GENERALIZED TRANSVERSALITY, EXCHANGE OF STABILITY
AND HOPF BIFURCATION

by

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1. INTRODUCTION

In Hopf bifurcation theory often an exchange of stability of an equilibrium gives rise to the creation of periodic orbits for a one parameter family of differential equations. In particular, let us consider the system in R^n given by

$$(1_\mu) \quad \begin{aligned} \dot{x} &= \alpha(\mu)x - \beta(\mu)y + X(\mu, x, y, z) \\ \dot{y} &= \alpha(\mu)y + \beta(\mu)x + Y(\mu, x, y, z) \\ \dot{z} &= A_\mu z + Z(\mu, x, y, z). \end{aligned}$$

where $\mu \in [0, \bar{\mu})$ for $\bar{\mu}$ sufficiently small, $\alpha(\mu)$, $\beta(\mu)$ and A_μ are C^∞ in μ with $\alpha(0) = 0$ and $\beta(0) = 1$. Assume $X, Y \in C^\infty[[0, \bar{\mu}) \times B^n(r_0)]$, $Z \in C^\infty[[0, \bar{\mu}) \times B^n(r_0), R^{n-2}]$ where $B^n(r_0) = \{w \in R^n, \|w\| < r_0\}$ and for each μ , X, Y, Z are of order greater than one at the origin. Finally, the eigenvalues $\{\lambda_j\}_{j=1}^{n-2}$ of the $(n-2) \times (n-2)$ matrix A_0 satisfy the non-resonance condition $\lambda_j \neq mi$, $m = 0, \pm 1, \dots$. We shall refer to the right hand sides of (1_0) and (1_μ) as $f_0(x, y, z)$ and $f_\mu(x, y, z)$ respectively.

In the case of R^2 the author [2] established a relationship between the maximum number of bifurcating periodic orbits in terms of the strength of attraction of the origin of (1_0) coupled with the "rate" with which the eigenvalues $\alpha(\mu) \pm i\beta(\mu)$ of (1_μ) cross the imaginary axis at $\mu = 0$. Another result was concerned with the case in which the instability of the origin of (1_μ) was not caused by the linear part of (1_μ) but by the higher order terms. In this case, the maximum number

of bifurcating periodic orbits was given in terms of the strength of attraction of the origin of (1_0) coupled with the strength of repulsion of the origin of (1_μ) .

In this paper, we wish to extend the results in [2] to the general case in R^n . In doing so, we shall be able to provide a more refined value on the maximum number of bifurcating orbits than those given by Flockerzi [5] and Kielhöfer [6]. Also, in contrast to the results in [5] and [6] we shall be able to state our results in terms of the "amount" of change of stability of the origin as the parameter μ crosses its critical value $\mu = 0$. In order to do so we need to discuss a structural property on the asymptotic stability of the origin of (1_0) . This type of structural property was used by Andronov et.al, [1] in his work on bifurcation theory and more recently by Negrini and Salvadori [7]. In particular, the origin of (1_0) is said to be h -asymptotically stable (h -completely unstable) if h is the smallest positive integer such that the origin of the perturbed system

$$\dot{w} = f_0(w) + g(w)$$

(where $w = (x, y, z)$) is asymptotically stable (completely unstable) for all g for which $g(w) = o(\|w\|^h)$.

Based on ideas given by Bernfeld, Negrini, and Salvadori [3], we shall obtain a two dimensional system in R^2 which is close in some sense to (1_μ) . The stability properties of the origin of this two dimensional system will be essential in finding the maximum number of

periodic orbits of (1_μ) whose period is near 2π .

2. REDUCTION PRINCIPLE

In order to construct the two dimensional system alluded to in the introduction, we consider an $(n-2)$ vector polynomial of some degree h , $h \geq 1$, in (x, y) and C^∞ in μ given by

$$\phi^{(h)}(\mu, x, y) = \phi_1(\mu, x, y) + \dots + \phi_h(\mu, x, y),$$

where $\phi_j(\mu, x, y)$ is homogeneous of degree j in (x, y) . We want to determine $\phi_1 \dots \phi_h$ in order to obtain along the solutions of (1_μ)

$$\left[\frac{d}{dt} (z - \phi^{(h)}(\mu, x, y)) \right]_{z=\phi^{(h)}(\mu, x, y)} = o(x^2 + y^2)^{h/2};$$

that is, we have to satisfy

$$\begin{aligned} & \frac{\partial \phi^{(h)}(\mu, x, y)}{\partial x} [\alpha(\mu)x - \beta(\mu)y + X(\mu, x, y, \phi^{(h)}(\mu, x, y))] \\ & + \frac{\partial \phi^{(h)}(\mu, x, y)}{\partial y} [\alpha(\mu)y + \beta(\mu)x + Y(\mu, x, y, \phi^{(h)}(\mu, x, y))] \\ & = A_\mu \phi^{(h)}(x, y) + Z(\mu, x, y, \phi^{(h)}(\mu, x, y)) + o(x^2 + y^2)^{h/2} \end{aligned}$$

This implies for every j , $1 \leq j \leq h$, ϕ_j has to satisfy the partial differential equation

$$\begin{aligned} & \frac{\partial \phi_j(\mu, x, y)}{\partial y} [\alpha(\mu)y + \beta(\mu)x] + \frac{\partial \phi_j(\mu, x, y)}{\partial x} [\alpha(\mu)x - \beta(\mu)y] \\ (2_\mu) \quad & = A_\mu \phi_j + U_j \end{aligned}$$

where U_j is a $n-2$ vector homogeneous polynomial of degree j depending on the functions $\phi_1 \dots \phi_{j-1}$. For $\mu = 0$, (2_0) has a unique solution

since the linear part of the system determining $\phi_j(0, x, y)$ has eigenvalues of the form $\lambda_s \pm mi$ where λ_s are the eigenvalues of A_0 and m an integer. The assumption on the eigenvalues of A_0 imply (2_0) has a unique solution. Moreover, there exists $\hat{\mu} \leq \bar{\mu}$ such that for $\mu < \hat{\mu}$ ($\hat{\mu}$ independent of h) (2_μ) has a solution. The solution can be solved recursively by observing that $\phi_1(\mu, x, y) \equiv 0$. The two dimensional surface $z = \phi^{(h)}(\mu, x, y)$ is tangent at the origin to the eigenspace corresponding to the eigenvalues $\alpha(\mu) \pm i\beta(\mu)$.

Given any $h > 0$ define the following two dimensional system for $\mu < \hat{\mu}$

$$(S_h^\mu) \quad \begin{aligned} \dot{x} &= \alpha(\mu)x - \beta(\mu)y + X(\mu, x, y, \phi^{(h)}(\mu, x, y)) \\ \dot{y} &= \alpha(\mu)y + \beta(\mu)x + Y(\mu, x, y, \phi^{(h)}(\mu, x, y)). \end{aligned}$$

We shall be considering the following possible cases: (1) there exists an h (and h must be odd) such that the origin of (S_h^0) is h -asymptotically stable; (2) case (1) does not hold. The h in case (1) can be found using linear algebraic techniques by computing an appropriate Lyapunov function. The procedure is due to Poincaré [8]. Our first result extends one of the results of Negrini and Salvadori [7] to the case in R^n .

Theorem 1. Assume there exists an $h > 1$ such that the origin of system (S_h^0) is h -asymptotically stable and that $\alpha(\mu)$ satisfies the transversality condition

$$(T) \quad \alpha'(0) > 0.$$

Then for each $\mu \in (0, \mu^*)$, $\mu^* \leq \hat{\mu}$, μ^* sufficiently small, there exists a unique periodic orbit of (1_μ) having period $T(\mu)$, where $T(\mu) \rightarrow 2\pi$ as $\mu \rightarrow 0$, which lies in a sufficiently small neighborhood of the origin.

3. GENERALIZED TRANSVERSALITY

In this section we remove the transversality condition (T) in Theorem 1 and assume a more generalized condition on the eigenvalues $\alpha(\mu) \pm i\beta(\mu)$ as they cross the imaginary axis at $\mu = 0$. Namely, assume there exists an integer $s \geq 1$ such that

$$(GT) \quad \alpha^{(j)}(0) = 0 \quad j = 0 \dots s-1, \quad \alpha^{(s)}(0) > 0.$$

The crossing is transversal in case $s = 1$. With no assumptions on the stability properties of the origin, Flockerzi [5] proved that up to s periodic solutions may exist for each μ . This work was expanded upon in several directions by Kielhöfer [6] using the Lyapunov-Schmidt method.

We look upon the number s as a type of measure of the instability of the origin of (S_h^μ) . We now wish to relate the number s and the number h associated with the asymptotic stability of the origin of (S_h^0) with the maximum number of bifurcating orbits of (1_μ) .

Theorem 2. Assume that there exists an $h > 1$ such that the origin of (S_h^0) is h -asymptotically stable and that $\alpha(\mu)$ satisfies (GT). Let $k = (h-1)/2$ and define

$$\ell = \min(s, k).$$

Then for each $\mu > 0$ and sufficiently small there exists at least one

bifurcating orbit and at most ℓ bifurcating orbits. In the remaining possible case in which there exists no h for which the origin of (S_h^0) is h -asymptotically stable then there are at most s bifurcating orbits.

Theorem 2 is sharp in the following sense: given the system $\dot{w} = f_0(w)$ and any integer j , $1 \leq j \leq \ell$ we can find a one-parameter family of differential equations $\dot{w} = f(\mu, w)$ with $f_0(w) = f(0, w)$ having j periodic orbits. This result extends the case in R^2 given in [2] and also addresses the case in which " $h = \infty$ ". Our result complements those given in [5] and [6] in that our additional assumption of h -asymptotic stability provides a more refined value on the maximum number of periodic orbits. When $k = \frac{h-1}{2}$ is less than s then there are at most k bifurcating orbits. In our discussion of bifurcating orbits, we are only considering the ones with period near 2π .

4. CHANGE OF STABILITY

The assumption that $\alpha(\mu) > 0$ for $\mu > 0$ implies that the origin of (S_h^μ) is completely unstable for any $h > 1$ and that it is recognizable by the linear part of the equation. There are examples in which $\alpha(\mu) \equiv 0$ for $\mu \geq 0$ and μ small such that the instability of the origin of (S_h^μ) is due to the higher order terms. A particular case of this in R^2 was considered by Caprino and Negrini [4] in which the right hand side of (1_μ) in R^2 begins with terms of order r , $r \geq 1$.

Let us assume that there exists an h such that the origin of (S_h^0) is h -asymptotically stable. For each $\mu > 0$ and sufficiently small assume there exists $\bar{h}(\mu)$ such that the origin of $(S_{\bar{h}(\mu)}^\mu)$ is $\bar{h}(\mu)$ completely

unstable. The special case $\bar{h}(\mu) \equiv 1$ (or equivalently $u(\mu) > 0$ for $\mu > 0$) is the usual assumption in Hopf bifurcation. We now give a relationship between $(h-\bar{h})$, which is a measure of the change of stability of the origin, and the maximum number of periodic orbits of (1_μ) . We shall thus assume

(A) There exists an h such that the origin of (S_h^0) is h -asymptotically stable. For $\mu > 0$ and sufficiently small there exists $\bar{h}(\mu)$ such that the origin of $(S_{\bar{h}}^\mu)$ is completely unstable.

We then have the existence of $\mu_0 > 0$ such that the following results hold under assumption (A).

Lemma 1. $\bar{h}(\mu) \equiv \bar{h}$ for all $\mu \in (0, \mu_0]$; that is $\bar{h}(\mu)$ is a fixed odd integer.

Lemma 2. $h > \bar{h}$.

Now define

$$(3) \quad m = (h - \bar{h})/2.$$

We now have the following relationship between m and the number of bifurcating orbits of (1_μ) .

Theorem 3. Assume condition (A). Then there exists an integer j , $1 \leq j \leq m$ such that the system (1_μ) has exactly j periodic orbits in a small neighborhood of the origin for all μ sufficiently small.

As before, this result is sharp in the sense that given system (1_0) and any integer j , $1 \leq j \leq m$ we can construct a one parameter system (1_μ) having j bifurcating periodic orbits.

The proofs of these results are based on the Poincaré map for (S_h^μ) and the corresponding displacement function and their relationship to the original systems (1_μ) . We do not include them here. In the case of the existence of a center manifold our results can be stated in terms of the stability of the origin of the original system (1_0) .

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