METHOD OF QUASI-UPPER AND LOWER SOLUTIONS
IN ABSTRACT CONES¹

by

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I. INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space with $\|\cdot\|$ and let E^* denote the dual of E . Let $K \subset E$ be a cone, that is, a closed convex subset such that $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap \{-K\} = \{0\}$. By means of K a partial order $\leq$ is defined as $v \leq u$ if $u - v \in K$. We let $K^* = \{\phi \in E^*: \phi(u) \geq 0 \text{ for all } u \in K\}$.

A cone K is said to be normal if there exists a real number $N > 0$ such that $0 \leq v \leq u$ implies $\|v\| \leq N\|u\|$ where N is independent of u, v . We shall always assume in this paper that K is a normal cone.

Let α denote the Kuratowski's measure of noncompactness, the properties of which may be found in [2,4].

For any $v_0, w_0 \in C[I, E]$ such that $v_0(t) \leq w_0(t)$ on I where $I = [0, T]$, we define the conical segment

$$[v_0, w_0] = \{u \in C[I, E]: v_0 \leq u \leq w_0\},$$

and the set

$$\Omega = \{(t, u) \in I \times E: t \in I \text{ and } v_0(t) \leq u \leq w_0(t)\}.$$

Let us consider the IVP

$$(1.1) \quad u' = f(t, u), \quad u(0) = u_0,$$

where $f \in C[\Omega, E]$. Suppose that $v_0, w_0 \in C^1[I, E]$ and

$$(1.2) \quad v_0' \leq f(t, v_0), \quad w_0' \geq f(t, w_0) \text{ on } I.$$

Then v_0, w_0 are called lower and upper solutions of (1.1) defined in a natural way.

A function f is said to be quasimonotone relative to K if $v \leq u$

and $\phi(v-u) = 0$, $\phi \in K^*$ implies $\phi(f(t,v)) \leq \phi(f(t,u))$. If $E = \mathbb{R}^n$ and $K = \mathbb{R}_+^n$, the standard cone, the inequalities induced by K are component-wise and the quasimonotonicity of f is reduced to

$$v \leq u \text{ and } v_i = u_i, 1 \leq i \leq n \text{ implies } f_i(t,v) \leq f_i(t,u).$$

In this special case one can prove the following result.

Theorem A. Let $E = \mathbb{R}^n$ and $K = \mathbb{R}_+^n$. Suppose that v_0, w_0 satisfy (1.2) with $v_0(t) \leq w_0(t)$ on I and that f is quasimonotone. Then there exists a solution $u(t)$ of (1.1) on I such that $v_0(t) \leq u(t) \leq w_0(t)$ on I provided $v_0(0) \leq u_0 \leq w_0(0)$.

If f is not known to be quasimonotone, we need to strengthen lower and upper solutions as follows: for each i , $1 \leq i \leq n$,

$$(1.3) \quad \begin{cases} v'_{0i} \leq f_i(t, \sigma) & \text{for all } \sigma \text{ such that } v_0(t) \leq \sigma \leq w_0(t) \text{ and } v_{0i}(t) = \sigma_i, \\ w'_{0i} \geq f_i(t, \sigma) & \text{for all } \sigma \text{ such that } v_0(t) \leq \sigma \leq w_0(t) \text{ and } w_{0i}(t) = \sigma_i. \end{cases}$$

We then have the following classical result of Müller.

Theorem B. Let $E = \mathbb{R}^n$ and $K = \mathbb{R}_+^n$. Suppose that v_0, w_0 satisfy (1.3). Then the conclusion of Theorem A holds. See for the details of proofs [1,7].

We observe that the proofs of Theorems A and B depend crucially on the modification of f , that is $\tilde{f}$ where $\tilde{f}(t,u) = f(t, p(t,u))$ and

$$p_i(t,u) = \max[v_{0i}(t), \min\{u_i, w_{0i}(t)\}], \text{ for each } i.$$

Clearly this modification makes sense only when $K = \mathbb{R}_+^n$.

If K is an arbitrary cone, the inequalities (1.2) need no change. On the other hand, the inequalities (1.3) can be formulated in terms of functionals from K^* , namely, for $\phi \in K^*$

$$(1.4) \quad \begin{cases} \phi(v'_0 - f(t, \sigma)) \leq 0 & \text{for all } \sigma \text{ such that } v_0(t) \leq \sigma \leq w_0(t) \text{ and } \phi(v_0(t) - \sigma) = 0, \\ \phi(w'_0 - f(t, \sigma)) \geq 0 & \text{for all } \sigma \text{ such that } v_0(t) \leq \sigma \leq w_0(t) \text{ and } \phi(w_0(t) - \sigma) = 0. \end{cases}$$

This version of condition (1.3) allows us to consider cones K other than the standard cone. The question is whether Theorems A and B hold even when K is an arbitrary cone. Theorem B may not be valid even in $\mathbb{R}^n$ as was shown by Volkmann [8]. Consider the example in $\mathbb{R}^3$. Let $K = [u \in \mathbb{R}^3: (u_1^2 + u_2^2)^{1/2} \leq u_3]$. Take $v_0 = (0, 0, 0)$, $w_0 = (2, 0, 2)$, $f_1 = f_3 = 0$ and

$$f_2 = \begin{cases} u_1 & \text{if } u_1 \in [0, 1], \\ 2 - u_1 & \text{if } u_1 \in [1, 2], \\ 0 & \text{otherwise.} \end{cases}$$

But the solution through $u_0 = (1, 0, 1)$ is $u(t) = (1, t, 1)$ which does not remain in the sector $[v_0, w_0]$. Note also that f is Lipschitzian. Thus it is evident that f being Lipschitzian is not sufficient to prove Theorem B in the set up corresponding to (1.4).

Recently results corresponding to Theorems A and B are proved in [3, 5]. In this paper, we consider a very general situation where f admits a mixed quasimonotone property and the results obtained include and unify several special cases.

2. MAIN RESULTS

To define appropriate classes of upper and lower solutions of (1.1) suitable to our needs, we shall suppose that f admits a decomposition of the form

$$(2.1) \quad f(t, u) = f_0(t, u) + f_1(t, u) + f_2(t, u),$$

where $f_0, f_1, f_2 \in C[\Omega, E]$.

Definition 2.1. Let $v_0, w_0 \in C^1[I, E]$. Then v_0, w_0 are said to be coupled

lower and upper quasi-solutions of (1.1) if

$$(2.2) \quad \begin{cases} v'_0 \leq f_0(t, v_0) + f_1(t, v_0) + f_2(t, w_0), & v(0) \leq u_0, \\ w'_0 \geq f_0(t, w_0) + f_1(t, w_0) + f_2(t, v_0), & w(0) \geq u_0. \end{cases}$$

If in (2.2), equalities hold, then v_0, w_0 are said to be coupled quasi-solutions of (1.1).

Clearly one can define, based on Definition 2.1, coupled maximal and minimal quasi-solutions of (1.1).

We also need a stronger form of coupled upper and lower quasi-solutions of (1.1).

Definition 2.2. Let $v_0, w_0 \in C^1[I, E]$ be such that $v_0(t) \leq w_0(t)$ on I . Then v_0, w_0 are said to be strongly coupled lower and upper quasi-solutions of (1.1) if there exists an $M > 0$ such that

$$(2.3) \quad \begin{cases} \phi[v'_0 - f_0(t, \sigma) - f_1(t, v_0) - f_2(t, w_0) + M(v_0 - \sigma)] \leq 0, \\ \phi[w'_0 - f_0(t, \sigma) - f_1(t, w_0) - f_2(t, v_0) + M(w_0 - \sigma)] \geq 0, \end{cases}$$

for all σ such that $v_0(t) \leq \sigma \leq w_0(t)$, $t \in I$ and $\sigma \in K^*$.

We list for convenience the following assumptions.

(A₁) For any bounded set B in E ,

$$\alpha(f(I \times B)) \leq L\alpha(B);$$

(A₂) $\|f(t, u_1) - f(t, u_2)\| \leq L\|u_1 - u_2\|$, $(t, u_1), (t, u_2) \in \Omega$;

(A₃) $f_1(t, u)$ is nondecreasing in u and $f_2(t, u)$ is nonincreasing in u relative to K ;

(A₄) $f_0(t, u_1) - f_0(t, u_2) \geq -M(u_1 - u_2)$ whenever $(t, u_1), (t, u_2) \in \Omega$ and $u_2 \leq u_1$;

(A₅) For any $\phi \in K^*$, $t \in I$,

$$\phi(f_0(t, u_1) - f_0(t, u_2)) = 0 \quad \text{if} \quad \phi(u_1 - u_2) = 0.$$

If f satisfies (2.1), (A₃), (A₄) and (A₅), we shall say that f is a mixed quasimonotone map. This decomposition then corresponds to a similar decomposition in finite dimensional case, see [6].

Remark 2.1. Suppose that f is uniformly continuous. Then condition (A₁) is redundant because in that case we have

$$\alpha(f(I \times B)) = \max_I \alpha(f(t, B))$$

and as a result (A₂) implies (A₁), see [2].

Let us consider the linear IVP

$$(2.4) \quad u' = F(t, u), \quad u(t) = u_0,$$

where $F(t, u) = f_0(t, \eta(t)) + f_1(t, \eta_1(t)) + f_2(t, \eta_2(t)) - M(u - \eta_1(t))$, and $\eta_1, \eta_2 \in C[I, E]$ such that $\eta_1, \eta_2 \in [v_0, w_0]$. Then we have

Lemma 2.1. Let (A₁) hold. Then for any $\eta_1, \eta_2 \in C[I, E]$ such that $\eta_1, \eta_2 \in [v_0, w_0]$, there exists a unique solution $u(t)$ on I of (2.4).

The proof of this lemma is similar to a corresponding result given in [3] with minor modifications.

For any $\eta_1, \eta_2 \in C[I, E]$ such that $\eta_1, \eta_2 \in [v_0, w_0]$, we define the mapping $A: [v_0, w_0] \rightarrow C[I, E]$ by

$$(2.5) \quad A[\eta_1, \eta_2] = u$$

where $u = u(t)$ is the unique solution of (2.4) on I . Concerning the map

A, we prove the following

Lemma 2.2. Let $v_0, w_0 \in C^1[I, E]$ with $v_0(t) \leq w_0(t)$ on I . Suppose that either (a) (2.2), (A_1) , (A_3) and (A_4) holds, or (b) (2.3), (A_1) and (A_3) holds. Then A maps the conical segment $[v_0, w_0]$ into itself. In case of (a), A possesses further a mixed monotone property on $[v_0, w_0]$, that is, $A[\eta_1, \eta_2] \leq A[\eta_2, \eta_1]$ whenever $\eta_1 \leq \eta_2$ and $\eta_1, \eta_2 \in [v_0, w_0]$.

Proof. Let $\eta_1, \eta_2 \in C[I, E]$ be such that $\eta_1, \eta_2 \in [v_0, w_0]$ and let $u = A[\eta_1, \eta_2]$ where $u = u(t)$ is the unique solution of (2.4) on I . We set $p = \phi(u - v_0)$ for $\phi \in K^*$ and note that $p(0) \geq 0$. If (a) holds, we have

$$\begin{aligned} p' &\geq \phi[f_0(t, \eta_1) + f_1(t, \eta_1) + f_2(t, \eta_2) - M(u - \eta_1) - f_0(t, v_0) \\ &\quad - f_1(t, v_0) - f_2(t, w_0)] \\ &\geq \phi[-M(\eta_1 - v_0) + f_1(t, v_0) + f_2(t, w_0) - M(u - \eta_1) \\ &\quad - f_1(t, v_0) - f_2(t, w_0)] = -Mp. \end{aligned}$$

If, on the otherhand, (b) is satisfied, one gets for all σ such that $v_0(t) \leq \sigma \leq w_0(t)$,

$$\begin{aligned} p' &\geq \phi[f_0(t, \eta_1) + f_1(t, \eta_1) + f_2(t, \eta_2) - M(u - \eta_1) - f_0(t, \sigma) \\ &\quad - f_1(t, v_0) - f_2(t, w_0) - M(v_0 - \sigma)]. \end{aligned}$$

choosing $\sigma = \eta_1$, we arrive at

$$\begin{aligned} p' &\geq \phi[f_0(t, \eta_1) + f_1(t, v_0) + f_2(t, w_0) - M(u - \eta_1) - f_0(t, \eta_1) \\ &\quad - f_1(t, v_0) - f_2(t, w_0) + M(v_0 - \eta_1)] = -Mp. \end{aligned}$$

Consequently, in both cases, we get $p(t) \geq p(0)e^{-Mt} \geq 0$ on I . Since $\phi \in K^*$ is arbitrary, this proves that $v_0 \leq u$ on I . A similar argument yields that $u \leq w_0$ on I . We thus have $A[v_0, w_0] \subset [v_0, w_0]$. It remains to prove that A is mixed monotone. Let $\eta_1 \leq \eta_2$, $u_1 = A[\eta_1, \eta_2]$ and $u_2 = A[\eta_2, \eta_1]$. We set $p = \phi(u_2 - u_1)$ and note $p(0) = 0$ where $\phi \in K^*$. Then, we obtain using (A_3)

$$\begin{aligned} p' &= \phi[f_0(t, \eta_2) + f_1(t, \eta_2) + f_2(t, \eta_1) - M(u_2 - \eta_2) - f_0(t, \eta_1) \\ &\quad - f_1(t, \eta_1) - f_2(t, \eta_2) + M(u_1 - \eta_1)] \\ &\geq \phi[-M(\eta_2 - \eta_1) - M(u_2 - \eta_2) + M(u_1 - \eta_1)] = -Mp. \end{aligned}$$

This shows as before $p(t) \geq 0$ on I proving $A[\eta_1, \eta_2] \leq A[\eta_2, \eta_1]$. The proof of lemma is complete.

In view of Lemma 2.2, we can define the sequences $\{v_n\}$, $\{w_n\}$ as follows:

$$v_{n+1} = A[v_n, w_n], \quad w_{n+1} = A[w_n, v_n] \quad \text{and} \quad v_n, w_n \in [v_0, w_0],$$

The following results relative to the sequences $\{v_n\}$, $\{w_n\}$ hold, whose proofs are similar to the corresponding results in [3,5].

Lemma 2.3. Let K be normal and let the assumptions of Lemma 2.2 be satisfied. Then the sequences $\{v_n\}$, $\{w_n\}$ are uniformly bounded, equicontinuous and relatively compact on I .

Lemma 2.4. Let K be normal and let the assumption (b) of Lemma 2.2 be satisfied. If (A_2) holds, then $m(t) = 0$ on I where either $m(t) = \limsup_{n \rightarrow \infty} \|v_n(t) - v_{n-1}(t)\|$ or $m(t) = \limsup_{n \rightarrow \infty} \|w_n(t) - w_{n-1}(t)\|$.

Proof. In view of foregoing Lemmas, because of (A_2) it follows, in case of (b), that $u(t) = \rho(t) = r(t)$ and $v_0(t) \leq u(t) \leq w_0(t)$ on I . In case (a), we have to show that (ρ, r) are coupled minimal and maximal quasi-solutions of (1.1) on I . To this end, let (u_1, u_2) be any coupled quasi-solutions of (1.1) such that $u_1, u_2 \in [v_0, w_0]$ on I . Assume that for some integer $k > 0$, we have $v_{k-1} \leq u_1, u_2 \leq w_{k-1}$ on I . Then setting $p = \phi(u_1 - v_k)$ and noting that $p(0) = 0$, we get

$$\begin{aligned} p' &= \phi[f_0(t, u_1) + f_1(t, u_1) + f_2(t, u_2) - f_0(t, v_{k-1}) - f_1(t, v_{k-1}) \\ &\quad - f_2(t, w_{k-1}) + M(v_k - v_{k-1})] \\ &\geq -Mp. \end{aligned}$$

This implies that $p(t) \geq 0$ and proves $v_k \leq u_1$ on I . A similar argument shows that $v_k \leq u_1, u_2 \leq w_k$ on I . Since $v_0 \leq u_1, u_2 \leq w_0$ on I by assumption, it follows by induction that $\rho \leq u_1, u_2 \leq r$ on I , proving that (ρ, r) are coupled minimal and maximal quasi-solutions of (1.1) on I . If (A_2) holds, in addition, one easily sets $r(t) = \rho(t) = u(t)$ and $v_0 \leq u \leq w_0$ on I . This completes the proof of the theorem.

Theorem 2.1 contains several special cases of importance. For example, consider the solution when (2.2) and (A_3) are satisfied. If (A_4) and (A_5) hold, then Theorem 2.1 contains an extension of the results in [6] to Banach spaces. If $f_1 \equiv 0$ and $f_2 \equiv 0$ or $f_2 \equiv 0$ and (A_5) holds, then we obtain the results of [3]. If $f_1 \equiv 0$ and (A_5) holds, then we have a new result. If $f_1 = 0$ one also obtains an extension of results in [6].

Consider next the solution when (2.3) and (A_3) hold. Then Theorem 2.1

In case (a), one can easily see that $\{v_n\}, \{w_n\}$ are monotone sequences such that

$$v_0 \leq v_1 \leq \dots \leq v_n \leq w_n \leq \dots \leq w_1 \leq w_0 \quad \text{on } I.$$

Hence, in this case, the whole sequences $\{v_n\}, \{w_n\}$ converge uniformly and monotonically to continuous functions ρ and r , that is,

$$\lim_{n \rightarrow \infty} v_n(t) = \rho(t), \quad \lim_{n \rightarrow \infty} w_n(t) = r(t) \quad \text{on } I.$$

In case (b) holds, Lemmas 2.3 and 2.4 and (A_2) permit us to conclude that the whole sequences $\{v_n\}, \{w_n\}$ converge uniformly to $\rho(t), r(t)$ respectively on I . It is then easy to show from (2.4) that $\rho(t)$ and $r(t)$ are coupled solutions of (1.1) on I in each case.

We can now state our main results.

Theorem 2.1. Let K be normal and let $v_0, w_0 \in C^1[I, E]$ with $v_0(t) \leq w_0(t)$ on I . If the assumptions (a) (2.2), (A_1) , (A_3) and (A_4) hold, then there exist sequences $\{v_n\}, \{w_n\}$ which converge uniformly and monotonically to coupled minimal and maximal quasi-solutions (ρ, r) of (1.1) on I ; that is, (u_1, u_2) is any coupled quasi-solutions such that $v_0 \leq u_1, u_2 \leq w_0$ on I , then

$$v_0 \leq v_1 \leq \dots \leq v_n \leq \rho \leq u_1, u_2 \leq r \leq w_n \leq \dots \leq w_1 \leq w_0 \quad \text{on } I,$$

provided $v_0(0) \leq u_0 \leq w_0(0)$. If, in addition to (a), the condition (A_2) is satisfied, then $\rho(t) = r(t) = u(t)$ and $v_0 \leq u \leq w_0$ on I . If, on the other hand, the assumptions (b) (2.3), (A_1) , (A_2) and (A_3) hold, then there exists a unique solution $u(t)$ of (1.1) on I such that $v_0(t) \leq u(t) \leq w_0(t)$ on I provided $v_0(0) \leq u_0 \leq w_0(0)$.

contains a very general extension of Müller's Theorem [1,7] to arbitrary abstract cones. If $f_1 \equiv 0$ and $f_2 \equiv 0$ or $f_2 \equiv 0$ and (A_5) holds, then we have the results of [5]. If $f_1 \equiv 0$ and (A_5) holds, then we obtain a new result.

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