EXISTENCE AND COMPARISON RESULTS FOR DIFFERENTIAL
EQUATIONS OF SOBOLEV TYPE

by

A. S. Vatsala
Department of Mathematics
University of Texas at Arlington

and

R. L. Vaughn*
Department of Mathematics
Texas Christian University
Fort Worth, Texas

UN

Technical Report No. 100
December, 1978

*Research partially supported by U.S. Army Research Grant DAAG29-77-G0062.

1. INTRODUCTION

Recently a new class of differential equations, called differential equations of Sobolev type was studied in [4] in which an existence theorem of Picards type was investigated as well as a variation of constants formula. In [5], existence and comparison results for a class of Volterra integral equation of Sobolev-type were discussed.

In this paper we recall the Peano type existence result from [5] for Sobolev-differential equations and show that solutions can be extended to the entire square under consideration. This result extends results found in [1] for nonlinear Volterra integral equations. Our results include a comparison result in addition to the usual type of differential inequalities, and a differential inequality theorem such as Müller's [6]. This in turn proves the existence of extremal solutions. For special cases of the above results see [2,3,7]

2. LOCAL EXISTENCE

We consider equations of the form

$$u'(t, x) = f(t, x, u(t, x), u(x, t)), \quad u(t_0, x) = u_0(x), \quad " ' = \frac{d}{dt} " \quad (2.1)$$

where $u_0 \in C[J, R^n]$, $J = [t_0, t_0 + \alpha]$ and $f \in C[J \times J \times R^n \times R^n, R^n]$.

We need the following assumptions and definitions:

$$(A_1) \quad ||f(t, x, u, v)|| \leq M \quad \text{for all } (t, x, u, v) \in J \times J \times R^n \times R^n.$$

$$(A_2) \quad \lim_{x_1 \rightarrow x_2} \left(\sup_{\phi} \left\{ \int_I ||f(s, x_1, \phi(s, x_1), \phi(x_1, s)) - f(s, x_2, \phi(s, x_2), \phi(x_2, s))|| ds : \right. \right. \\ \left. \left. \phi \in C[J \times J, R^n] \right\} \right) = 0$$

$$(A_3) \quad \lim_{x_1 \rightarrow x_2} \left(\sup_{\phi} \left(\sup_{\psi} \left\{ \int_I ||f(s, x_1, \phi(s, x_1), \psi(x_1, s)) - f(s, x_2, \phi(s, x_2), \psi(x_2, s))|| ds : \right. \right. \right. \\ \left. \left. \left. \phi, \psi \in C[J \times J, R^n] \right\} \right) \right) = 0$$

$$(A_4) \quad |f(t, x, u, v) - f(t, x, \bar{u}, v)| \leq L|u - \bar{u}|$$

We now prove the following existence result.

Theorem 2.1. Suppose that $u_0 \in C[J, R^n]$, $f \in C[J \times J \times R^n \times R^n, R^n]$ satisfying the assumptions (A_1) and (A_2) . Then a solution to (2.1) exists on $[t_0, t_0 + \alpha]$ for some $\alpha > 0$.

Proof: Since u_0 is continuous on J , $u_0(J)$ is bounded and uniformly continuous. Thus $\exists N > 0$ so that

$$|u_0(x) - u_0(\bar{x})| < N \quad \text{for every } x, \bar{x} \in J.$$

Let $\alpha = \min\{\alpha, \frac{N}{M}\}$ and let $J_\alpha = [t_0, t_0 + \alpha]$.

Define $A \subseteq C[J_0 \times J_0, R^n]$ by

$$A = \{\phi \in C[J_\alpha \times J_\alpha, R^n] : \sup_{t, x \in J_\alpha} \|\phi(t, x) - u_0(x)\| \leq N\}.$$

clearly A is closed, bounded and convex.

For any $\phi \in A$, define the function $T\phi$ by

$$(T\phi)(t, x) = u_0(x) + \int_{t_0}^t \|f(s, x, \phi(s, x), \phi(x, s))\| ds.$$

$$\begin{aligned} \text{Then } \|(T\phi)(t, x) - u_0(x)\| &\leq \int_{t_0}^t \|f(s, x, \phi(s, x), \phi(x, s))\| ds \\ &\leq \alpha M \leq N. \end{aligned}$$

Thus $TA \subseteq A$.

Also $\|(T\phi)(t, x)\| \leq \sup_{x \in J_\alpha} \|u_0(x)\| + N$. Thus TA is uniformly bounded.

We now show that TA is equicontinuous. Let $\epsilon > 0$ be given and let $t_1, x_1, t_2, x_2 \in J_\alpha$, then

$$\|(T\phi)(t_1, x_1) - (T\phi)(t_2, x_2)\| \leq \|u_0(x_1) - u_0(x_2)\| + \int_{t_1}^{t_2} \|f(s, x_2, \phi(s, x_2), \phi(x_2, s))\| ds$$

$$+ \int_{t_0}^{t_1} || f(s, x_2, \phi(s, x_2), \phi(x_2, s)) - f(s, x_1, \phi(s, x_1), \phi(x_1, s)) || ds$$

$$= I_1 + I_2 + I_3.$$

Since $u_0(x)$ is uniformly continuous, we can choose δ_1 , so that $|x_1 - x_2| < \delta_1 \Rightarrow I_1 < \epsilon/3$. Also $I_2 \leq (t_2 - t_1)M$, thus if $|t_2 - t_1| < \frac{\epsilon}{3M} = \delta_2$, $I_2 < \epsilon/3$. Now using (A_2) we can choose δ_3 so that $|x_1 - x_2| < \delta_3 \Rightarrow I_3 < \epsilon/3$.

Thus if $\max\{|t_1 - t_2|, |x_1 - x_2|\} < \min\{\delta_1, \delta_2, \delta_3\} = \delta$

$$|| (T\phi)(t_1, x_1) - (T\phi)(t_2, x_2) || < \epsilon.$$

Thus TA is equicontinuous, and $\overline{TA}$ is compact.

Now let $\{\phi_n\} \in A$ be a sequence converging to ψ . Since f is continuous

$$\int_{t_0}^t f(s, x, \phi_n(s, x), \phi_n(x, s)) ds \rightarrow \int_{t_0}^t f(s, x, \psi(s, x), \psi(x, s)) ds.$$

Thus $T\phi_n \rightarrow T\psi$ and therefore T is continuous. Now applying the Schauder fixed point Theorem, the proof is complete.

Our next result provides conditions for the extension of solutions to equation (2.1).

Theorem 2.2. Let $u_0 \in C[J, R^n]$, and $f \in C[J \times J \times R^n \times R^n, R^n]$, and suppose that assumptions (A_1) , (A_3) and (A_4) hold. Then any solution,

u , of (2.1) which exists on $J_\alpha \times J_\alpha$ can be extended to $J_\beta \times J_\beta$ where $\beta = \min(2\alpha, a)$.

Proof: Let u be a solution of (1.1). Let $\gamma = \min(a/2, \alpha)$. Restrict u to $J_\gamma \times J_\gamma$.

Consider the equation

$$U'(t, x) = F(t, x, U(t, x), U(x, t)), \quad U(t_0, x) = U_0(x), \quad (2.2)$$

where $U_0(x) = (u(t_0 + \gamma, x), u_0(x + \gamma))$,

and $F(t, x, V, W) = (f(t + \gamma, x, u_1, w_2), f(t, x + \gamma, w_1, u_2))$

where $U = (u_1, u_2)$, $W = (w_1, w_2)$ with $u_i, w_i \in R^n$ for $i = 1, 2$.

It is clear that $U_0 \in C[J_\gamma, R^{2n}]$ and that $F \in C[J_\gamma \times J_\gamma \times R^{2n} \times R^{2n}, R^{2n}]$ and it is easy to verify that

$$\|U_0(x) - U_0(\bar{x})\| \leq \sqrt{10} N \quad \text{and} \quad \|F(t, x, V, W)\| \leq \sqrt{2} M, \quad \text{for} \\ t, x, \bar{x} \in J_\gamma.$$

Thus there exists a solution $U(t, x) = (u_1(t, x), u_2(t, x))$ to (2.2) on $J_\beta \times J_\beta$ where $\beta = \min(\gamma, \sqrt{5} N/M) = \gamma$.

Note that $u_1'(t, x) = f(t + \gamma, x, u_1(t, x), u_2(x, t)), u_2'(t, x) = f(t, x + \gamma, u_2(t, x), u_1(x, t))$, and $u_1(x_0, s) \equiv u(x_0 + \gamma, s)$.

Now let $m(t) = |u_2(t, x_0) - u(t, x_0 + \gamma)|$, then $m(0) = 0$, and by assumption (A_4) $m'(t) \leq |f(t, x_0 + \gamma, u_2(t, x_0), u_1(x_0, t)) - f(t, x_0 + \gamma, u(t, x_0 + \gamma), u(x_0 + \gamma, t))| \leq L m(t)$. Thus $m(t) \equiv 0$, and so $u_2(t, x_0) \equiv u(t, x_0 + \gamma)$.

Now consider the equation

$$\tilde{u}'(t, x) = \tilde{f}(t, x, \tilde{u}(t, x), \tilde{u}(x, t)), \quad \tilde{u}(t_0, x) = u_2(t_0 + \gamma, x), \quad (2.3)$$

where $\tilde{f}$ is defined by $\tilde{f}(t, x, u, w) = f(t + \gamma, x + \gamma, u, w)$.

Using Theorem 2.1 we conclude that there exists a solution $\tilde{u}(t, x)$ to (2.3) on $J_\gamma \times J_\gamma$, and using (A_4) , as above we find that $\tilde{u}(t, x_0) \equiv u_1(t, x_0 + \gamma)$.

Now define the function $\bar{u}(t, x)$ on $J_\beta \times J_\beta$ as follows:

$$\bar{u}(t, x) = \begin{cases} u(t, x) & t, x \in J_\gamma \\ u_1(t - \gamma, x) & t \in [t_0 + \gamma, t_0 + \beta], \quad x \in J_\gamma \\ u_2(t, x - \gamma) & x \in [t_0 + \gamma, t_0 + \beta], \quad t \in J_\gamma \\ \tilde{u}(t - \gamma, x - \gamma) & t, x \in [t_0 + \gamma, t_0 + \beta]. \end{cases}$$

We need only establish that $\bar{u}$ is an extension of u . We verify one case, suppose that $t \in [t_0 + \gamma, t_0 + \beta]$, $x \in J_\gamma$, then

$$\begin{aligned} \bar{u}'(t, x) &= u_1'(t - \gamma, x) = f(t, x, u_1(t - \gamma, x), u_2(x, t - \gamma)) \\ &= f(t, x, \bar{u}(t, x), \bar{u}(x, t)). \end{aligned}$$

The other cases are similar. Thus $\bar{u}$ extends the solution u to $J_\beta \times J_\beta$.

Remark: The above theorem can be used to extend solution of (2.1) to $J \times J$ as long as $a < +\infty$. One can easily see that if $\alpha \geq a/2$ that the value of β in Theorem 2.2 is α . For values of $\alpha < a/2$ one needs only to repeat the above argument a finite number of times to extend u to $J \times J$.

Corollary 2.3. Let u_0 and f be as in Theorem 2.2. Then solutions to (2.1) can be extended to $[t_0, t_0 + a] \times [t_0, t_0 + a]$ as long as a is finite.

3. DIFFERENTIAL INEQUALITIES

In this section, we develop the theory of differential inequalities.

Consider the following system of differential inequalities

$$D_- u(t, x) \leq f(t, x, u(t, x), u(x, t)) \quad (3.1)$$

$$D_- v(t, x) \geq f(t, x, v(t, x), v(x, t)) \quad (3.2)$$

where $D_- v(t, x) = \liminf_{h \rightarrow 0} \frac{v(t+h, x) - v(t, x)}{h}$

Definition 3.1. A function $f(t, x, u, v) \in C[J \times J \times R^n \times R^n, R^n]$ is said to be quasimonotone nondecreasing where $u, v \in C[J \times J, R^n]$ whenever $f_j(t, x, u, v) \leq f_j(t, x, \bar{u}, v)$ where $u_i \leq \bar{u}_i$ and $u_j = \bar{u}_j$ for every j , $j = 1, 2, \dots, n$.

Theorem 3.1. Let f be (i) quasimonotone nondecreasing in $u(t, x)$ and non-decreasing in $u(x, t)$ on $J \times J$, then if further if one of the inequalities above is strict and $u(t_0, x) < v(t_0, x)$, then $u(t, x) < v(t, x)$ on $J \times J$.

Proof: If the conclusion is not true, consider the set

$Z(t, x) = \{(t, x) | u(t, x) \geq v(t, x), u(x, t) \geq v(x, t)\}$ which is nonempty.

Let Z_t be the projection of Z on the t axis. Let $t_1 = \inf Z_t$.

Certainly $t_1 > t_0$. It then follows that there is an index j ,
 $1 \leq j \leq n$ and for $i = 1, 2, \dots, n$

$$u_i(s, x) < v_i(s, x) \quad \text{for } s, x \in [t_0, t_1] \times [t_0, t_0 + \alpha]$$

$$u_i(x, s) < v_i(x, s) \quad \text{for } x, s \in [t_0, t_0 + \alpha] \times [t_0, t_1].$$

and either

$$u_j(t_1, x) \leq v_j(t_1, x)$$

or

$$u_j(x, t_1) \leq v_j(x, t_1) \quad \forall x \in J.$$

Consequently there is an $x_1 \in J$ such that either

$$u_j(t_1, x_1) = v_j(t_1, x_1) \tag{3.3}$$

or

$$u_j(x_1, t_1) = v_j(x_1, t_1) . \tag{3.4}$$

Let x_1 be the minimum value of x for which (3.3) or (3.4) happens.

Certainly $x_1 > t_0$.

If (3.3) happens, then

$$\begin{aligned} D_- u_j(t_1, x_1) &= \liminf_{h \rightarrow 0^-} \frac{u_j(t_1 + h, x_1) - u_j(t_1, x_1)}{h} \\ &> \liminf_{h \rightarrow 0^-} \frac{v_j(t_1 + h, x_1) - v_j(t_1, x_1)}{h} = D_- v_j(t_1, x_1) \end{aligned}$$

But by hypothesis

$$\begin{aligned} D_- u_j(t_1, x_1) &\leq f_j(t_1, x_1, u(t_1, x_1), u(x_1, t_1)) \\ &\leq f_j(t_1, x_1, v(t_1, x_1), v(x_1, t_1)) < D_- v_j(t_1, x_1) \end{aligned}$$

which leads to a contradiction.

If (3.4) happens we have $u_j(x_1, t_1) = v_j(x_1, t_1)$. Let $x_1 = \tilde{t}$, $t_1 = \tilde{x}$ ie $u_j(\tilde{t}, \tilde{x}) = v_j(\tilde{t}, \tilde{x})$ and $u_j(\tilde{t}+h, \tilde{x}) < v_j(\tilde{t}+h, \tilde{x})$, for $h < 0$ by definition of x_1 and t_1 . Therefore

$$\begin{aligned} D_- u_j(\tilde{t}, \tilde{x}) &= \liminf_{h \rightarrow 0^-} \frac{u_j(\tilde{t}+h, \tilde{x}) - u_j(\tilde{t}, \tilde{x})}{h} \\ &> \liminf_{h \rightarrow 0^-} \frac{v_j(\tilde{t}+h, \tilde{x}) - v_j(\tilde{t}, \tilde{x})}{h} = D_- v_j(t_1, x_1) , \end{aligned}$$

but by hypothesis

$$\begin{aligned} D_- u_j(\tilde{t}, \tilde{x}) &\leq f_j(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u(\tilde{x}, \tilde{t})) \\ &\leq f_j(\tilde{t}, \tilde{x}, v(\tilde{t}, \tilde{x}), v(\tilde{x}, \tilde{t})) < D_- v_j(\tilde{t}, \tilde{x}) \end{aligned}$$

hence a contradiction and the theorem is complete.

Remark 3.1. The above theorem is true if the Dini derivative is replaced by any fixed Dini derivative. See for details [3].

If one of the inequalities (3.1), (3.2) is not assumed strict, the conclusion of theorem (3.1) fails to hold. However if f satisfies

a onesided Lipschitz condition, we get the following result.

Theorem 3.2. Let the assumption (1) of Theorem (3.1) hold. Suppose further that

$$f(t, x, v_1, w_1) - f(t, x, v_2, w_2) \leq L[(v_1 - v_2) + (w_1 - w_2)] \quad (3.5)$$

Whenever $v_1 \geq v_2, w_1 \geq w_2$. Then $u(t_0, x) \leq v(t_0, x)$ for $x \in J$ implies $u(t, x) \leq v(t, x)$ on $J \times J$.

Proof: Let $\tilde{v}(t, x) = v(t, x) + \epsilon e^{3L(t+x)}$ where $\epsilon > 0$ is sufficiently small vector in R^n .

Then $\tilde{v}'(t, x) = v'(t, x) + 3\epsilon L e^{3L(t+x)}$. That is

$$\tilde{v}'(t, x) = f(t, x, v(t, x), v(x, t)) + 3\epsilon L e^{3L(t+x)}.$$

$$\geq f(t, x, \tilde{v}(t, x), \tilde{v}(x, t)) + \epsilon L e^{3L(t+x)}$$

consequently we have

$$\tilde{v}'(t, x) > f(t, x, \tilde{v}(t, x), \tilde{v}(x, t)) \quad (3.6)$$

By Remark 3.1 we now get $u(t, x) < \tilde{v}(t, x)$ on $J \times J$. Taking the limit as $\epsilon \rightarrow 0$ we conclude $u(t, x) \leq v(t, x)$ on $J \times J$ which proves the stated result. We now obtain bounds for solutions of (2.1) based on the classical result of Müller's [6].

Theorem 3.2. Let $v, w \in C^1[J \times J, R^n]$ satisfy the following inequalities

$$v_i'(t, x) < f_i(t, x, Z(t, x), Z(x, t)) \text{ whenever } v(t, x) \leq Z(t, x) \leq w(t, x)$$

$$\text{and } v_i(t, x) = Z_i(t, x) \quad (3.7)$$

$$w_i'(t, x) > f_i(t, x, Z(t, x), Z(x, t)) \text{ whenever } v(t, x) \leq Z(t, x) \leq w(t, x)$$

$$\text{and } Z_i(t, x) = w_i(t, x) \quad (3.8)$$

Then if $v(t_0, x) < u(t_0, x) < w(t_0, x)$ then $v(t, x) < u(t, x) < w(t, x)$

where $u(t, x)$ is the solution to (2.1).

Proof: If not, consider the set

$$P(t, x) = \bigcup_{i=1}^n \{ (t, x) \mid v_i(t, x) \geq u_i(t, x) \geq w_i(t, x) \text{ or } v_i(x, t) \geq u_i(x, t) \geq w_i(x, t) \}.$$

Let P_t be the projection of P on the t axis. Let $t_1 = \inf P_t$.

Certainly $t_1 > t_0$ then there exists a j and an x_1 such that

$$v_i(t_1, x) < u_i(t_1, x) < w_i(t_1, x)$$

$$v_i(x, t_1) < u_i(x, t_1) < w_i(x, t_1) \text{ for } i \neq j$$

$$\text{and either } v_j(t_1, x) \leq u_j(t_1, x) \leq w_j(t_1, x) \text{ for all } x$$

or

$$v_j(x, t_1) \leq u_j(x, t_1) \leq w_j(x, t_1) \text{ for all } x.$$

Let x be the minimum value of x for which either $v_j(t_1, x_1)$

$$= u_j(t_1, x_1)$$

$$\text{or} \quad v_j(t_1, x_1) = u_j(t_1, x_1)$$

$$\text{or} \quad u_j(t_1, x_1) = w_j(t_1, x_1)$$

$$\text{or} \quad v_j(x_1, t_1) = u_j(x_1, t_1)$$

$$\text{or} \quad v_j(x_1, t_1) = w_j(x_1, t_1) \quad .$$

If $v_j(t_1, x_1) = u_j(t_1, x_1)$ then by (3.7) $v'_j(t_1, x_1) < f_j(t_1, x_1, u(t_1, x_1), u(x_1, t_1)) = u'_j(t_1, x_1)$ but $v_j(t_1 - h, x_1) < u_j(t_1 - h, x_1)$ hence $v'_j(t_1, x_1) \geq u'_j(t_1, x_1)$. This implies $u'_j(t_1, x_1) \leq v'_j(t_1, x_1) < u'_j(t_1, x_1)$ which is absurd. Similarly we can arrive at a contradiction when $u_j(t_1, x_1) = w_j(t_1, x_1)$.

Suppose $v_j(x_1, t_1) = u_j(x_1, t_1)$. Let $x_1 = \tilde{t}$, $\tilde{t}_1 = \tilde{x}$. Then $v_j(\tilde{t}, \tilde{x}) = u_j(\tilde{t}, \tilde{x})$ also $v_j(\tilde{t} - h, \tilde{x}) < u_j(\tilde{t} - h, \tilde{x})$ by the definition of x_1 .

$$u_j(\tilde{t}, \tilde{x}) \leq v_j(\tilde{t}, \tilde{x}) < f_j(\tilde{t}, \tilde{x}, u(\tilde{t}, \tilde{x}), u(\tilde{x}, \tilde{t})) = u_j(\tilde{t}, \tilde{x})$$

because $v(\tilde{x}, \tilde{t}) \leq u(\tilde{x}, \tilde{t}) < w(\tilde{x}, \tilde{t})$ and $v_j(\tilde{t}, \tilde{x}) = u_j(\tilde{t}, \tilde{x})$

which leads to a contradiction. In a similar way we can arrive at a contradiction when $v_j(x_1, t_1) = w_j(x_1, t_1)$.

Remark 3.2. The conclusion of the above theorem is not true if the inequalities (3.7), (3.8) are not strict. However if f satisfies a one-sided Lipschitz condition, of the following type

$$\begin{aligned} & f_i(t, x, (w_1(t, x), \dots, w_i(t, x), \dots, w_n(t, x)), (w_1(x, t), \dots, w_i(x, t), \dots, w_n(x, t))) \\ & - f_i(t, x, (w_1(t, x), \dots, v_i(t, x), w_n(t, x)), (w_1(x, t), \dots, v_i(x, t), \dots, w_n(x, t))) \\ & \leq L_i [(w_i(t, x) - v_i(t, x)) + (w_i(x, t) - v_i(x, t))] \end{aligned} \quad (3.9)$$

Whenever $w_i(t, x) \geq u_i(t, x)$ on $J \times J$ for $i = 1, \dots, n$ the conclusion is valid.

Proof: Consider $v_i(t, x) = v_i(t, x) - \epsilon \in e^{3L_i(t+x)}$

and $w_i(t, x) = w_i(t, x) + \epsilon \in e^{3L_i(t+x)}$

where $\epsilon > 0$ is arbitrarily small vector

now $\tilde{v}_i(t, x) = v_i(t, x) - 3L_i \in e^{L_i(t+x)}$

$$\leq f_i(t, x, (s_1(t, x), \dots, v_i(t, x), \dots, s_n(t, x)), (s_1(x, t), \dots, v_i(x, t), \dots, s_n(x, t)))$$

$$- \epsilon \in L_i e^{L_i(t+x)}$$

$$\leq f_i(t, x, s_1(t, x), \dots, \tilde{v}_i(t, x), \dots, s_n(t, x)), (s_1(x, t), \dots, \tilde{v}_i(x, t), \dots, s_n(x, t)))$$

$$- \epsilon \in L_i e^{L_i(t+x)}$$

That is

$$\tilde{v}_i(t, x) < f_i(t, x, s_1(t, x), \dots, \tilde{v}_i(t, x), \dots, s_n(t, x)), (s_1(x, t), \dots, \tilde{v}_i(x, t), \dots, s_n(x, t)))$$

$$\leq f_i(t, x, s(t, x), s(x, t)) \text{ whenever } \tilde{v}(t, x) \leq s(t, x) \leq \tilde{w}(t, x) \text{ and}$$

$$\tilde{v}_i(t, x) = s_i(t, x).$$

Similarly we can prove

$$\tilde{w}_i(t, x) > f_i(t, x, s(t, x), s(x, t)) \text{ whenever } \tilde{v}_i(t, x) \leq s_i(t, x) \leq \tilde{w}_i(t, x) \text{ and}$$

$$\tilde{w}_i(t, x) = s_i(t, x).$$

Now from Theorem 3.2 it follows that

$$\tilde{v}_i(t, x) < u_i(t, x) < \tilde{w}_i(t, x)$$

Taking the limit as $\epsilon \rightarrow 0$ it follows that

$$v_i(t, x) \leq u_i(t, x) \leq w_i(t, x) \quad \text{on } J \times J.$$

Remark: The existence of extremal solutions for (2.1) can be proved by using the results of Theorem 2.1, 2.2 and by Remark 3.1. The proof follows the same lines as in [3,5].

REFERENCES

- [1] T. L. Herman, "Existence and continuation properties of solutions of a nonlinear Volterra integral equation", (1976), Vol. 2 Dynamical Systems, An International Symposium, Academic Press (307-310).
- [2] H. H. Katojwada and R. E. Kalaba, "Extension of S. L. Sobolev's initial value method to nonlinear integral equations", *J. Nonlinear Analysis*, to be published.
- [3] V. Lakshmikantham and S. Leela, "Differential and integral inequalities", Academic Press, New York (1969).
- [4] V. Lakshmikantham, A. S. Vatsala and R. L. Vaughn, "Existence and comparison results for a class of Volterra integral equations of Sobolev type", to appear in *Journal of Integral Equations*.
- [5] V. Lakshmikantham and M. Lord, "Sobolev type Differential equations", *Appl. Math. and Comp.*, Vol. 4, No. 2 (1978), pp. 147-155.
- [6] M. Müller, "Über das Fundmental theorem in der theorie der gewöhnlichen Differentialgleichungen", *Math. z* 26(1926), 619-645.
- [7] Sobolev, S. L., (1954) *Uspelic Math Nauk* 9, No. 3, pp. 234-235.