \frac{\Delta W}{W_m} \right| \leq \frac{\max_{x \in I} \langle |u(x, \omega) - m(x)| \rangle}{\int_0^1 m(s) ds} \quad (31)$$

Furthermore, by noting (28), the unit mean-square deviation can be derived, similarly.

We recall that our discussion in sections 2, 3 and 4, is with respect to the transformed SBVP (6) - (7). However, our main objective is to study SBVP (1) - (2). Therefore, we need to obtain the estimates on the deviation of $p(x, \omega)$ from the smooth case, that is,

$$(h^3(x)p_0 p_0')' = \Lambda(h(x)p_0)' \quad (32)$$

and

$$p_0(0) = p_0(1) = 1. \quad (33)$$

From the transformation that was discussed in section 1, the solution processes $p(x, \omega)$, $u(x, \omega)$, $m(x)$ and $p_0(x)$ of (1) - (2), (6) - (7), (12) - (13) and (32) - (33), respectively, are related by

$$|p(x, \omega) - p_0(x)| \leq |u(x, \omega) - m(x)|. \quad (34)$$

For details about the validity of (34), see Appendix-IV. From (34), we get

$$\langle |p(x, \omega) - p_0(x)| \rangle \leq \langle |u(x, \omega) - m(x)| \rangle. \quad (35)$$

From (34) and (22) the sample bounds for quadratic deviation of $p(x, \omega)$ from $p_0(x)$ is given by

$$|p(x, \omega) - p_0(x)|^2 \leq \int_0^1 G^*(x, s, \omega) \lambda(s, \omega) ds \quad \text{w.p. 1.} \quad (36)$$

Similarly, from (27) the sample estimate for deviation is

$$|p(x, \omega) - p_0(x)| \leq \Lambda \int_0^1 \left[|H_2(s, \omega) - h(s)| \sqrt{2\bar{\beta}(s)} + |H_1^3(s, \omega) - h^3(s)| h^{-2}(s) \right. \\ \left. v(h(s), \bar{\beta}(s), \bar{N}) H_1^{-3}(s, \omega) \exp \left[\int_s^x q_1(\theta, \omega) H_1^{-3}(\theta, \omega) d\theta \right] ds \right]. \quad (37)$$

From (36) and (37), the estimates for the absolute mean and mean-square deviation of $p(x, \omega)$ with $p_0(x)$ are given by (29) and (28), respectively. From (29), (30), (31), (34), the estimate for the unit mean-deviation is given by

$$\rho = \frac{\int_0^1 \langle |p(s, \omega) - p_0(s)| \rangle ds}{\int_0^1 p_0(s) ds} \\ \leq \frac{\max_{x \in I} \langle |p(x, \omega) - p_0(x)| \rangle}{\int_0^1 p_0(s) ds}$$

$$\rho \leq \frac{\sqrt{2} \Lambda \left[2v^2 \int_0^1 R_\eta(s) \bar{\beta}(s) ds + 49 \int_0^1 R_\xi(s) v^2(h(s), \bar{\beta}(s), \bar{N}) ds \right]}{\int_0^1 p_0(s) ds}.$$

5. APPENDIX:

In this section, we present detail discussion about the derivation of some of the estimates and relations that are given in the main body of the paper.

APPENDIX-I: We note the SBVP (6) - (7) is equivalent to

$$u''(x, \omega) = g(x, u(x, \omega), u'(x, \omega), \omega) \quad (\text{A.I. 1})$$

with

$$u(0, \omega) = u(1, \omega) = \frac{1}{2}, \quad (\text{A.I. } 2)$$

where

$$\begin{aligned} g(x, u, u', \omega) = & H_1^{-3}(x, \omega) \left[\Lambda H_2(x, \omega) \left[2u \right]^{-\frac{1}{2}} - 3H_1'(x, \omega) H_1^2(x, \omega) \right] u' \\ & + \Lambda H_1^{-3}(x, \omega) H_2'(s, \omega) \sqrt{2u}. \end{aligned} \quad (\text{A.I. } 3)$$

As we mentioned in Section 2,

$$\alpha(x, \omega) = \frac{1}{2} \quad (\text{A.I. } 4)$$

and

$$\beta(x, \omega) = \frac{1}{2} \left[1 + \Lambda \int_0^x H_2(s, \omega) H_1^{-3}(s, \omega) ds \right]^2 \quad (\text{A.I. } 5)$$

can be selected as lower and upper solution processes of (A.I. 1) - (A.I. 2). From the nature of stochastic processes H_2 and H_1 , for the above choice of α and β , we have

$$\alpha(x, \omega) \leq \beta(x, \omega) \quad \text{w.p. } 1 \quad x \in I. \quad (\text{A.I. } 6)$$

From the definition of $g(x, u, u', \omega)$, it is obvious that $g(x, u, u', \omega)$ satisfies a Nagumo condition [1] with respect to α and β with a Nagumo function being linear (say, $(L_1(\omega)|u'| + L_2(\omega))$). As a consequence of the Nagumo condition, one can find a positive random number $N(\omega)$ such that

$$|u'(x, \omega)| \leq N(\omega) \quad \text{w.p. } 1 \quad x \in I. \quad (\text{A.I. } 7)$$

We further note that the partial derivative of $g(x, u, u', \omega)$ with respect to u is equal to

$$g_u(x, u, u', \omega) = \Lambda H_1^{-3}(x, \omega) \left[H_2'(x, \omega) (2u)^{-1/2} - H_2(x, \omega) (2u)^{-3/2} u' \right]$$

and moreover

$$|g_u(x, u, u', \omega)| \leq Q(C, \omega) \quad \text{w.p. 1} \quad (\text{A.I. 8})$$

whenever $u \in [\alpha(x, \omega), \beta(x, \omega)]$ w.p. 1 and $|u'| \leq C$

where $Q(C, \omega) = \max_{x \in I} \{ \Lambda H_1^{-3}(x, \omega) [|H_2'(x, \omega)| + |H_2(x, \omega)| C] \}$. From the regu-

larity condition on H_1 and H_2 , we conclude that

$$\frac{Q(C, \omega)}{C^2} \rightarrow 0 \quad \text{w.p. 1 as } C \rightarrow \infty \quad (\text{A.I. 9})$$

$$\frac{\text{linear Nagumo function in } s}{s^2} \rightarrow \text{finite w.p. 1 as } s \rightarrow \infty \quad (\text{A.I. 10})$$

Consider the auxiliary linear SBVP

$$u'' = H(x, u, u', \omega) \quad (\text{A.I. 11})$$

with

$$u(0, \omega) = u(0, 1) = \frac{1}{2}, \quad (\text{A.I. 12})$$

$$\begin{aligned} H(x, u, u', \omega) = & H_1^{-3}(x, \omega) [\Lambda H_2(x, \omega) (2z(x, \omega))^{-1/2} - 3H_1'(x, \omega) H_1^2(x, \omega)] u' \\ & + \Lambda H_1^{-3}(x, \omega) H_2'(x, \omega) \sqrt{2z(x, \omega)} + Q(C, \omega) (u - z(x, \omega)), \end{aligned} \quad (\text{A.I. 13})$$

where

$$z(x, \omega) \in [\alpha(x, \omega), \beta(x, \omega)] \quad \text{w.p. 1, } x \in I.$$

From (A.I. 4) - (A.I. 10), one can conclude that the SBVP (A.I. 11) -

(A.I. 12) has unique solution process [2]. Moreover, the solution process of (A.I. 11) - (A.I. 12) lies in the segment $[\alpha(x, \omega), \beta(x, \omega)]$ w.p. 1 for $x \in I$. The iterates $\alpha_n(x, \omega)$ and $\beta_n(x, \omega)$ are defined by the solution process of the auxiliary SBVP (A.I. 11) - (A.I. 12) with $z(x, \omega)$ in $H(x, u, u', \omega)$ is replaced by $z_{n-1}(x, \omega)$, where $z_{n-1}(x, \omega)$ stand for either $\alpha_{n-1}(x, \omega)$ or $\beta_{n-1}(x, \omega)$ with $\alpha_0(x, \omega) = \alpha(x, \omega)$ and $\beta_0(x, \omega) = \beta(x, \omega)$.

For the deterministic boundary value problem (12) - (13) corresponding to the smooth surfaces, one can verify the fact that

$$\bar{\alpha}(x) = \frac{1}{2} \quad (\text{A.I. 14})$$

and

$$\bar{\beta}(x) = \frac{1}{2} \left[1 + \Lambda \int_0^x h^{-2}(s) ds \right]^2 \quad (\text{A.I. 15})$$

are lower and upper solutions of the problem (12) - (13). For more details, see [1]. We further note that $\bar{N}$ being the Nagumo number for BVP (12) - (13), that is, $|m'(t)| \leq \bar{N}$ on I for any solution $m(t)$ of (12) - (13).

APPENDIX-II: Derivation of the Random Differential Inequality:

By differentiating $V(x, u-m)$ in (14) with respect to x , we obtain

$$V'(x, u-m) = n(u-m)^{n-1}(u'-m') \quad (\text{A.II. 1})$$

multiplying the equation (A.II. 1) by $H_1^3(x, \omega)$, we get

$$H_1^3(x, \omega) V' = H_1^3(x, \omega) (u'-m') n(u-m)^{n-1} \quad (\text{A.II 2})$$

Again, by differentiating the above equation both sides with respect to x , we have

$$(H_1^3(x, \omega) v')' = (H_1^3(x, \omega) (u' - m'))' n(u-m)^{n-1} + n(n-1) H_1^3(x, \omega) (u' - m')^2 (u-m)^{n-2}. \quad (\text{A.II. 3})$$

We note that in the above discussion, it is assumed that $n \geq 2$. Otherwise, for $n = 1$, the second term in the right-hand side of (A.II. 3) is assumed to be zero. Our objective is to complete the expression in the right-hand side with respect to problems (6) - (7) and (12) - (13), that is, we will replace $(H_1^3(x, \omega) u')'$ and $(h^3(x) m')'$ by $\Lambda \left[H_2(x, \omega) \sqrt{2u} \right]'$ and $\Lambda \left[h(x) \sqrt{2m} \right]'$, respectively. For this purpose, by adding $\Lambda \left[H_2(x, \omega) \sqrt{2m} \right]' - \Lambda \left[H_2(x, \omega) \sqrt{2u} \right]' = 0$ and $(h^3(x) m')' - \Lambda \left[h(x) \sqrt{2m} \right]' = 0$ to the right-hand side of (A.II. 3) and rearranging the terms, we get

$$\begin{aligned} (H_1^3(x, \omega) v')' &= \left[(H_1^3(x, \omega) u')' - \Lambda \left[h(x) \sqrt{2m} \right]' + (h^3(x) m')' - (H_1^3(x, \omega) m')' \right. \\ &\quad \left. + \Lambda \left[H_2(x, \omega) \sqrt{2m} \right]' - \Lambda \left[H_2(x, \omega) \sqrt{2u} \right]' \right] n(u-m)^{n-1} \\ &\quad + n(n-1) (u-m)^{n-2} (u' - m')^2 H_1^3(x, \omega) \\ &= \left[\Lambda \left[H_2(x, \omega) \sqrt{2u} \right]' - \Lambda \left[H_2(x, \omega) \sqrt{2m} \right]' + \Lambda \left[[H_2(x, \omega) - h(x)] \sqrt{2m} \right]' \right. \\ &\quad \left. - \left[[H_1^3(x, \omega) - h^3(x)] m' \right] \right] n(u-m)^{n-1} \\ &\quad + n(n-1) (u-m)^{n-2} (u' - m')^2 H_1^3(x, \omega) \end{aligned}$$

$$\begin{aligned}
&= \left[\Lambda \left(H_2(x, \omega) \left[\sqrt{2u} - \sqrt{2m} \right] \right)' + \Lambda \left(\left[H_2(x, \omega) - h(x) \right] \sqrt{2m} \right)' \right. \\
&\quad \left. - \left(\left[H_1^3(x, \omega) - h^3(x) \right] m' \right)' \right] n(u-m)^{n-1} \\
&\quad + n(n-1)(u-m)^{n-2} (u'-m')^2 H_1^3(x, \omega). \quad (\text{A.II. 4})
\end{aligned}$$

Since (see, [6])

$$\sqrt{2u} - \sqrt{2m} = \left(\int_0^1 \left[2(su + (1-s)m) \right]^{-1/2} ds \right) (u-m), \quad (\text{A.II. 5})$$

from (A.II. 4) and (A.II. 5), we get

$$\begin{aligned}
(H_1^3(x, \omega) V')' &= \left[\Lambda \left(H_2(x, \omega) \left(\int_0^1 \left[2(su + (1-s)m) \right]^{-1/2} ds \right) (u-m) \right)' \right] n(u-m)^{n-1} \\
&\quad + F_0(x, \omega), \quad (\text{A.II. 6})
\end{aligned}$$

where

$$\begin{aligned}
F_0(x, \omega) &= \left[\Lambda \left(\left[H_2(x, \omega) - h(x) \right] \sqrt{2m} \right)' - \left(\left[H_1^3(x, \omega) - h^3(x) \right] m' \right)' \right] n(u-m)^{n-1} \\
&\quad + n(n-1)(u-m)^{n-2} (u'-m')^2 H_1^3(x, \omega) \\
&= n f'(x, \omega) (u-m)^{n-1} + n(n-1)(u-m)^{n-2} (u'-m')^2 H_1^3(x, \omega); \\
f(x, \omega) &= \Lambda \left(H_2(x, \omega) - h(x) \right) \sqrt{2m} - \left(H_1^3(x, \omega) - h^3(x) \right) m'. \quad (\text{A.II. 7})
\end{aligned}$$

We observe that $F_0(x, \omega)$ and $f'(x, \omega)$ depend on the parameters u, u', m

and m' . By performing the differentiating operation in the right-hand side of (A.II. 6) and using (A.II. 1), we have

$$(H_1^3(x, \omega)V')' = q_1(x, \omega)V' + n(q_1(x, \omega))'V + F_0(x, \omega), \quad (\text{A.II. 8})$$

where

$$q_1(x, \omega) = \Lambda H_2(x, \omega) \int_0^1 \left[2(su + (1-s)m) \right]^{-\frac{1}{2}} ds. \quad (\text{A.II. 9})$$

We note that $q_1(x, \omega)$ depends on u and m . Let $\gamma(x, \omega)$ be a stochastic process defined by

$$\sup_{(u, u', m, m') \in \bar{B}} \left\{ n | (q_1(x, \omega))' | \right\} < \gamma(x, \omega), \quad (\text{A.II. 10})$$

where

$$\bar{B} = \left\{ (u, u', m, m') \in \mathbb{R}^4 : (u, u', m, m') \in [\alpha(x, \omega), \beta(x, \omega)] \times [-N(\omega), N(\omega)] \times [\bar{\alpha}(x), \bar{\beta}(x)] \times [-\bar{N}, \bar{N}], x \in I \right\};$$

$\alpha(x, \omega), \beta(x, \omega), \bar{\alpha}(x), \bar{\beta}(x), N(\omega)$, and $\bar{N}$ are as defined in Appendix-I.

Now by adding and subtracting $\gamma(x, \omega)V$ to the right-hand side of (A.II. 8),

we get

$$(H_1^3(x, \omega)V')' = q_1(x, \omega)V' + \left[\gamma(x, \omega) + n(q_1(x, \omega))' \right] V + F_0(x, \omega) - \gamma(x, \omega)V. \quad (\text{A.II. 11})$$

Now, multiplying both sides of (A.II. 11) by $\exp \left[-\int_0^x q_1(\theta, \omega) H_1^{-3}(\theta, \omega) d\theta \right]$,

rewriting and rearranging some of the terms, we obtain

$$(H(x, \omega)V')' \geq q(x, \omega)V + \lambda(x, \omega) \quad (\text{A.II. 12})$$

where

$$H(x, \omega) = H_1^3(x, \omega) \exp \left[-\int_0^x K(\theta, \omega) d\theta \right] \quad (\text{A.II. 13})$$

$$q(x, \omega) = \left[\gamma(x, \omega) + n(q_1(x, \omega))' \right] \exp \left[-\int_0^x K(\theta, \omega) d\theta \right], \quad (\text{A.II. 14})$$

$$\lambda(x, \omega) = \left[F_0(x, \omega) - \gamma(x, \omega) (u-m)^n \right] \exp \left[-\int_0^x K(\theta, \omega) d\theta \right], \quad (\text{A.II. 15})$$

$$K(x, \omega) = q_1(x, \omega) H_1^{-3}(x, \omega), \quad (\text{A.II. 16})$$

and $q_1(x, \omega)$ is as defined in (A.II. 9). From (A.II. 6), (A.II. 9), (A.II. 14), (A.II. 15) and (A.II. 16), it is obvious that the coefficients $H(x, \omega)$, $q(x, \omega)$ and $\lambda(x, \omega)$ depend on the parameters (u, u', m, m') . Moreover, they are sample continuous on $\bar{B}$ in (A.II. 10). Hence completing the validity of random differential inequality (15).

As noted before, for $n = 1$ in (14), the second term in (A.II. 3) is zero. Therefore, in this case $F_0(x, \omega)$ in (A.II. 6) is equal to $f'(x, \omega)$ in (A.II. 7) which depends on only two parameters m and m' . In the light of this remark, equation (A.II. 6) reduces to (25).

APPENDIX-III: Derivations of (26), (27) and (29):

By sample integrating (25), we arrive at

$$H_1^3(x, \omega) V' = q_1(x, \omega) V + f(x, \omega) + c_0 \quad (\text{A.III. 1})$$

where c_0 is a constant of integration. By employing a method of variation of parameter, the general solution process of (A.III. 1) is given by

$$V(x, \omega) = \exp \left[\int_0^x K(\theta, \omega) d\theta \right] c_1 + \int_0^x H_1^{-3}(s, \omega) \left[f(s, \omega) + c_0 \right] \exp \left[\int_0^s K(\theta, \omega) d\theta \right] ds, \quad (\text{A.III.2})$$

where $K(x, \omega)$ is as defined in (A.II. 16). This together with boundary conditions in (24), yields

$$V(0, \omega) = c_1 = 0$$

and

$$V(1, \omega) = 0 = c_0 \int_0^1 H_1^{-3}(s, \omega) \exp \left[\int_s^1 K(\theta, \omega) d\theta \right] ds \\ + \int_0^1 f(s, \omega) H_1^{-3}(s, \omega) \exp \left[\int_0^1 K(\theta, \omega) d\theta \right] ds.$$

Hence

$$c_0 = - \frac{\int_0^1 f(s, \omega) H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds}{\int_0^1 H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds}.$$

From this, (A.III. 2) reduces to

$$V(x, \omega) = \int_0^x f(s, \omega) H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \\ - \frac{\left[\int_0^x H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] \left[\int_0^1 f(s, \omega) H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right]}{\int_0^1 H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds}.$$

Splitting each of the integrals from 0 to 1 into two parts, from 0 to x and x to 1, the above equation reduces to

$$V(x, \omega) = \left[\int_0^x f(s, \omega) H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] \left[\int_x^1 H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right]$$

$$- \left[\int_0^x H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] \left[\int_x^1 f(s, \omega) H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] /$$

$$\left(\int_0^1 H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right)$$

which implies that

$$|V(x, \omega)| \leq \left[\int_0^x |f(s, \omega)| H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] \left[\int_x^1 H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right]$$

$$+ \left[\int_0^x H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] \left[\int_x^1 |f(s, \omega)| H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right] /$$

$$\left(\int_0^1 H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds \right)$$

$$\leq \int_0^1 |f(s, \omega)| H_1^{-3}(s, \omega) \exp \left[\int_s^x K(\theta, \omega) d\theta \right] ds.$$

This together with (23) establishes the validity of inequality (26).

To establish the validity of (27), we first note from the definition of $f(x, \omega)$ in (A.II. 7) that

$$|f(x, \omega)| \leq \Lambda |H_2(x, \omega) - h(x)| \sqrt{2m(x)} + |H_1^3(x, \omega) - h^3(x)| |m'(x)|, \quad (\text{A.III. 3})$$

where $m(x)$ is the solution of (12) - (13). From Appendix-I, we further

note that

$$\bar{\alpha}(x) \leq m(x) \leq \bar{\beta}(x) \quad \text{for } x \in I \quad (\text{A.III. 4})$$

where $\bar{\alpha}$ and $\bar{\beta}$ are as defined in (A.I. 14) and (A.I. 15), respectively. Furthermore, by integrating (12) from 0 to x and using (13), we obtain

$$h^3(x)m'(x) - h^3(0)m'(0) = \Lambda h(x)\sqrt{2m(x)} - \Lambda h(0)$$

from which we have

$$\begin{aligned} |m'(x)| &\leq \Lambda h^{-2}(x) \left[\frac{h(x)\sqrt{2\bar{\beta}(x)} - h(0) + \Lambda^{-1} h^3(0)\bar{N}}{h(x)} \right] \\ &\leq \Lambda h^{-2}(x) v(h(x), \bar{\beta}(x), \bar{N}), \end{aligned} \quad (\text{A.III. 5})$$

in view of the fact that $|m'(x)| \leq \bar{N}$ for all $x \in I$ and (A.III. 4), where $v(h(x), \bar{\beta}(x), \bar{N})$ is defined by (27) and $\bar{N}$ is defined in Appendix I. From (A.III. 3) and (A.III. 5), inequality (26) reduces to (27).

From (27), by taking first mean then applying Schwartz's inequality, we obtain

$$\begin{aligned} \langle |u(x, \omega) - m(x)| \rangle &\leq \Lambda \int_0^1 \left(|H_2(s, \omega) - h(s)| \sqrt{2\bar{\beta}(s)} + |H_1^3(s, \omega) - h^3(s)| h^{-2}(s) \right. \\ &\quad \left. v(h(s), \bar{\beta}(s), \bar{N}) \right)^2 ds \int_0^1 \left(H_1^{-6}(s, \omega) \exp \left[2 \int_s^x K(\theta, \omega) d\theta \right] \right)^{1/2} ds \\ &\leq \sqrt{2} \Lambda \left[\int_0^1 \left(H_1^{-6}(s, \omega) \exp \left[\int_s^1 K(\theta, \omega) d\theta \right] \right)^{1/2} ds \right]^{1/2} \left[2 \int_0^1 \langle H_2(s, \omega) - h(s) \rangle^2 \right. \\ &\quad \left. \bar{\beta}(s) ds + \int_0^1 \left(H_1^3(s, \omega) - h^3(s) \right)^2 h^{-4}(s) v^2(h(s), \bar{\beta}(s), \bar{N}) ds \right]^{1/2} \quad (\text{A.III. 6}). \end{aligned}$$

Since

$$H_1^3(x, \omega) - h^3(x) \leq 7 |\delta_1(x, \omega) + \delta_2(x, \omega)| h^2(x) \quad (\text{A.III. 7})$$

whenever $|\delta_1(x, \omega) + \delta_2(x, \omega)| < h(x)$. From (A.III. 7), we have

$$\langle (H_1^3(x, \omega) - h^3(x))^2 \rangle \leq 49 h^4(x) \langle (H_1(x, \omega) - h(x))^2 \rangle \quad (\text{A.III. 8})$$

From (3), (4), (A.III. 7), (A.III. 8) and definitions of $R_\eta(x)$ and $R_\xi(x)$ in (29), inequality (A.III. 6) reduces to

$$\langle |u(x, \omega) - m(x)| \rangle \leq \sqrt{2} A \left[2v^2 \int_0^1 R_\eta(s) \bar{\beta}(s) ds + 49 \int_0^1 R_\xi(s) v^2 (h(s), \bar{\beta}(s), \bar{N}) ds \right]^{\frac{1}{2}}$$

where

$$A = \Lambda \left[\int_0^1 \langle H_1^{-6}(s, \omega) \exp \left[\int_s^1 K(\theta, \omega) d\theta \right] \rangle ds \right]^{\frac{1}{2}}.$$

This completes the verification of (29).

APPENDIX-IV: Derivation of (34):

Let $p(x, \omega)$, $u(x, \omega)$, $m(x)$ and $p_0(x)$ be solution processes of (1) - (2), (6) - (7), (12) - (13) and (32) - (33), respectively. From the transformation defined in Section 1, we have

$$|p(x, \omega) - p_0(x)| = |\sqrt{2u(x, \omega)} - \sqrt{2m(x)}| \quad \text{for } x \in I. \quad (\text{A.IV. 1})$$

On the other hand, from the generalized mean-value theorem [6], we get

$$\sqrt{2u(x, \omega)} - \sqrt{2m(x)} = \int_0^1 \left[2 \left(su(x, \omega) + (1-s)m(x) \right) \right]^{-\frac{1}{2}} ds (u(x, \omega) - m(x)). \quad (\text{A.IV 2})$$

Further, we note that

$$\left| \int_0^1 \left[2 \left(s u(x, \omega) + (1-s) m(x) \right) \right]^{-\frac{1}{2}} ds \right| \leq 1 \quad \text{w.p. 1} \quad (\text{A.IV } 3)$$

Whenever $u(x, \omega) \in [\alpha(x, \omega), \beta(x, \omega)]$, $m(x) \in [\bar{\alpha}(x), \bar{\beta}(x)]$

for $x \in I$; $\alpha, \beta, \bar{\alpha}$, and $\bar{\beta}$ as defined in Appendix-I. From (A.IV. 1), (A.IV 2) and (A.IV. 3), we have

$$|p(x, \omega) - p_0(x)| \leq |u(x, \omega) - m(x)| \quad \text{w.p. 1 for } x \in I.$$

This completes the proof of inequality (34).

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