**ERROR ESTIMATES OF SOLUTIONS AND MEAN
OF SOLUTIONS OF STOCHASTIC
DIFFERENTIAL SYSTEMS**

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ABSTRACT

Stochastic differential equations are considered. Estimates in terms of statistical properties are given for the difference between the solutions and solutions of the mean of stochastic differential systems. For this purpose necessary theorems are developed and sufficient conditions are given to obtain error estimates. A few examples are worked out to demonstrate the usefulness of the results.

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1. INTRODUCTION:

Mathematical modeling of several real world problems lead us to differential equations. In formulating the mathematical model one can ignore the randomness in the system and obtain a deterministic model. Such a deterministic model of a dynamic system can be described by systems of deterministic differential equations. However, if one incorporates the inherent randomness of a system into the mathematical model, then the dynamic of the system will be described by a stochastic differential equation with random parameters. In general, the laws governing the random phenomena and the corresponding parameters, are not precisely known. Therefore, one is interested in approximating stochastic model by means of deterministic model. Such an approximation will lead us to the study of estimating the error response corresponding to stochastic and deterministic models.

The objective of this paper is to estimate the error between the solution and the solution of the mean of a random differential equation. Very recently [1,2] problems of this nature with regard to roots of random polynomials have been investigated. Furthermore, certain relationships between the random eigenvalue problem and its corresponding mean problem, have been discussed in [3]. The present study provides a tool that verifies to what extent the deterministic mathematical model differs with the corresponding stochastic model. The mathematical conditions are given in terms of statistical properties of rate coefficients and the initial data of the system. In addition to this, we have obtained some sufficient conditions to guarantee the boundedness of solution with probability one (w.p. 1). The purpose of obtaining such result is to develop suitable and more feasible conditions to estimate the difference between the solution of the mean and the solution.

Several remarks and examples are given to indicate the usefulness of the result. Some related deterministic results can be found in [4] - [6].

We organize our article as follows: In section 2 we prove a few auxiliary results. In section 3 some results for the boundedness of the solutions with probability one are developed. Main theorems are proved in section 4, and to illustrate the theory a few examples are worked out in section 5.

2. GENERALIZED VARIATION OF CONSTANT FORMULA:

For our discussion we consider the initial value problem

$$(2.1) \quad y'(t, \omega) = F(t, y(t, \omega), \omega), \quad y(t_0, \omega) = y_0(\omega)$$

and the corresponding mean system of differential equations

$$(2.2) \quad m'(t) = f(t, m(t)), \quad m(t_0) = m_0 = E(y_0(\omega)),$$

where $f(t, z) = E[F(t, z, \omega)]$. From (2.1) and (2.2) we have

$$(2.3) \quad y'(t, \omega) = f(t, y(t, \omega)) + R(t, y(t, \omega), \omega), \quad y(t_0, \omega) = y_0(\omega),$$

where $R(t, y(t, \omega), \omega) = F(t, y(t, \omega), \omega) - E[F(t, y(t, \omega), \omega)]$.

In our presentation we will be using the following initial value problem also:

$$(2.4) \quad x' = f(t, x), \quad x(t_0, \omega) = y_0(\omega) = x_0(\omega)$$

where f is as defined in (2.2).

Hereafter the notations and definitions are adopted from [7]. Without further mention we assume that all the inequalities and relations involving random quantities are valid with probability 1 (w.p. 1).

Now we assume the following hypotheses:

(H₁) $R \in M[R_+ \times R^n, R[\Omega, R^n]]$ and R is almost sure sample continuous in

x for fixed $t \in R_+$; $f \in C[R_+ \times R^n, R^n]$, f_x exists and $f_x \in C[R_+ \times R^n, R^{n^2}]$.

(H₂) The random function F in (2.1) satisfies suitable regularity conditions so that the initial value problem (2.1) has sample solution process existing for $t \geq t_0$.

The above conditions imply that $x(t, \omega) = x(t, t_0, x_0(\omega))$ is a unique solution of (2.4), and further that $x(t, \omega)$ is sample continuously differentiable with respect to (t_0, x_0) .

Let $\phi(t, t_0, x_0(\omega))$ be the fundamental solution of the variational system associated with (2.4). Further assume that

(H₃) $V(t, x) \in C[R_+ \times R^n, R^m]$, V_x exists and is continuous for $(t, x) \in R_+ \times R^n$.

We now state and prove a few lemmas.

Lemma 2.1: Let the hypotheses (H₁) - (H₃) be satisfied and $x(t, \omega) = x(t, t_0, x_0(\omega))$ be the sample solution of (2.4) and $y(t, \omega) = y(t, t_0, y_0(\omega), \omega)$ be the sample solution of (2.3) for $t \geq 0$. Then

$$(2.5) \quad V(t, y(t, \omega)) = V(t, x(t, \omega)) + \int_{t_0}^t V_x(t, x(t, s, y(s, \omega))) \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds.$$

Proof: Let $x(t, s, y(s, \omega))$ be the sample solution processes of (2.4) through $(s, y(s))$, and $y(s, \omega) = y(s, t_0, y_0(\omega), \omega)$ be the sample solution processes of (2.3) through (t_0, y_0) . From the hypotheses (H₁) - (H₃) and [7: Theorem 2.6.4] we obtain

$$\begin{aligned} \frac{dV}{ds}(t, x(t, s, y(s, \omega))) &= V_x(t, x(t, s, y(s, \omega))) \left[\frac{d}{ds} x(t, s, y(s, \omega)) \right. \\ &\quad \left. + \frac{d}{dy} x(t, s, y(s, \omega)) \frac{d}{ds} y(s, \omega) \right] \\ &= V_x(t, x(t, s, y(s, \omega))) \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) \\ &\quad \text{w.p. 1.} \end{aligned}$$

Integrating (in the sample sense) both the sides from t_0 to t , and noting that $x(t, t, y(t, t_0, y_0(\omega), \omega)) = y(t, t_0, y_0(\omega), \omega)$, we obtain

$$V(t, y(t, \omega)) = V(t, x(t, \omega)) + \int_{t_0}^t V_x(t, x(t, s, y(s, \omega))) \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds.$$

This completes the proof of the lemma.

Some particular cases of Lemma 1.1 will be illustrated in the following remarks.

Remark 2.1: Let $V(t, x(t, \omega)) = x(t, \omega)$. Then in Lemma 2.1, (2.5) reduces to

$$y(t, \omega) = x(t, \omega) + \int_{t_0}^t \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds$$

This gives the result of Theorem 2.7.3 [7].

Remark 2.2: If $v(t, x(t, \omega)) = \|x(t, \omega)\|^2$ then in Lemma 2.1, (2.5) reduces to

$$\|y(t, \omega)\|^2 = \|x(t, \omega)\|^2 + 2 \int_{t_0}^t x(t, s, y(s, \omega)) \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds.$$

Now we prove another lemma which gives the expressions for the difference between the solutions of the perturbed system (2.3) and the initial value problem (2.4).

Lemma 2.2: Suppose all the hypotheses of Lemma 2.1 hold. Then

$$v(t, y(t, \omega) - x(t, \omega)) = V(t, 0) + \int_{t_0}^t V_x[t, x(t, s, y(s, \omega)) - x(t, s, x(s, \omega))] \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds.$$

Proof: By following the proof of Lemma 2.1 we have the relation

$$(2.7) \quad \frac{dV}{ds}(t, x(t, s, y(s, \omega)) - x(t, s, x(s, \omega))) = V_x[t, x(t, s, y(s, \omega)) - x(t, s, x(s, \omega))] \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega)$$

Integrating (2.7) from t_0 to t , and noting that $x(t, t, y(t, t_0, y_0(\omega), \omega)) = y(t, t_0, y_0(\omega), \omega)$, we complete the proof of the lemma.

The next lemma gives the expression for the difference between the solution of the perturbed system (2.3) and the solution of the mean system of equations (2.2)

Lemma 2.3: Suppose that all the hypotheses of Lemma 2.1 be satisfied.

Then

$$(2.8) \quad \begin{aligned} V(t, y(t, \omega) - m(t)) &= V(t, x(t, \omega) - m(t)) \\ &+ \int_{t_0}^t V_x[t, x(t, s, y(s, \omega)) - x(t, s, m(s))] \\ &\quad \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds, \end{aligned}$$

where $m(t) = m(t, t_0, m_0)$ is a solution of (2.2).

Proof: By noting the fact that $x(t, s, m(s)) = m(t)$, the proof of the lemma follows from Lemma 2.2.

A particular case of Lemma 2.3 is illustrated in the following remark which will be very much useful in studying the statistical properties of solution processes.

Remark 2.3: If $V(t, x) = \|x\|^2$, then from (2.8) we obtain

$$\begin{aligned} \|y(t, \omega) - m(t)\|^2 &= \|x(t, \omega) - m(t)\|^2 + \int_{t_0}^t [x(t, s, y(s, \omega)) \\ &\quad - x(t, s, m(s))] \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds. \end{aligned}$$

3. BOUNDEDNESS SOLUTIONS:

In this section we consider the boundedness of the solution processes w.p. 1 of (2.1). The obtained result will justify some of the assumption that will be made in the following sections in order to obtain estimates of

the difference between the solution of the mean (2.2) and the solution of (2.1).

Assume that F in (2.1) satisfies assumption (H_2) . Then the differential system (2.1) is said to be

(SB_1) bounded w.p. 1 (or almost surely sample bounded) if for each $\alpha \geq 0$, $t_0 \in J$, there exists a positive function $\beta = \beta(t_0, \alpha)$ which is continuous in t_0 for each α such that the inequality

$$\|y_0(\omega)\| \leq \alpha, \text{ w.p. 1}$$

implies $\|y(t, \omega)\| < \beta$, w.p. 1, $t \geq t_0$

(SB_2) Uniformly bounded w.p. 1 if β in (SB_1) is independent of t_0 .

Consider the following random comparison differential systems:

$$(3.1) \quad u'(t, \omega) = g(t, u(t, \omega), \omega), \quad u(t_0, \omega) = u_0(\omega),$$

where $g \in M[J \times \mathbb{R}^n, R[\Omega, \mathbb{R}^n]]$ is such that $g(t, u, \omega)$ satisfies the Caratheodory conditions in (t, u) w.p. 1 and $g(t, u, \omega)$ is quasi-monotonic, non-decreasing in u for fixed t w.p. 1.

The differential system (3.1) is said to be

(SB_1^*) bounded w.p. 1 if given $\alpha \geq 0$, $t_0 \in J$ there exists a positive function $\beta = \beta(t_0, \alpha)$ such that

$$\sum_{i=1}^n u_{i0}(\omega) \leq \alpha, \text{ w.p. 1}$$

implies

$$\sum_{i=1}^n u_i(t, \omega) < \beta, \quad t \geq t_0 \text{ w.p. 1.}$$

(SB_2^*) uniformly bounded w.p. 1 if β in (SB_1^*) is independent of t_0 .

We now prove the following theorem which is a probabilistic version of [8: Theorem 3.13.1].

Theorem 3.1: Assume that

(i) $g \in M[J \times R^m, R[\Omega, R^m]]$ and $g(t, u, \omega)$ is sample continuous and quasi-monotone nondecreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times R^n, R[\Omega, R^m]]$ satisfies a local Lipschitz condition in y w.p. 1 and for $(t, y) \in J \times R^n$

$$D_{(2.1)}^+ V(t, y(t, \omega)) \leq g(t, V(t, y(t, \omega))).$$

(iii) For $(t, y) \in J \times R^n$, $V(t, 0) \equiv 0$ and

$$\sum_{i=1}^m V_i(t, y(t, \omega)) \geq b(\|y\|),$$

where $b \in K$ on the interval $0 \leq u < \infty$ and $b(u) \rightarrow \infty$ as $u \rightarrow \infty$.

Then

$$SB_1^* \Rightarrow SB_1.$$

Proof: Let $\alpha \geq 0$ and $t_0 \in J$ be given and let $\|y_0(\omega)\| \leq \alpha$. From the hypothesis on $V(t, y(t, \omega))$, there exists a number $\alpha_1 = \alpha_1(t_0, \alpha)$ such that

$$\|y_0(\omega)\| \leq \alpha \text{ and } V(t_0, y_0(\omega)) \leq \alpha,$$

are satisfied, simultaneously. From the equiboundedness of (3.1), we have

$$(3.2) \quad \sum_{i=1}^m r_i(t, t_0, u_0(\omega), \omega) \leq \beta_1, \quad t \geq t_0,$$

$$\text{if} \quad \sum_{i=1}^m u_{i0}(\omega) \leq \alpha_1.$$

Further as $b(u) \rightarrow \infty$ as $u \rightarrow \infty$ we can choose a $\beta = \beta(t_0, \alpha)$, such that

$$(3.3) \quad b(\beta) \geq \beta_1(t_0, \alpha).$$

Now let

$$u_0(\omega) = V(t_0, y_0(\omega)).$$

Then from [3: Theorem 2.8.1] we obtain

$$(3.4) \quad V(t, y(t, \omega)) \leq r(t, t_0, u_0(\omega)), \quad t \geq t_0,$$

where $r(t, t_0, u_0(\omega), \omega)$ is the maximal solution of (3.1).

Suppose that there exists a solution

$y(t, t_0, y_0(\omega))$ with $\|y_0(\omega)\| \leq \alpha$, $\Omega_1 \subset \Omega$, $P(\Omega) > 0$ and $t \geq t_0$ such that

$$\|y(t_1, t_0, y_0(\omega), \omega)\| = \beta, \quad \omega \in \Omega$$

and

$$\|y(t, t_0, y_0(\omega), \omega)\| < \beta, \quad t \in (t_0, t_1).$$

Then from (iii), (3.2) - (3.4), we obtain

$$\begin{aligned} b(\beta) &\leq \sum_{i=1}^m V_i(t, y(t_1, t_0, y_0(\omega), \omega)) \\ &\leq \sum_{i=1}^m r_i(t_1, t_0, u_0(\omega), \omega) < \beta_1(t_0, \alpha) \leq b(\beta) \end{aligned}$$

This establishes the fact that SB_1 holds.

4. MAIN RESULTS:

To develop theorems on the estimations of solutions we assume the following.

$$\begin{aligned} (H_4) \quad & \|V_x[t, x(t, s, y(s, \omega)) - x(t, s, m(s))]\| \\ & \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) \| \\ & \leq C(\|y(s, \omega) - m(s)\|) g(s, \omega), \end{aligned}$$

where $C \in C[R_+, R_+]$ and nondecreasing on R_+ , $g \in M[R_+, R[\Omega, R_+]]$ and is sample Lebesgue integrable.

$$(H_5) \quad b(\|x\|) \leq \|V(t, x)\| \leq a(\|x\|),$$

where $a, b \in C[R_+, R_+]$; a is differentiable on R_+ ; b^{-1} exists and it is

non-decreasing and continuous on R_+ .

Theorem 4.1: Suppose the hypothesis $(H_1) - (H_5)$ are satisfied. Then

$$b(\|y(t, \omega) - m(t)\|) \leq h^{-1} \left[\int_{t_0}^t g(s, \omega) ds + H(N(t, \omega)) \right], \quad \text{w.p. 1,}$$

where

$$N(t, \omega) = a(\|x(t, \omega) - m(t)\|),$$

$$\frac{dh(s)}{ds} = \frac{1}{h(s)}$$

and

$$h(s) = C(b^{-1}(s)).$$

Moreover, if h^{-1} is a concave function, then

$$E[b(\|y(t, \omega) - m(t)\|)] \leq h^{-1} \left[E \left(\int_{t_0}^t g(s, \omega) ds \right) + E(H(N(t, \omega))) \right].$$

Proof: From Lemma 2.3, (H_4) and (H_5) we obtain

$$(4.1) \quad b(\|y(t, \omega) - m(t)\|) \leq a(\|x(t, \omega) - m(t)\|) + \int_{t_0}^t C(\|y(s, \omega) - m(s)\|) g(s, \omega) ds.$$

Let

$$r(t, \omega) = \int_{t_0}^t C(\|y(s, \omega) - m(s)\|) g(s, \omega) ds.$$

Therefore

$$(4.2) \quad r'(t, \omega) = C(\|y(t, \omega) - m(t)\|) g(t, \omega).$$

From (4.1) we obtain

$$(4.3) \quad b(\|y(t, \omega) - m(t)\|) \leq N(t, \omega) + r(t, \omega).$$

From (4.2) and (4.3) we obtain

$$(4.4) \quad r'(t, \omega) \leq h(N(t, \omega) + r(t, \omega)) g(t, \omega).$$

Let

$$(4.5) \quad N(t, \omega) + r(t, \omega) = u(t, \omega).$$

Differentiating (4.5) on both the sides we obtain

$$(4.6) \quad N'(t, \omega) + r'(t, \omega) = u'(t, \omega),$$

$$u(t_0, \omega) = N(t_0, \omega).$$

Using (4.6) in (4.4) we get

$$(4.7) \quad u'(t, \omega) \leq N'(t, \omega) + h(u(t, \omega)) g(t, \omega).$$

Now we consider the following comparison differential equation

$$(4.8) \quad v'(t, \omega) = N'(t, \omega) + h(v(t, \omega)) g(t, \omega)$$

with
$$v(t_0, \omega) = u(t_0, \omega).$$

Since $h(\cdot)$ is nondecreasing and $g(t, \omega)$ is nonnegative, by an application of

[5: corollary 2.1] we obtain

$$(4.9) \quad v(t, \omega) \leq H^{-1} \left[\int_{t_0}^t g(s, \omega) ds + H(N(t, \omega)) \right].$$

From (4.5)-(4.9) and an application of a comparison theorem [4; Theorem 2.9.3],

(4.7) reduces to

$$(4.10) \quad r(t, \omega) \leq -N(t, \omega) + H^{-1} \left[\int_{t_0}^t g(s, \omega) ds + H(N(t, \omega)) \right].$$

From (4.3) and (4.10) we obtain

$$(4.11) \quad b(\|y(t, \omega) - m(t)\|) \leq H^{-1} \left[\int_{t_0}^t g(s, \omega) ds + H(N(t, \omega)) \right].$$

This proves the first part of the theorem.

If H^{-1} is concave, then taking expectation on both the sides of (4.11)

and using Jensen's inequality [4], we obtain

$$E(b(\|y(t, \omega) - m(t)\|)) \leq H^{-1} \left[E \int_{t_0}^t g(s, \omega) ds + E H(N(t, \omega)) \right].$$

This proves the second part of the theorem.

In the following corollary we prove another inequality which gives the estimates in terms of the initial conditions for (2.2) and (2.4).

Corollary 4.1: Suppose that all the hypotheses of Theorem 4.1 are satisfied and

$$\|\phi(t, s, y(s, \omega))\| \leq K,$$

where K is a positive constant. Then

$$(4.12) \quad b(\|y(t, \omega) - m(t)\|) \leq H^{-1} \left[H(N(t_0, \omega)) + \int_{t_0}^t g(s, \omega) ds \right],$$

where $N(t_0, \omega) = a(K\|x_0(\omega) - m_0\|)$.

Moreover if H^{-1} is concave then

$$(4.13) \quad E(b(\|y(t, \omega) - m(t)\|)) \leq H^{-1} \left[E(H(N(t_0, \omega))) + E \left(\int_{t_0}^t g(s, \omega) ds \right) \right].$$

Proof: We prove the first part of the corollary only. The second part follows from Theorem 4.1.

Application of Theorem 4.1 gives

$$b(\|y(t, \omega) - m(t)\|) \leq a(\|x(t, \omega) - m(t)\|) + \int_{t_0}^t C(\|y(s, \omega) - m(s)\|) \cdot g(s, \omega) ds.$$

From Theorem 2.7.4 [3], we can derive

$$x(t, t_0, x_0(\omega)) - x(t, t_0, m(t_0)) = \int_0^1 \phi(t, t_0, m_0 + s(x_0(\omega) - m_0)) ds (x_0(\omega) - m_0).$$

Using this inequality and the hypotheses of the corollary, we obtain

$$(4.14) \quad \|x(t, t_0, x_0(\omega)) - x(t, t_0, m_0)\| \leq K\|x_0(\omega) - m_0\|.$$

From (4.14), and an application of Theorem 4.1, the proof of the corollary is complete.

To illustrate the feasibility of assumption (H_4) , and the fruitfulness of the above result, we present a particular example.

Illustration 4.1. Let

$$(4.15) \quad V(t, x) = \|x\|^2 \quad \text{and} \quad \|\phi(t, s, y(s, \omega))\| \leq K.$$

Note that (H_4) is feasible provided that the solution processes of (2.1) are bounded w.p.1. This can be tested by using the boundedness results in Section 3. From (4.15) we obtain

$$(4.16) \quad \|V(t, x)\| \leq 2\|x\|.$$

From the boundedness assumption on the solution processes (2.1), together with (4.15), (4.16), one can find g such that

$$(4.17) \quad \|V_x(t, x(t, s, y(s, \omega)) - x(t, s, m(s)))\phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega)\| \\ \leq 2\|x(t, s, y(s, \omega)) - x(t, s, m(s))\|g(s, \omega),$$

where $g(s, \omega) \in M[R_+, R[\Omega, R]]$. Now by an application of Corollary 4.1, we get

$$(4.18) \quad b(\|y(t, \omega) - m(t)\|) \leq H^{-1}[K\|x_0(\omega) - m_0\| + \int_{t_0}^t g(s, \omega) ds] \\ = [K\|x_0(\omega) - m_0\| + \int_{t_0}^t g(s, \omega) ds]^2,$$

since $H(s) = s^{1/2}$. By noting the fact that $b(u) = u^2 = a(u)$, (4.18) reduces to the following form:

$$(4.19) \quad \|y(t, \omega) - m(t)\|^2 \leq [K\|x_0(\omega) - m_0\| + \int_{t_0}^t g(s, \omega) ds]^2.$$

Taking expectation on both sides of (4.19) we get

$$(4.20) \quad E\|y(t, \omega) - m(t)\|^2 \leq E[K\|x_0(\omega) - m_0\| + \int_{t_0}^t g(s, \omega) ds]^2 \\ = E[K^2\|x_0(\omega) - m_0\|^2 + 2K\|x_0(\omega) - m_0\| \int_{t_0}^t g(s, \omega) ds \\ + (\int_{t_0}^t g(s, \omega) ds)^2].$$

Remark 4.1: We remark that by selecting various conditions on the process $g(s, \omega)$ and $\|x_0(\omega) - m_0\|$ one can obtain more attractive estimates. For example:

(1) Taking the square root and then expectation on both the sides of (4.19) we obtain

$$(4.21) \quad E\|y(t, \omega) - m(t)\| \leq E[K\|x_0(\omega) - m_0\|] + E\left[\int_{t_0}^t g(s, \omega) ds\right].$$

(ii) When $\|x_0(\omega) - m_0\|$ and $g(s, \omega)$ are independent random processes, from (4.20) we get

$$\begin{aligned} E\|y(t, \omega) - m(t)\|^2 &\leq K^2 E(\|x_0(\omega) - m_0\|^2) + 2K E(\|x_0(\omega) - m_0\|) E\left(\int_{t_0}^t g(s, \omega) ds\right) \\ &\quad + E\left[\int_{t_0}^t \int_{t_0}^t (g(s, \omega) g(u, \omega)) ds du\right]. \end{aligned}$$

Further, if $g(s, \omega)$ is any stationary gaussian process and $E g(s, \omega)$ is a constant (which we can take to be zero) then $E(g(s, \omega)g(u, \omega))$ depends only on $s-u$. Therefore, if $E g(s, \omega) = 0$ then in this case (4.22) reduce to

$$E\|y(t, \omega) - m(t)\|^2 \leq K^2 E(\|x_0 - m_0\|)^2 + \int_{t_0}^t \int_{t_0}^t c(u-s) du ds,$$

where $E(g(s, \omega)g(u, \omega)) = c(u-s)$.

So far we considered the right hand side of (H_4) to be a product of two random functions. If the right hand side of (H_4) is a product of a deterministic function and a random function we get another interesting inequality which we state in the following corollary.

Corollary 4.2: Suppose that all the hypotheses of Theorem 4.1 are satisfied;

and instead of (H_4) we assume the following:

$$(H_6) \quad \|V_x(t, x(t, s, y(s, \omega)) - x(t, s, m(s)))\phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega)\| \\ \leq w(s)g(s, \omega),$$

where $w(s) \in C[R_+, R_+]$. Then

$$(4.23) \quad E(b(\|y(t, \omega) - m(t)\|)) \leq E[a(\|x(t, \omega) - m(t)\|)] \\ + \left[\int_{t_0}^t w^2(s) ds \right]^{1/2} E \left[\int_{t_0}^t g^2(s, \omega) ds \right]^{1/2}$$

Moreover, if $\|\phi(t, s, y(s, \omega))\| \leq K$, Then

$$E(b(\|y(t, \omega) - m(t)\|)) \leq E(a(K\|x_0(\omega) - m_0\|)) \\ + \left[\int_{t_0}^t w^2(s) ds \right]^{1/2} E \left[\int_{t_0}^t g^2(s, \omega) ds \right]^{1/2}.$$

Proof: From (H_6) , an application of the proof of the Theorem 4.1, Corollary 4.1, and Holder's inequality the proof of the corollary follows.

Now we present another illustration to exhibit the feasibility of (H_6) , as well as the applications of Corollary 4.2.

Illustration 4.2: Let

$$(4.27) \quad V(t, x) = \|x\| \quad \text{and} \quad \|\phi(t, s, y(s, \omega))\| \leq K.$$

Then

$$\|V_x(t, x)\| \leq 1.$$

Again by the boundedness assumption of the solution process of (2.1) it is easy to see that

$$(4.28) \quad V_x(t, x(t, s, y(s, \omega)) - x(t, s, m(s)))\phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega) \\ \leq g(s, \omega),$$

where $g \in M[R_+, R[\Omega, R]]$. Now take $b(u) = a(u) = u$. From (4.27), (4.28) and an application of Corollary 4.1 we have

$$(4.29) \quad \|y(t, \omega) - m(t)\| \leq K \|x_0(\omega) - m_0\| + \int_{t_0}^t g(s, \omega) ds,$$

since $H(s) = s$. Taking expectation on both sides we get

$$(4.30) \quad E\|y(t, \omega) - m(t)\| \leq K E(\|x_0(\omega) - m_0\|) + E\left(\int_{t_0}^t g(s, \omega) ds\right).$$

Remark 4.2: We remark that from (4.29) we can also get the mean square estimates. For example, squaring (4.29) and using the known inequality, $(a+b)^2 \leq 2(a^2+b^2)$ and taking expectation on both the sides we get

$$(4.31) \quad E\|y(t, \omega) - m(t)\|^2 \leq 2[K^2 E\|x_0(\omega) - m_0\|^2 + E\left(\int_{t_0}^t g(s, \omega) ds\right)^2].$$

From the estimates (i) (4.21), (4.30) and (ii) (4.20), (4.22), (4.31), one can immediately note the role of different types of functions V .

In the next theorem we consider the right hand side of (H_4) to be a function of $V(\cdot)$. Its immediate application is illustrated in Remark 4.3.

Theorem 4.2: Let the hypotheses $(H_1)-(H_3)$ be satisfied. Further assume that

$$(H_7) \quad \begin{aligned} & \|V_x[t, x(t, s, y(s, \omega)) - x(t, s, m(s))] \cdot \phi(t, s, y(s, \omega)) \\ & \cdot R(s, y(s, \omega), \omega)\| \leq C[V(y(s, \omega) - m(s))]g(s, \omega). \end{aligned}$$

Then

$$(4.32) \quad V(t, y(t, \omega) - m(t)) \leq H^{-1} \left[\int_{t_0}^t g(s, \omega) ds + H(M(t, \omega)) \right], \quad \text{w.p.1,}$$

where

$$M(t, \omega) = V(t, x(t, \omega) - m(t)) \quad \text{and} \quad H(s) = \int \frac{ds}{C(s)}.$$

Moreover, if H^{-1} is a concave function then

$$(4.33) \quad E(V(t, y(t, \omega) - m(t))) \leq H^{-1} \left[E \int_{t_0}^t g(s, \omega) ds + E\{H(M(t, \omega))\} \right].$$

Proof: From Lemma 2.3 we obtain

$$(4.34) \quad \begin{aligned} V(t, y(t, \omega) - m(t)) &= V(t, x(t, \omega) - m(t)) \\ &+ \int_{t_0}^t V_x[t, x(t, s, y(s, \omega)) - x(t, s, m(s))] \\ &\quad \cdot \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega) ds \\ &\leq V(t, x(t, \omega) - m(t)) \\ &+ \int_{t_0}^t C[V(t, y(s, \omega) - m(s))] g(s, \omega) ds. \end{aligned}$$

Let

$$(4.35) \quad r(t, \omega) = \int_{t_0}^t C[V(t, y(s, \omega) - m(s))] g(s, \omega) ds.$$

Differentiation (4.35) on both sides we get

$$r'(t, \omega) = C[V(t, y(t, \omega) - m(t))] g(t, \omega).$$

Now the proof of the theorem follows by applying the method used in Theorem 4.1.

Remark 4.3: Suppose $V(t, x) = \|x\|^p$. Then one can compute that

$$(4.36) \quad \begin{aligned} \|V_x(t, x(t, s, y(s, \omega)) - x(t, s, m(s))) \phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega)\| \\ \leq p \|x(t, s, y(s, \omega)) - x(t, s, m(s))\|^{p-1} g(s, \omega) \\ = C[V(t, x(t, s, y(s, \omega)) - x(t, s, m(s))] p g(s, \omega), \end{aligned}$$

where $C(r) = r^{\frac{p-1}{p}}$. Application of Theorem 4.2 gives

$$(4.37) \quad \begin{aligned} \|y(t, \omega) - m(t)\|^p &\leq H^{-1} \left[p \int_{t_0}^t g(s, \omega) ds + p \|x(t, \omega) - m(t)\| \right] \\ &\leq \left[\int_{t_0}^t g(s, \omega) ds + \|x(t, \omega) - m(t)\| \right]^p. \end{aligned}$$

Taking expectation on both sides of (4.37) we obtain

$$(4.38) \quad E\|y(t, \omega) - m(t)\|^p \leq E \left[\int_{t_0}^t g(s, \omega) ds + \|x(t, \omega) - m(t)\| \right]^p.$$

Using the known inequality, $(a+b)^p \leq 2^p(a^p + b^p)$, (4.38) reduces to

$$(4.39) \quad E\|y(t, \omega) - m(t)\|^p \leq 2^p \left[E \left(\int_{t_0}^t g(s, \omega) ds \right)^p + E\|x(t, \omega) - m(t)\|^p \right].$$

5. Examples: In this section we present a few simple and illustrative examples.

Example 5.1: Consider the following differential equations.

$$(5.1) \quad y'(t, \omega) = a(t, \omega)y(t, \omega), \quad y(t_0, \omega) = y_0(\omega),$$

where $a \in M[R, R[\Omega, R]]$ and satisfies enough regularity conditions for existence of a solution process for $t \geq t_0$.

$$(5.2) \quad y'(t, \omega) = E(a(t, \omega))y(t, \omega) + [a(t, \omega) - E(a(t, \omega))]y(t, \omega), \quad y(t_0, \omega) = y_0(\omega)$$

$$(5.3) \quad m'(t) = E(a(t, \omega))m(t), \quad m(t_0) = E(y_0(\omega))$$

$$x' = E(a(t, \omega))x, \quad x(t_0, \omega) = x_0(\omega)$$

with $y_0(\omega) = x_0(\omega)$.

Solving (5.4) we get

$$(5.5) \quad x(t, \omega) = x_0(\omega) \exp\left[\int_{t_0}^t E(a(s, \omega)) ds\right]$$

and

$$(5.6) \quad \phi(t, t_0, x_0(\omega)) = \exp\left[\int_{t_0}^t E(a(s, \omega)) ds\right].$$

(i) Suppose that $V(t, x) = x$ and let $a(z) = b(z) = z$. Then we get

$$(5.7) \quad |V_x(t, x(t, s, y(s, \omega)) - x(t, s, m(s)))\phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega)| \leq g(s, \omega),$$

where

$$g(s, \omega) = |a(s, \omega) - E a(s, \omega)| \exp\left[\int_s^t E a(u, \omega) du\right].$$

From Theorem 4.1 we get

$$(5.8) \quad |y(t, \omega) - m(t)| = H^{-1}\left[\int_{t_0}^t g(s, \omega) + H(N(t, \omega))\right],$$

where

$$H(x) = H^{-1}(z) = z,$$

$$N(t, \omega) = |x(t, \omega) - m(t)|.$$

From (5.5), (5.6) and an application (5.7) and (5.8), we have

$$(5.9) \quad \begin{aligned} |y(t, \omega) - m(t)| &\leq |x_0(\omega) - m_0| \exp\left[\int_{t_0}^t E a(s, \omega) ds\right] \\ &+ \int_{t_0}^t \{|a(s, \omega) - E(a(s, \omega))| \exp\left[\int_s^t E a(u, \omega) du\right] ds\}. \end{aligned}$$

Taking expectation on both sides we get

$$\begin{aligned}
 E|y(t, \omega) - m(t)| &\leq E(|x_0(\omega) - m_0|) \exp\left[\int_{t_0}^t E a(s, \omega) ds\right] \\
 &+ E\left[\int_{t_0}^t \{|a(s, \omega) - E a(s, \omega)| \exp\left[\int_s^t E(a(u, \omega)) du\right] ds\right].
 \end{aligned}
 \tag{5.10}$$

If $|\phi(t, s, y(s, \omega))| \leq K$, from (5.10) we obtain

$$E|y(t, \omega) - m(t)| \leq K[E|x_0(\omega) - m_0| + E\int_{t_0}^t |a(s, \omega) - E a(s, \omega)| ds].
 \tag{5.11}$$

(11) Suppose $V(t, x) = x^2$ and let $a(z) = z^2 = b(z)$. Then from (4.17) we get

$$\begin{aligned}
 &|V_x(t, x(t, s, y(s, \omega)) - x(t, s, m(s)))\phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega)| \\
 &\leq 2|x(t, s, y(s, \omega)) - x(t, s, m(s))|g(s, \omega),
 \end{aligned}
 \tag{5.12}$$

where

$$g(s, \omega) = |a(s, \omega) - E(a(s, \omega))| \exp\left[\int_s^t E(a(u, \omega)) du\right].$$

From Theorem 4.1 we get

$$|y(t, \omega) - m(t)|^2 \leq [|x(t, \omega) - m(t)| + \int_{t_0}^t g(s, \omega) ds]^2.
 \tag{5.13}$$

Taking expectation on both sides of (5.13) we obtain

$$E|y(t, \omega) - m(t)|^2 \leq E[|x(t, \omega) - m(t)| + \int_{t_0}^t g(s, \omega) ds]^2.
 \tag{5.14}$$

If $|\phi(t, s, y(s, \omega))| \leq K$, (5.14) reduces to

$$E|y(t, \omega) - m(t)|^2 \leq K^2 E[|x_0(\omega) - m_0| + \int_{t_0}^t |a(s, \omega) - E(a(s, \omega))| ds]^2.
 \tag{5.15}$$

If the random processes $|x_0(\omega) - m_0|$ and $|a(s, \omega) - E a(s, \omega)|$ are uncorrelated and $|a(s, \omega) - E a(s, \omega)|$ is a stationary gaussian process then (5.15) reduces to

$$E|y(t, \omega) - m(t)|^2 \leq K^2 [E|x_0(\omega) - m_0|^2 + \int_{t_0}^t \int_{t_0}^t C(s-u) ds du],$$

where $C(s-u) = E[(a(s, \omega) - E a(s, \omega))(a(u, \omega) - E a(u, \omega))]$.

Note that from (5.13) and (5.14) we can get estimates similar to (5.11) and (4.21).

In the next example we consider a nonlinear equation.

Example 5.2: Consider the following differential equations.

$$(5.16) \quad y'(t, \omega) = -a(t, \omega)y^3(t, \omega), \quad y(t_0, \omega) = y_0(\omega),$$

where $a(t, \omega) \in M[R, R[\Omega, R_+]]$ and satisfies enough regularity conditions for the existence of a solution process for $t \geq t_0$.

$$(5.17) \quad y'(t, \omega) = -E(a(t, \omega))y^3(t, \omega) + [E(a(t, \omega)) - a(t, \omega)]y^3(t, \omega), \quad y(t_0, \omega) = y_0(\omega)$$

$$(5.18) \quad m'(t) = -E(a(t, \omega))m^3(t), \quad m(t_0) = E(x_0(\omega)),$$

$$(5.19) \quad x' = -E(a(t, \omega))x^3, \quad x(t_0, \omega) = x_0(\omega),$$

with $x_0(\omega) = y_0(\omega)$.

We assume that the solution of (5.16) is bounded w.p.1. That is, we assume that $y^2(t, \omega) \leq \alpha^{-2}$ w.p.1, $\alpha \neq 0$. From (5.19) we obtain

$$(5.20) \quad x(t, \omega) = \frac{x_0(\omega)}{[1 + 2x_0^2(\omega) \int_{t_0}^t E a(s, \omega) ds]^{1/2}}.$$

From (5.20) we get

$$\phi(t, t_0, x_0(\omega)) = \frac{1}{[1 + 2x_0^2(\omega) \int_{t_0}^t a(s, \omega) ds]^{3/2}}.$$

From (5.17) and (5.21) we get the following inequality:

$$(5.22) \quad |\phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega)| \leq \frac{|a(s, \omega) - E a(s, \omega)|}{[\alpha^2 + 2 \int_s^t E a(u, \omega) du]^{3/2}},$$

since $y^2 \leq \alpha^{-2}$, w.p.1. Further, we note that $\alpha^2 > 0$ and $2 \int_s^t a(u, \omega) du$ is positive. Therefore, (5.22) reduces to

$$(5.23) \quad |\phi(t, s, y(s, \omega)) R(s, y(s, \omega), \omega)| \leq K_1 |a(s, \omega) - E a(s, \omega)|,$$

where K_1 is a suitable positive constant.

Now we obtain a few inequalities for different choice of $V(t, x)$.

(i) Let $b(z) = z = a(z)$ and

$$(5.24) \quad V(t, x) = x.$$

(5.24) together with (5.22) verifies the assumption (H_6) . Moreover, we note that $|\phi(t, s, y(s, \omega))| \leq 1$ and $H(s) = s$. From (4.28), (4.30), (5.22) and (5.24) we get

$$(5.25) \quad E|y(t, \omega) - m(t)| \leq E|x_0(\omega) - m_0| + E \left\{ \int_{t_0}^t \frac{|a(s, \omega) - E a(s, \omega)| ds}{[\alpha^2 + 2 \int_s^t E a(u, \omega) du]^{3/2}} \right\}.$$

In the above discussion, if instead of (5.22) we use (5.23) we get

$$(5.26) \quad E|y(t, \omega) - m(t)| \leq E|x_0(\omega) - m_0| + E \left\{ \int_{t_0}^t |a(s, \omega) - E a(s, \omega)| ds \right\}.$$

(11) Let $b(z) = z^2 = a(z)$ and

$$(5.27) \quad V(t, x) = x^2.$$

(5.27) together with (5.22) verifies the assumption (H_4) . Since

$|\phi(t, s, y(s, \omega))| \leq 1$ and $H(s) = s^{1/2}$, from (4.17), (4.20), (5.22) and (5.27),

we obtain

$$(5.28) \quad E|y(t, \omega) - m(t)|^2 \leq E \left[|x_0(\omega) - m_0| + \int_{t_0}^t \frac{|a(s, \omega) - E a(s, \omega)| du}{[\alpha^2 + 2 \int_{t_0}^s E a(u, \omega) du]^{3/2}} \right]^2.$$

In the above discussion if instead of (5.22) we consider (5.23), we get

$$(5.29) \quad E|y(t, \omega) - m(t)|^2 \leq E[|x_0(\omega) - m_0| + K_1 \int_{t_0}^t |a(s, \omega) - E a(s, \omega)| du]^2.$$

In particular if the random processes $|x_0(\omega) - m_0|$ and $|a(s, \omega) - E a(s, \omega)|$ follow certain additional conditions we get the following interesting inequalities.

(a) If $|x_0(\omega) - m_0|$ and $|a(s, \omega) - E a(s, \omega)|$ are independent processes, then (5.29) reduces to

$$(5.30) \quad E|y(t, \omega) - m(t)|^2 \leq E[x_0(\omega) - m_0]^2 + K_1 E|x_0(\omega) - m_0| \\ \times E \left[\int_{t_0}^t (|a(s, \omega) - E a(s, \omega)|) du \right] \\ + K^2 E \int_{t_0}^t \int_{t_0}^t (|a(s, \omega) - E a(s, \omega)| |a(u, \omega) - E a(u, \omega)|) du ds.$$

(b) If $|a(s, \omega) - E a(s, \omega)|$ and $|x_0(\omega) - m_0|$ are uncorrelated and $|a(s, \omega) - E a(s, \omega)|$ is a stationary gaussian process, (5.29) reduces to

$$(5.31) \quad E|y(t, \omega) - m(t)|^2 \leq E(x_0(\omega) - m_0)^2 + K^2 \int_{t_0}^t \int_{t_0}^s c(s-u) du ds,$$

where $c(s-u) = E|a(s, \omega) - E a(s, \omega)| |a(s, \omega) - E a(s, \omega)|$.

(iii) Let

$$(5.32) \quad v(t, x) = x^p.$$

The conditions (5.32) together with (5.22) verifies (4.36) with $C(r) = r^{(p-1)/p}$.

Application of (4.38) and (5.22) gives

$$(5.33) \quad E|y(t, \omega) - m(t)|^p \leq E \left[|x_0(\omega) - m_0| + \int_{t_0}^t \frac{|a(s, \omega) - E a(s, \omega)|}{[\alpha^2 + 2 \int_{t_0}^s E a(u, \omega) du]^{3/2}} ds \right]^p.$$

In the above discussions, if in the place of (5.22) we use (5.23) we ~~obtain~~

$$(5.34) \quad E|y(t, \omega) - m(t)|^p \leq E[|x_0(\omega) - m_0| + K_1 \int_{t_0}^t |a(s, \omega) - E a(s, \omega)| ds]^p.$$

Using the known inequality, $(a+b)^p \leq 2^p(a^p + b^p)$, (5.34) can be further simplified to the following form:

$$(5.35) \quad E|y(t, \omega) - m(t)|^p \leq 2^p |x_0(\omega) - m_0|^p + K_1^p E \left[\int_{t_0}^t |a(s, \omega) - E a(s, \omega)| ds \right]^p.$$

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