ON THE FUNDAMENTAL THEORY OF NONLINEAR SECOND ORDER
STOCHASTIC BOUNDARY VALUE PROBLEMS*

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ABSTRACT

A Study of nonlinear second order stochastic boundary value problems (SBVP for short) is initiated through sample calculus approach. A basic existence result for bounded nonlinearities is established. The method of upper and lower solution processes and a general existence theorem are established. After proving the stochastic version of the needed maximum principle, the monotone iterative technique is developed which yields existence of multiple solution processes of SBVP. Finally, by developing a stochastic comparison result, the important problem of finding error estimates between the sample solutions of SBVP and the solutions of corresponding mean BVP, is considered.

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1. INTRODUCTION:

Due to the entirely different nature of the underlying physical processes, the study of nonlinear boundary value problems (BVP for short) is substantially more difficult than that of initial value problem. However, this fascinating field of differential equations has developed significantly, in recent years, and a variety of techniques that are employed in the development of the theory are well recorded [3]. One of the techniques, which has attracted much attention because of its immense value in applications such as reaction-diffusion equations, is the method of upper and lower solutions [10,14,17,18,21,22]. Furthermore, monotone iterative techniques that offer constructive methods, have been fruitfully utilized to obtain multiple solutions, two sided bounds on solutions and asymptotic behaviour of solutions [4,6,8,9,15,17,18,23].

Although stochastic BVP result in several applications, the corresponding development of the theory of SBVP Has lagged behind considerably. Recently, linear random BVP have been investigated successfully in [5,19,20] by employing the method of moments. Also, an indepth study of stochastic Reynolds equation arising in hydrodynamic theory of lubrication has been made in [11,12,13]. One of the reasons for the lack of development of the theory of nonlinear SBVP is perhaps due to the fact that the interplay of different modes of probabilistic analysis is not well understood, the essential ingredients of sample calculus are not sufficiently well-known, and the crucial role played by the method of upper and lower solutions is not realized.

In this paper, we initiate a systematic study of the theory of nonlinear second order SBVP through sample calculus approach [16] and

obtain some fundamental results. We first prove an existence result for bounded nonlinearities and then develop the method of upper and lower sample solution processes by means of which the restrictive boundedness assumption can be dropped. This existence result being of theoretical value, we then proceed to extend the constructive monotone iterative technique to SBVP which yields monotone sequences that converge to extremal sample solution processes. Since the success of this technique rests heavily on the availability of stochastic maximum principle, we first prove a general result concerning random differential inequalities, a special case of which is the desired stochastic maximum principle, and then discuss the monotone technique thoroughly. Finally, the important problem of finding the error estimates between the sample solutions of SBVP and the solutions of corresponding mean BVP is considered. This necessitates proving a general stochastic comparison result for SBVP, the deterministic special case of which is itself of very recent origin [9]. We utilize Lyapunov-like functions in obtaining this error estimate from which interesting special cases can be obtained. The present investigation is motivated by a specific problem arising in gas lubrication and the results obtained in that direction appear elsewhere [7]. We do hope that our investigation of nonlinear SBVP will pave the way for future development of the theory utilizing other modes of approach and techniques [3,16].

2. EXISTENCE OF SAMPLE SOLUTIONS:

Consider the stochastic boundary-value problem (SBVP for short)

$$(2.1) \quad x''(t, \omega) = f(t, x(t, \omega), x'(t, \omega), \omega), \quad B_\mu(\omega)x(\mu, \omega) = b_\mu(\omega),$$

where $f \in M[I \times R \times R, R[\Omega, R]]$; $I = [0, 1]$, $M[I \times R \times R, R[\Omega, R]]$

is a set of real-valued random functions such that $f(t, x(t, \omega), y(t, \omega), \omega)$ is product-measurable whenever real-valued random functions $x(t, \omega)$ and $y(t, \omega)$ are product-measurable; $R[\Omega, R]$ denotes a collection of all random variables defined on a complete probability space (Ω, F, P) into R ;

$$B_\mu(\omega)x(\mu, \omega) \equiv \alpha_\mu(\omega)x(\mu, \omega) + (-1)^{\mu+1}\beta_\mu(\omega)x'(\mu, \omega) \quad \text{for } \mu = 0, 1;$$

$\alpha_\mu, \beta_\mu, b_\mu \in R[\Omega, R]$ for $\mu = 0, 1$, $\alpha_0, \alpha_1 \geq 0$, $\beta_0, \beta_1 > 0$ w.p. 1, and

prime stands for the sample derivative. We shall always assume that

$b_\mu \in B(\rho)$ w.p. 1 for some $\rho > 0$, $f \in M[I \times R \times R, R[\Omega, R]]$ and $f(t, x, y, \omega)$

is sample continuous in (x, y) for each $t \in I$. A random process $x(t) \equiv$

is said to be a sample solution process of SBVP (2.1) if,

- (i) $x(t)$ is sample continuously differentiable;
- (ii) $x(t)$ and $x'(t)$ are product-measurable processes, which satisfy (2.1) w.p. 1.

The proposed method for establishing the existence of a sample solution of (2.1) depends on its equivalent integral equation. This simple fact we state in the form of a lemma whose proof we omit since it is a consequence of the method of variation of parameters.

Lemma: Assume that $f \in M[I \times R \times R, R[\Omega, R]]$ and that $f(t, x(t, \omega), x'(t, \omega), \omega)$ is sample Lebesgue integrable on I , whenever $x(t, \omega)$ is sample continuously differentiable process. Then the SBVP (2.1), is equivalent to the stochastic integral equation

$$x(t, \omega) = \phi(t, \omega) + \int_0^1 G(t, s, \omega) f(s, x(s, \omega), x'(s, \omega), \omega) ds,$$

where the integral represents the sample Lebesgue integral, $\phi(t, \omega) = b_0(\omega) + (b_0(\omega) - b_1(\omega))t$, $t \in I$, and $G(t, s, \omega)$ is a stochastic Green's function of the corresponding scalar problem $x'' = 0$ and $B_\mu(\omega)x(\mu, \omega) = 0$.

We shall prove a basic existence theorem on which our further discussion depends.

Theorem 2.1: If (A) $K \in IB[I, R[\Omega, R_+]]$, that is, K is a product-measurable R_+ -value random function whose sample Lebesgue integral is bounded w.p. 1 and

$$|f(t, x, y, \omega)| \leq K(t, \omega) \quad \text{for } (t, x, y) \in I \times R \times R \quad \text{w.p. 1,}$$

then, the SBVP (2.1), has at least one sample solution process $x(t, \omega)$ on I .

Proof: Because of (A) and the nature of boundary conditions, one can pick deterministic positive number M sufficiently large so that

$$|b_\mu(\omega)| \leq M \quad \text{for } \mu = 0, 1, \quad |b_0(\omega) - b_1(\omega)| \leq M,$$

$$\max_{s, t \in I} |G(t, s, \omega)| \int_0^1 K(s, \omega) ds \leq M \quad \text{and} \quad \max_{s, t \in I} |G_t(t, s, \omega)| \int_0^1 K(s, \omega) ds \leq M.$$

Define

$$B(M) = \{x \in C^1[[0, 1], R[\Omega, R]] : |x(t, \omega)| \leq 2M, |x'(t, \omega)| \leq 2M \quad \text{w.p. 1 on } I\} \text{ and for } x \in B(M),$$

$$T(\omega)x(t, \omega) = \int_0^1 G(t, s, \omega) f(s, x(s, \omega), x'(s, \omega), \omega) ds + \phi(t, \omega) \quad \text{where}$$

$C^1[[0, 1], R[\Omega, R]]$ is a collection of all real-valued separable and sample continuously differentiable random functions on I with a state space (R, F^1) . We note that $B(M)$ is a closed, bounded and convex subset of a Banach space $C^1[[0, 1], L^\infty[\Omega, R]]$. The choice of M now

assures that the set $T(\omega)(B(M)) = \{T(\omega)x(t, \omega) : x(t, \omega) \in B(M)\}$ is equibounded w.p. 1 and also $T(\omega)(B(M)) \subset B(M)$ w.p. 1. Because of the fact

$$|(T(\omega)x(t, \omega))''| \leq K(t, \omega),$$

there results the inequality

$$|(T(\omega)x(t_1, \omega))' - (T(\omega)x(t_2, \omega))'| \leq \left| \int_{t_1}^{t_2} K(s, \omega) ds \right|,$$

which, in view of the sample integrability of $K(t, \omega)$, shows $T(\omega)(B(M))$ is equicontinuous on I w.p. 1. Thus by Ascoli-Arzelà's Theorem we see that $T(\omega)(B(M))$ is a relatively compact subset of $C^1[[0, 1], L^\infty[\Omega, R]]$. Thus by the stochastic version of Schauder's fixed point theorem [1, 2], it follows that there exists a solution of the SBVP (2.1), proving the theorem.

Remark 2.1: In the foregoing discussion we have assumed that $K \in IB[I, R[\Omega, R_+]]$. Instead of this, one can assume that K satisfies the conditions of De La Vallée-Poussin's theorem [16].

In order to remove the boundedness assumption (A) of Theorem 2.1, we need certain assumptions which we list below for convenience:

(A₀) $\alpha, \beta \in C^2[I, R[\Omega, R]]$, $\alpha(t, \omega) \leq \beta(t, \omega)$ on I w.p. 1,

$$\alpha''(t, \omega) \geq f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega), \quad B_\mu(\omega)\alpha(\mu, \omega) \leq b_\mu(\omega),$$

$$\beta''(t, \omega) \leq f(t, \beta(t, \omega), \beta'(t, \omega), \omega), \quad B_\mu(\omega)\beta(\mu, \omega) \geq b_\mu(\omega);$$

(A₁) f satisfies a Nagumo condition with respect to α, β , namely,

there exists a $h \in C[R_+, R[\Omega, (0, \infty)]]$ such that

$$|f(t, x, x', \omega)| \leq h(|x'|, \omega), \text{ w.p. 1,}$$

for $t \in I$, $\alpha(t, \omega) \leq x \leq \beta(t, \omega)$ and there exists a $N(\omega) > 0$ such that

$$\int_{\lambda(\omega)}^{N(\omega)} \frac{s ds}{h(s, \omega)} > \max_{t \in I} \alpha(t, \omega) - \min_{t \in I} \beta(t, \omega), \text{ where}$$

$$\lambda(\omega) = \max \left[|\beta(0, \omega) - \alpha(1, \omega)|, |\beta(1, \omega) - \alpha(0, \omega)| \right]$$

$$(A_2) \quad |f(t, x, x', \omega)| \leq K(t, \omega) \quad \text{for } t \in I,$$

$$\alpha(t, \omega) \leq x \leq \beta(t, \omega) \text{ and } |y| \leq C(\omega), \text{ w.p. 1, where } K \in IB[I, R[\Omega, R_+]].$$

We can now prove the following result which removes condition

(A) of Theorem 2.1.

Theorem 2.2: Consider the SBVP (2.1) and let the conditions (A_0) , (A_1) and (A_2) hold. Then there exists at least one sample solution process for the SBVP (2.1).

Remark 2.2: The assumption (A_1) implies that there exists $N \in R[\Omega, R_+]$ with $N(\omega) > 0$ w.p. 1, which depends only on α, β and the Nagumo function h such that any sample solution x of $x''(t, \omega) = f(t, x(t, \omega), x'(t, \omega), \omega)$ satisfying $\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)$ on I also verifies the estimate $|x'(t, \omega)| \leq N(\omega)$ on I , w.p. 1. The proof is analogous to the classical result, see [3] and uses probabilistic analysis.

Proof of Theorem 2.2: By (A_1) , there exists, in view of Remark 2.2, an $N(\omega) > 0$ such that $|x'(t, \omega)| \leq N(\omega)$ on I w.p. 1, where $x(t, \omega)$ is any sample solution of $x''(t, \omega) = f(t, x(t, \omega), x'(t, \omega), \omega)$. Choose a $C(\omega) > 0$ such that $C(\omega) > \max_{t \in I} \{N(\omega), \max_{t \in I} |\alpha'(t, \omega)|, \max_{t \in I} |\beta'(t, \omega)|\}$

and define a modified function F relative to the triple (α, β, C) such that

$$F(t, x, x', \omega) = f(t, p(t, x, \omega), q(x', \omega), \omega) + r(t, x, \omega),$$

where

$$p(t, x, \omega) = \max [\alpha(t, \omega), \min \{x, \beta(t, \omega)\}],$$

$$q(x', \omega) = \max [-C(\omega), \min \{x', C(\omega)\}]$$

$$r(t, x, \omega) = \begin{cases} \frac{x - \beta(t, \omega)}{1 + x^2}, & \text{if } x > \beta(t, \omega) \\ 0, & \text{if } \alpha(t, \omega) \leq x \leq \beta(t, \omega) \\ \frac{x - \alpha(t, \omega)}{1 + x^2}, & \text{if } x < \alpha(t, \omega). \end{cases}$$

Clearly, in view of (A_2) , F satisfies an assumption of the type (A) of Theorem 2.1, $F \in M[I \times R \times R, R[\Omega, R]]$ and $F(t, x, x', \omega)$ is sample continuous in (x, y) for $t \in I$. Hence by Theorem 2.1 there exists a sample solution x of SBVP.

$$(2.2) \quad x''(t, \omega) = F(t, x(t, \omega), x'(t, \omega), \omega), \quad B_\mu(\omega)x(\mu, \omega) = b_\mu(\omega), \quad \mu = 0, 1.$$

We shall now show that $\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)$ on I w.p. 1, where $x(t, \omega)$ is a sample solution of SBVP (2.2). We shall only prove $x(t, \omega) \leq \beta(t, \omega)$ on I w.p. 1 since a similar argument can be used to show $\alpha(t, \omega) \leq x(t, \omega)$ on I w.p. 1. Assume that the inequality $x(t, \omega) \leq \beta(t, \omega)$ on I w.p. 1 is false. Then there exists an event $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) > 0$ such that for $\omega \in \Omega_1$, $x(t, \omega) - \beta(t, \omega)$ has a positive maximum at a point $t_0 \in [0, 1]$. We note that t_0 may depend on $\omega \in \Omega_1$. If $t_0 \in (0, 1)$, then we have $\beta(t_0, \omega) < x(t_0, \omega)$ and $x'(t_0, \omega) = \beta'(t_0, \omega)$. Hence $F(t_0, x(t_0, \omega), x'(t_0, \omega), \omega) = \beta'(t_0, \omega)$ and $|x'(t_0, \omega)| \leq C$. Consequently, for $\omega \in \Omega_2$ with $\Omega_2 \subseteq \Omega_1$ and $P(\Omega_2) = P(\Omega_1)$,

$$x''(t_0, \omega) - \beta''(t_0, \omega) \geq F(t_0, x(t_0, \omega), x'(t_0, \omega), \omega) - f(t_0, \beta(t_0, \omega), \omega),$$

$$\beta'(t_0, \omega, \omega) = \frac{x(t_0, \omega) - \beta(t_0, \omega)}{1 + x^2(t_0, \omega)} > 0,$$

in view of the definition of $F(t, x, x', \omega)$. This is impossible at a point maximum of $x(t, \omega) - \beta(t, \omega)$.

If $t_0 = 0$, we have $x(0, \omega) - \beta(0, \omega) > 0$, whenever $\omega \in \Omega_1$. Be-

cause of sample continuity of the solution process of (2.2) $x(0, \omega) - \beta(0, \omega) > x(t, \omega) - \beta(t, \omega) > 0$ for sufficiently close to 0, provided that $\omega \in \Omega_3 \subseteq \Omega_1$ with $P(\Omega_3) = P(\Omega_1)$. From this, it follows that $x'(0, \omega) - \beta'(0, \omega) \leq 0$, whenever $\omega \in \Omega_3$ with $P(\Omega_3) > 0$. By the boundary conditions $B_0(\omega) \beta(0, \omega) \geq b_0(\omega)$ and $B_0(\omega) x(0, \omega) = b_0(\omega)$, we get $\beta_0(\omega) (x'(0, \omega) - \beta'(0, \omega)) \geq 0$ w.p. 1. Since $\beta_0 > 0$ w.p. 1, we must have $x'(0, \omega) = \beta'(0, \omega)$ whenever $\omega \in \Omega_3$. Hence, as before, we obtain $x''(0, \omega) - \beta''(0, \omega) > 0$ with probability $P(\Omega_3) > 0$. This contradicts the sample maximum property of $x(t, \omega) - \beta(t, \omega)$. A similar contradiction holds in the case of $t_0 = 1$. Thus, we have established $\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)$ on I w.p. 1. It therefore follows that $x''(t, \omega) = f(t, x(t, \omega), q(x'(t, \omega)), \omega)$. Noting that $C(\omega) > N(\omega)$, w.p. 1 and employing the argument similar to the classical case, see [3] one can easily show that $|x'(t, \omega)| \leq N(\omega)$ on I w.p. 1. Hence $x(t, \omega)$ is also a sample solution of SBVP (2.1) and the proof is complete.

3. MONOTONE ITERATIVE TECHNIQUE:

In this section, we wish to develop monotone iterative technique for SBVP (2.1), which provides a constructive method for the existence of sample extremal solutions. Since this method depends crucially on stochastic maximum principle, we first prove a general comparison result which includes the required maximum principle.

Theorem 3.1: Let $f(t, x, y, \omega)$ of (2.1) be increasing in x for each $(t, y) \in I \times \mathbb{R}$ and satisfy the sample Lipschitz condition

$$|f(t, x, y_1, \omega) - f(t, x, y_2, \omega)| \leq L(\omega) |y_1 - y_2| \quad \text{w.p. 1,}$$

where $L(\omega) > 0$ w.p. 1. Assume that $\alpha, \beta \in C^2[I, \mathbb{R}[\Omega, \mathbb{R}]]$ and

$\alpha''(t, \omega) \geq f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega)$, $\beta''(t, \omega) \leq f(t, \beta(t, \omega), \beta'(t, \omega), \omega)$
on I , w.p. 1. Then for $\mu = 0, 1$, $B_\mu(\omega) \alpha(\mu, \omega) \leq B_\mu(\omega) \beta(\mu, \omega)$ implies
that $\alpha(t, \omega) \leq \beta(t, \omega)$ on I , w.p. 1.

Proof: We shall first show that the desired inequality is valid at the
end points of I . Assume that this is false. Then there exists an event
 $\Omega_\mu \subseteq \Omega$ with $P(\Omega_\mu) > 0$ such that $\alpha(\mu, \omega) > \beta(\mu, \omega)$ for $\mu = 0, 1$, which
implies, in view of the assumption $B_\mu(\omega) > 0$, that $(-1)^\mu (\alpha'(\mu, \omega) -$
 $\beta'(\mu, \omega)) \geq 0$ for $\mu = 0, 1$ whenever $\omega \in \Omega_\mu$. We now consider the case
 $\mu = 0$, since, the proof concerning the case $\mu = 1$ is similar. Because
of sample continuity of the process $\alpha(t, \omega) - \beta(t, \omega)$, one can find a
 $\delta > 0$ such that $\alpha(t, \omega) - \beta(t, \omega) > 0$ on $[0, \delta)$. Consequently, the sam-
ple monotonicity of f , gives

$$\begin{aligned} (3.1) \quad \alpha''(t, \omega) - \beta''(t, \omega) &\geq f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega) - f(t, \beta(t, \omega), \beta'(t, \omega), \omega) \\ &> f(t, \beta(t, \omega), \alpha'(t, \omega), \omega) - f(t, \beta(t, \omega), \beta'(t, \omega), \omega) \\ &\geq -L(\omega) |\alpha'(t, \omega) - \beta'(t, \omega)|. \end{aligned}$$

We claim that there exists a $t_0 \in [0, \delta)$ such that $\alpha'(t_0, \omega) - \beta'(t_0, \omega) > 0$.
Suppose that this is not true. Then there exists an event $\bar{\Omega}_0 \subseteq \Omega_0$
such that $P(\bar{\Omega}_0) > 0$, and $\alpha'(t, \omega) - \beta'(t, \omega) \leq 0$ on $[0, \delta)$. Thus, by
the theory of random differential inequalities [16], we have

$$\alpha'(t, \omega) - \beta'(t, \omega) \geq (\alpha'(0, \omega) - \beta'(0, \omega)) \exp [L(\omega)t].$$

This together with $\alpha'(t, \omega) - \beta'(t, \omega) \leq 0$ implies that $\alpha'(t, \omega) \equiv \beta'(t, \omega)$
on $[0, \delta)$ w.p. 1. Now, in view of (3.1), we then arrive at the contra-
diction $\alpha''(t, \omega) - \beta''(t, \omega) > 0$ on $[0, \delta)$, $\omega \in \bar{\Omega}_0$. This contradiction
establishes the claim $\alpha'(t_0, \omega) - \beta'(t_0, \omega) > 0$ for some $t_0 \in [0, \delta)$.
Again, we employ the differential inequality (3.1) to conclude $\alpha'(t, \omega) -$
 $\beta'(t, \omega) > 0$ on $[t_0, \delta)$. This argument may be extended to show that

$\delta = 1$ which implies that $\alpha(1, \omega) > \beta(1, \omega)$ whenever $\omega \in \Omega_0$. From this it follows that $\alpha'(1, \omega) \leq \beta'(1, \omega)$, which contradicts the inequality $\alpha'(1, \omega) - \beta'(1, \omega) > 0$. Thus $\alpha(0, \omega) > \beta(0, \omega)$ is impossible.

Now, it remains to show that $\alpha(t, \omega) \leq \beta(t, \omega)$ on $(0, 1)$. Suppose that it is false. Then there exist $t_0, t_1, t_2 \in (0, 1)$ and positive random numbers ϵ_1 and ϵ_0 , $\Omega_0 \subseteq \Omega$ such that (a) $P(\Omega_0) > 0$; (b) $0 < \epsilon_1(\omega) < \epsilon_0(\omega)$, $\omega \in \Omega_0$; (c) $t_1 < t_0 < t_2$; (d) $\alpha(t_\mu, \omega) - \beta(t_\mu, \omega) = \epsilon_1(\omega)$, $\omega \in \Omega_0$, $\mu = 0, 1$; (e) $\max_{t_1 \leq t \leq t_2} [\alpha(t, \omega) - \beta(t, \omega)] = \alpha(t_0, \omega) - \beta(t_0, \omega) = \epsilon_0(\omega)$, $\omega \in \Omega_0$. Hence, we have

$$\alpha(t_0, \omega) > \beta(t_0, \omega), \alpha'(t_0, \omega) = \beta'(t_0, \omega)$$

and

$$\alpha''(t_0, \omega) \leq \beta''(t_0, \omega)$$

for $\omega \in (\Omega_0 - N)$ and $N \subset \Omega_0$ with $P(N) = 0$.

Consequently we arrive at a contradiction,

$$\begin{aligned} \alpha''(t_0, \omega) - \beta''(t_0, \omega) &\geq f(t_0, \alpha(t_0, \omega), \alpha'(t_0, \omega), \omega) - f(t_0, \beta(t_0, \omega), \beta'(t_0, \omega), \omega) \\ &> f(t_0, \beta(t_0, \omega), \beta'(t_0, \omega), \omega) - f(t_0, \beta(t_0, \omega), \beta'(t_0, \omega), \omega) \\ &> 0, \end{aligned}$$

proving the theorem.

The following corollary of Theorem 3.1 will be used often in our discussion.

Corollary 3.1: Let $q, r \in M[1, R[\Omega, R]]$ and $r(t, \omega) > 0$ w.p. 1. Assume that $\phi \in C^2[1, R[\Omega, R]]$,

$$\phi''(t, \omega) + q(t, \omega) \phi'(t, \omega) - r(t, \omega) \phi(t, \omega) \leq 0$$

and

$$B_{\mu}(\omega) \phi(\mu, \omega) \geq 0 \quad \text{w.p. 1, } \mu = 0, 1.$$

Then $\phi(t, \omega) \geq 0$ for $t \in I$ w.p. 1.

We are now in a position to develop the monotone technique for SBVP (2.1). For this purpose, we need further assumptions on f .

(A₃) $f(t, x, y, \omega)$ sample continuously differentiable in (x, y) for each $t \in I$.

In view of (A₃), there exists a $Q \in M[R_+, R[\Omega, R_+]]$ such that

$$|F_x(t, x, y, \omega)| \leq Q(C(\omega), \omega)$$

for $t \in I$, $\alpha(t, \omega) \leq x \leq \beta(t, \omega)$ and $|y| \leq C(\omega)$. Let us consider the auxiliary SBVP.

$$(3.2) \quad x''(t, \omega) = H(t, x(t, \omega), x'(t, \omega), \omega), \quad B_{\mu}(\omega)x(\mu, \omega) = b_{\mu}(\omega), \quad \mu = 0, 1,$$

where $H(t, x, x', \omega) = f(t, \eta(t, \omega), x', \omega) + Q(C(\omega), \omega)(x - \eta(t, \omega))$,

for any $\eta \in C[I, R[\Omega, R]]$ such that $\alpha(t, \omega) \leq \eta(t, \omega) \leq \beta(t, \omega)$ on I w.p. 1. Let us also list another assumption.

$$(A_4) \quad \frac{Q(C, \omega)}{C^2} \rightarrow 0 \quad \text{w.p. 1 as } C \rightarrow \infty \quad \text{and the limit of } \frac{h(s, \omega)}{s^2}$$

exists and bounded w.p. 1, as $s \rightarrow \infty$.

Then we have the following result

Theorem 3.2: Assume that the assumptions (A₀), (A₁), (A₂), (A₃) and

(A₄) hold. Then there exist sequences $\{\alpha_n(t, \omega)\}$, $\{\beta_n(t, \omega)\}$ which

converge uniformly and monotonically w.p. 1 to the sample minimal and

maximal solution processes $p(t, \omega)$ and $r(t, \omega)$, respectively, of

SBVP (2.1), that is,

$$\alpha(t, \omega) \leq \alpha_1(t, \omega) \leq \dots \leq \alpha_n(t, \omega) \leq p(t, \omega) \leq r(t, \omega) \leq \beta_n(t, \omega) \leq \dots$$

$$\dots \beta_1(t, \omega) \leq \beta(t, \omega) \quad \text{on } I \quad \text{w.p. 1.}$$

Proof: Let us first show that there exists a unique sample solution for SBVP (3.2). For this purpose, we prove that H satisfies the assumptions of Theorem 2.2 which guarantees existence and then Stochastic maximum principle assures the uniqueness.

To prove (A_0) for H , consider

$$\begin{aligned} & f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega) - H(t, \alpha(t, \omega), \alpha'(t, \omega), \omega) \\ &= f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega) - f(t, \eta(t, \omega), \alpha'(t, \omega), \omega) \\ &\quad + Q(C(\omega), \omega)(\alpha(t, \omega) - \eta(t, \omega)) \\ &= [f_x(t, \xi, \alpha'(t, \omega), \omega) - Q(C(\omega), \omega)](\alpha(t, \omega) - \eta(t, \omega)), \end{aligned}$$

where $\alpha(t, \omega) \leq \xi \leq \eta(t, \omega)$. Since $C(\omega) \geq \max \left\{ \max_{t \in I} |\alpha'(t, \omega)|, \max_{t \in I} |\beta'(t, \omega)| \right\}$, the definition of Q shows that α satisfies (A_0) . Similarly one shows that β does.

To show H satisfies (A_1) , defining

$$\ell(C(\omega), \omega) = Q(C(\omega), \omega) \max_{t \in I} |\alpha(t, \omega) - \beta(t, \omega)|, \text{ we see that}$$

$$\begin{aligned} |H(t, x, x', \omega)| &= |f(t, \eta(t, \omega), x', \omega) + Q(C(\omega), \omega)(x - \eta(t, \omega))| \\ &\leq |f(t, \eta(t, \omega), x', \omega)| + Q(C(\omega), \omega)|x - \eta(t, \omega)| \\ &\leq h(|x'|, \omega) + \ell(C(\omega), \omega) \end{aligned}$$

for $t \in I$ and $\alpha(t, \omega) \leq x \leq \beta(t, \omega)$ w.p. 1. Our objective is to pick positive random number $N(\omega)$, so large that

$$\int_{\lambda}^N \frac{s ds}{h(s, \omega) + \ell(C(\omega), \omega)} \geq \max_{t \in I} \beta(t, \omega) - \min_{t \in I} \alpha(t, \omega),$$

where λ is given by

$$\lambda \geq \max \{ |\alpha(0, \omega) - \beta(1, \omega)|, |\alpha(1, \omega) - \beta(0, \omega)| \}.$$

By assumption (A_4) , there exists random numbers, $\tau(\omega) > 0$, $p(\omega) > 0$ such that $h(s, \omega) \leq p(\omega)s^2$ for $s \geq \tau(\omega)$ and random function $k(m, \omega)$ such that $k(m, \omega) \rightarrow 0$ w.p. 1 as $m \rightarrow \infty$ satisfying $\ell(C(\omega), \omega) \leq k(m, \omega)$ w.p. 1 provided $N(\omega) \geq m(\omega)$. For some arbitrary positive random number z pick $N(\omega) \geq z(\omega)m(\omega)$. Then, since we can assume $m(\omega) > \lambda(\omega)$ and $m(\omega) \geq \tau(\omega)$,

$$\begin{aligned} \int_{\lambda(\omega)}^{N(\omega)} \frac{s ds}{(h(s, \omega) + \ell(C(\omega), \omega))} &\geq \int_{m(\omega)}^{N(\omega)} \frac{s ds}{(h(s, \omega) + (C(\omega), \omega))} \\ &\geq \int_{m(\omega)}^{N(\omega)} \frac{s ds}{(p(\omega)s^2 + k(m(\omega), \omega)N^2(\omega))} \\ &\geq \frac{1}{2p(\omega)} \ln \left[\frac{p(\omega)N^2(\omega) + k(m(\omega), \omega)N^2(\omega)}{p(\omega)m^2(\omega) + k(m(\omega), \omega)N^2(\omega)} \right] \\ &\geq \frac{1}{2p(\omega)} \left[\ln z^2(\omega) + \ln \left[\frac{p(\omega) + k(m(\omega), \omega)}{p(\omega) + z^2(\omega)k(m(\omega), \omega)} \right] \right] \\ &\geq \max_{t \in I} \beta(t, \omega) - \min_{t \in I} \alpha(t, \omega), \end{aligned}$$

provided $z(\omega) = \exp [2p(\omega)(\max_{t \in I} \beta(t, \omega) - \min_{t \in I} \alpha(t, \omega))]$ and $m(\omega)$ is such

that $k(m(\omega), \omega) = p(\omega) \exp [-2p(\omega)(\max_{t \in I} \beta(t, \omega) - \min_{t \in I} \alpha(t, \omega))]$. For this

choice of $z(\omega)$ and $m(\omega)$, $N(\omega)$ can be determined. This proves (A_1) .

Clearly (A_2) holds.

To show that the problem (3.2) has a unique sample solution, let,

if possible, $x(t, \omega)$ and $y(t, \omega)$ be sample solutions of (3.2). Then setting

$$\phi(t, \omega) = x(t, \omega) - y(t, \omega),$$

we get

$$\begin{aligned} \phi''(t, \omega) &= f(t, \eta(t, \omega), x'(t, \omega), \omega) - f(t, \eta(t, \omega), y'(t, \omega), \omega) \\ &\quad + Q(C(\omega), \omega)(x(t, \omega) - y(t, \omega)) \\ &= f_y(t, \eta(t, \omega), \sigma, \omega) \phi'(t, \omega) + Q(C(\omega), \omega) \phi(t, \omega), \end{aligned}$$

where $\sigma = s x'(t, \omega) + (1-s)y'(t, \omega)$ for some $s \in (0, 1)$ with $|\sigma| \leq C(\omega)$.

Also $B_\mu(\omega)\phi(\mu, \omega) = 0$ w.p. 1. Hence by corollary 3.1, we can conclude

$\phi(t, \omega) \equiv 0$ w.p. 1 proving the uniqueness of sample solution of SBVP (3.2).

Let us define the mapping $A: [\alpha, \beta] \longrightarrow C[I, R[\Omega, R]]$ by $A\eta = x$, where $x = x(t, \omega)$ is the sample solution process of SBVP (3.2) for every $\eta \in [\alpha, \beta] = \{\eta \in C[I, R[\Omega, R]]: \alpha(t, \omega) \leq \eta(t, \omega) \leq \beta(t, \omega) \text{ on } I \text{ w.p. 1}\}$. We shall show that

$$(i) \quad \alpha \leq A\alpha, \quad \beta \geq A\beta;$$

$$(ii) \quad A \text{ is monotone mapping on } [\alpha, \beta].$$

Let $A\alpha = \alpha_1$ where α_1 is the sample solution of SBVP (3.2) corresponding to $\eta = \alpha$. Setting $\phi = \alpha_1 - \alpha$, we get, as before,

$$\phi'' \leq f_y(t, \alpha, \sigma, \omega)\phi' + Q\phi \quad \text{and} \quad B_\mu(\omega)\phi(\mu, \omega) \geq 0, \quad \text{where}$$

$\sigma = s \alpha_1' + (1-s)\alpha'$ for some $s \in (0, 1)$. Hence by Corollary 3.1, it follows that $\alpha \leq \alpha_1$ on I w.p. 1. One can show similarly $\beta \geq A\beta$ on I w.p. 1 proving (i).

To prove (ii), let $\eta_1, \eta_2 \in [\alpha, \beta]$ be such that $\eta_1 \leq \eta_2$ on I w.p. 1. Suppose that $A\eta_1 = x_1$ and $A\eta_2 = x_2$ where x_1, x_2 are the

sample solutions of SBVP (3.2) relative to η_1, η_2 respectively.

Setting $\phi = x_2 - x_1$, we see that $B_\mu(\omega) \phi(\mu, \omega) = 0$ and

$$\begin{aligned} \phi'' &= f(t, \eta_2, x_2', \omega) - f(t, \eta_1, x_1', \omega) \\ &\quad + Q(x_2 - x_1 + \eta_1 - \eta_2) \\ &= f(t, \eta_2, x_2', \omega) - f(t, \eta_2, x_1', \omega) \\ &\quad + f(t, \eta_2, x_1', \omega) - f(t, \eta_1, x_1', \omega) \\ &\quad + Q(\eta_1 - \eta_2) + Q\phi \\ &\leq f_y(t, \eta_2, \sigma, \omega) \phi' + Q\phi, \end{aligned}$$

where $s x_2' + (1-s)x_1'$ for some $s \in (0,1)$. Hence it follows corollary 3.1 that $x_1 \leq x_2$ proving (11).

It is now easy to define sequences $\{\alpha_n(t, \omega)\}$ and $\{\beta_n(t, \omega)\}$ such that

$$\alpha_n = A\alpha_{n-1}, \beta_n = A\beta_{n-1}, \alpha_0 = \alpha, \beta_0 = \beta,$$

and conclude that they are monotone sequences. We note that for each

fixed $t \in I$ $\rho(t, \omega) = \sup_{n \geq 0} \alpha_n(t, \omega)$ and $r(t, \omega) = \inf_{n \geq 0} \beta_n(t, \omega)$, respectively, are real-valued random variables. Moreover, for fixed $t \in I$

$$\lim_{n \rightarrow \infty} \alpha_n(t, \omega) = \sup_{n \geq 0} \alpha_n(t, \omega) \quad \text{and} \quad \lim_{n \rightarrow \infty} \beta_n(t, \omega) = \inf_{n \geq 0} \beta_n(t, \omega) \quad \text{w.p. 1.}$$

Therefore $\rho(t, \omega)$ and $r(t, \omega)$ are random functions defined on I . It

is obvious that $\rho(t, \omega) \leq r(t, \omega)$ on I w.p. 1. Furthermore, if $x(t, \omega)$

is any sample solution process of SBVP (2.1) in $[\alpha, \beta]$, then

$A\alpha \leq Ax = x$. This leads us to the conclusion that $\rho(t, \omega) \leq x(t, \omega)$,

Similarly the inequality $x(t, \omega) \leq r(t, \omega)$ can be verified. Finally, we

need to verify that $\rho(t, \omega)$ and $r(t, \omega)$ are sample solution processes

of SBVP (2.1). We shall only prove $\rho(t, \omega)$ is a sample solution

process of SBVP (2.1), since the proof for $r(t, \omega)$ is similar. By

SBVP (3.2) and the definition of $\{\alpha_n(t, \omega)\}$ we have

$$\begin{aligned}\alpha_n(t, \omega) &= \phi(t, \omega) + \int_0^1 G(t, s, \omega) H(s, \alpha_n(s, \omega), \alpha_n'(s, \omega), \omega) ds \\ &= \phi(t, \omega) + \int_0^1 G(t, s, \omega) \left[F(s, \alpha_{n-1}(s, \omega), \alpha_n'(s, \omega), \omega) \right. \\ &\quad \left. + Q(C(\omega), \omega) (\alpha_n(s, \omega) - \alpha_{n-1}(s, \omega)) \right] ds.\end{aligned}$$

The fact that α_n' is almost surely sample bounded and equicontinuous shows, by the application of Lebesgue dominated convergence theorem [16], that

$$\lim_{n \rightarrow \infty} \alpha_n(t, \omega) = \phi(t, \omega) + \int_0^1 G(t, s, \omega) F(s, \rho(s, \omega), \rho'(s, \omega), \omega) ds$$

which implies that

$$\rho(t, \omega) = \phi(t, \omega) + \int_0^1 G(t, s, \omega) F(s, \rho(s, \omega), \rho'(s, \omega), \omega) ds.$$

It therefore follows that $\rho(t, \omega)$ is a sample minimal solution process of SBVP (2.1). We have thus completed the proof of Theorem 3.2.

4. COMPARISON PRINCIPLE:

When dealing with stochastic differential equations, an important problem is to find error estimates between the solutions of SBVP (2.1) and the solutions of corresponding mean BVP.

$$(4.1) \quad z'' = R(t, z, z'), \quad \bar{B}_\mu z(\mu) = \bar{b}_\mu,$$

where $R(t, x, x') = E[f(t, x, x', \omega)]$, $\bar{B}_\mu z(\mu) \equiv \bar{\alpha}_\mu z(\mu) + (-1)^{\mu+1} \bar{\beta}_\mu z'(\mu)$,

$\bar{\alpha}_\mu = E[\alpha_\mu(\omega)]$, $\bar{\beta}_\mu = E[\beta_\mu(\omega)]$, $\bar{b}_\mu = E[b_\mu(\omega)]$, and E stands for average

or mean value of random variable or function. For this purpose, we need

to develop a comparison result for SBVP. We shall state such a result below, since its proof follows from the argument that is used in [9] and in Theorem 3.1 with appropriate modifications.

Theorem 4.1: Assume that

$$(1) \quad m \in C^2[I, R[\Omega, R]],$$

$$m''(t, \omega) \geq g(t, m(t, \omega), m'(t, \omega), \omega), \quad 0 < t < 1,$$

and

$$D_\mu(\omega) m(\mu, \omega) \leq d_\mu(\omega) \quad \text{w.p. } 1,$$

where

$$D_\mu(\omega) m(\mu, \omega) = \delta_\mu(\omega) m(\mu, \omega) + (-1)^{\mu+1} \gamma_\mu(\omega) m'(\mu, \omega), \quad \mu = 0, 1;$$

(ii) $r(t, \omega)$ is the maximal sample solution process of

$$(4.2) \quad u''(t, \omega) = g(t, u(t, \omega), u'(t, \omega), \omega), \quad D_\mu(\omega) u(\mu, \omega) = d_\mu(\omega);$$

(iii) for every lower solution process $v(t, \omega)$, the modified

$$u''(t, \omega) = \tilde{g}(t, u(t, \omega), u'(t, \omega), \omega), \quad D_\mu(\omega) u(\mu, \omega) = d_\mu(\omega)$$

admits a solution, where $\tilde{g}(t, u, u', \omega) = g(t, p(t, u, \omega), u', \omega) + \frac{u - p(t, u, \omega)}{1 + u^2}$.

Then $m(t, \omega) \leq r(t, \omega)$ on I w.p. 1.

Remark 4.1: A comparison theorem dual to Theorem 4.1 holds with appropriate modifications yielding $p(t, \omega) \leq m(t, \omega)$ on I where $p(t, \omega)$ is the sample minimal solution process of (4.2) and $m(t, \omega)$ is an upper solution process.

Also it is clear from Theorem 3.1 that condition (iii) becomes fluous if $g(t, u, u', \omega)$ is nondecreasing in u and satisfies a sample Lipschitz condition in u' .

Now, we are ready to formulate a very general result that gives an error estimate on solution processes of (2.1) and solutions of the corresponding mean BVP (4.1).

Theorem 4.2: Assume that

$$(1) \quad V \in C^2[I \times R, R_+] \quad \text{and}$$

$$V''_{(f-R)}(t, x-z) \geq g(t, V(t, x-z), V'(t, x-z), \omega) \quad \text{w.p. } 1$$

where $V'(t,y) = V_t(t,y) + V_x(t,y)y'$, $V''_f(t,y) = U(t,y,y') + V_x(t,y)f(t,y,y',\omega)$; $U(t,y,y') = V_{tt}(t,y) + 2V_{tx}(t,y)y' + V_{xx}(t,y)y'^2$; g is as defined in (4.2);

(ii) the boundary conditions in (2.1) imply

$$D_\mu(\omega) V(\mu, x(\mu, \omega) - z(\mu)) \leq d_\mu(\omega) \quad \text{w.p. 1.}$$

(iii) $x(t, \omega)$, $r(t, \omega)$ and $z(t)$ are sample solution process of (2.1), the maximal sample solution process of (4.2) and solution of (4.1), respectively.

Then

$$V(t, x(t, \omega) - z(t)) \leq r(t, \omega) \quad \text{on } I \quad \text{w.p. 1.}$$

Proof: Setting $m(t, \omega) = V(t, x(t, \omega) - z(t))$ and using assumptions (1), (ii) and (iii), we obtain

$$m''(t, \omega) \geq g(t, m(t, \omega), m'(t, \omega), \omega) \quad \text{w.p. 1}$$

and

$$D_\mu(\omega) m(\mu, \omega) \leq d_\mu(\omega) \quad \text{w.p. 1}$$

Now an application of Theorem 4.1 yields the desired conclusion.

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