MONOTONE METHOD FOR EQUATIONS
DESCRIBING TRANSPORT PHENOMENA IN A BANACH SPACE

by

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1. INTRODUCTION

Existence of extremal solutions of initial and boundary value problems (B.V.P. for short) of differential equations in a Banach space has been recently considered in [2,7] by utilizing monotone iterative method. A special type of B.V.P. of the form

$$x' = f(t, x, y), \quad x(0) = x_0$$

$$-y' = g(t, x, y), \quad y(1) = y_1$$

in finite dimensional spaces have been recently investigated in [6]. It is of interest to investigate (1.1) in a Banach space since special forms of them occur in transport processes [1,8,9]. In order to develop monotone technique for (1.1) it becomes necessary to develop a comparison Theorem (see Theorem 2.1) which is of interest in itself. The advantage of using monotone iterative technique to such equations is that the iterates are solutions of linear initial value problems which are easily computable. In this paper we develop monotone technique for (1.1).

2. PRELIMINARIES AND BASIC COMPARISON PRINCIPLE

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the set of continuous linear functionals on E . A proper subset K of E is said to be a cone if $\lambda K \subset K$ for $\lambda \geq 0$, $K + K \subset K$, $K = \bar{K}$ and $K \cap (-K) = \{0\}$. Here $\bar{K}$ denote the closure of K . The cone K induces a partial ordering on E defined by $u \leq v$ if $v - u \in K$. Given a cone K , we let $K^* = \{\phi \in E^* : \phi(u) \geq 0 \text{ for all } u \in K\}$. A cone K is said to be normal if there exists a real number $N > 0$ such that $0 \leq u \leq v$ implies $\|u\| \leq N\|v\|$ where N is independent of u and v . We shall assume in this paper that K is a normal cone.

Let α denote the Kuratowski's measure of noncompactness, the properties of which may be found in [4,5].

For any $v, w \in C[I, E]$ such that $v(t) \leq w(t)$ on I where $I = [0, 1]$, we define the conical segment

$$[v, w] = \{x \in C[I, E]: v(t) \leq x \leq w(t), t \in I\}.$$

Let us consider the boundary value problem (BVP) of the form

$$\begin{aligned} x' &= f(t, x, y) \quad , \quad x(0) = x_0, \\ -y' &= g(t, x, y) \quad , \quad y(1) = y_1, \end{aligned} \quad (2.1)$$

where $f, g \in C[I \times E \times E, E]$ and x_0 and y_1 are in E .

We list the following assumptions for convenience.

(A₁) For any bounded sets A, B in E

$$\alpha(f(I \times A \times B)) \leq L [\alpha(A) + \alpha(B)],$$

$$\alpha(g(I \times A \times B)) \leq L [\alpha(A) + \alpha(B)],$$

where $L \geq 0$ is a constant;

(A₂) $v_{0,i}, w_{0,i} \in C^1[I, E]$ with $v_{0,i}(t) \leq w_{0,i}(t)$ for $i = 1, 2$ on I , such that

$$\begin{aligned} v_{0,1}' &\leq f(t, v_{0,1}, v_{0,2}) \quad , \quad v_{0,1}(0) \leq x_0, \\ -v_{0,2}' &\leq g(t, v_{0,1}, v_{0,2}) \quad , \quad v_{0,2}(1) \leq y_1, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} w_{0,1}' &\geq f(t, w_{0,1}, w_{0,2}) \quad , \quad w_{0,1}(0) \geq x_0, \\ -w_{0,2}' &\geq g(t, w_{0,1}, w_{0,2}) \quad , \quad w_{0,2}(1) \geq y_1; \end{aligned} \quad (2.3)$$

(A₃) $f(t, x_1, y) - f(t, x_2, y) \geq -M(x_1 - x_2)$ whenever $v_{0,1} \leq x_2 \leq x_1 \leq w_{0,1}$
 and $v_{0,2} \leq y \leq w_{0,2}$ and $g(t, x, y_1) - g(t, x, y_2) \geq -M(y_1 - y_2)$ whenever
 $v_{0,1} \leq x \leq w_{0,1}$ and $v_{0,2} \leq y_2 \leq y_1 \leq w_{0,2}$ for some $M > 0$;

(A₄) $f(t, x, y)$ is monotone nondecreasing in y and $g(t, x, y)$ is monotone nondecreasing in x .

Remark 2.1 We note that the pair of functions $(v_{0,1}, v_{0,2})$ and $(w_{0,1}, w_{0,2})$ satisfying the inequalities (2.2) and (2.3) are called the lower and upper solutions respectively of the BVP (2.1)

We next state and prove the following comparison theorem needed for our main results. See [3] for a similar comparison Theorem for usual B.V. problems.

Theorem 2.1 Assume that

(i) $m_1, m_2 \in C^1[I, R], f_1, f_2 \in C[R \times R, R]$ and

$$(2.4) \quad \begin{aligned} m_1'(t) &\leq f_1(m_1(t), m_2(t)) \quad , \quad m_1(0) \leq r_1(0) = r_{0,1} \quad , \\ -m_2'(t) &\leq f_2(m_1(t), m_2(t)) \quad , \quad m_2(1) \leq r_2(1) = r_{0,2} \quad ; \end{aligned}$$

(ii) $f_1(m_1, m_2), f_2(m_1, m_2)$ are nondecreasing in m_2 and m_1 respectively and $v(t) = (r_1(t), r_2(t))$ is the maximal solution of the BVP

$$(2.5) \quad \begin{aligned} r_1'(t) &= f_1(r_1(t), r_2(t)) \quad , \quad r_1(0) = r_{0,1} \quad , \\ -r_2'(t) &= f_2(r_1(t), r_2(t)) \quad , \quad r_2(1) = r_{0,2} \quad ; \end{aligned}$$

(iii) for every lower solution $v(t) = (v_1(t), v_2(t))$ of (2.5) the modified BVP

$$(2.6) \quad u_1' = f_1(Pu_1, Pu_2) - \frac{u_1 - Pu_1}{1+u_1^2}, \quad u_1(0) = r_{0,1},$$

$$-u_2' = f_2(Pu_1, Pu_2) - \frac{u_2 - Pu_2}{1+u_2^2}, \quad u_2(1) = r_{0,2},$$

has a solution $u(t) = (u_1(t), u_2(t))$.

Then $m(t) \leq r(t)$ on I , that is $m_1(t) \leq r_1(t)$, $m_2(t) \leq r_2(t)$ on I , where $Pu_1 = \max\{v_1, u_1\}$, $Pu_2 = \max\{v_2, u_2\}$.

Proof. Let the modified functions defined on the right sides of the equations in (2.6) be relative to $m_1(t)$ and $m_2(t)$ respectively. Let $u(t) = (u_1(t), u_2(t))$ be a solution of the modified BVP (2.6) guaranteed by (iii). We shall first show that $m(t) \leq u(t)$ on I , i.e. $m_1(t) \leq u_1(t)$ and $m_2(t) \leq u_2(t)$ on I . If this is false, then there exists a point $t_0 \in I$ and a smallest $\epsilon > 0$ such that

$$(2.7) \quad m_1(t_0) = u_1(t_0) + \epsilon \quad \text{and} \quad m_2(t_0) \leq u_2(t_0),$$

or

$$(2.8) \quad m_1(t_0) = u_1(t_0) + \epsilon \quad \text{and} \quad m_2(t_0) > u_2(t_0),$$

The other possibilities are that there exists a point $t_1 \in I$ such that

$$(2.9) \quad m_2(t_1) = u_2(t_1) + \epsilon \quad \text{and} \quad m_1(t_1) \leq u_1(t_1),$$

or

$$(2.10) \quad m_2(t_1) = u_2(t_1) + \epsilon \quad \text{and} \quad m_1(t_1) > u_1(t_1).$$

We prove that (2.7) and (2.8) cannot happen. Similarly, we can prove that (2.9) and (2.10) cannot happen. Certainly $t_0 > 0$ and $t_1 < 1$.

If $t_0 \in (0,1)$, then $m_1'(t_0) = u_1'(t_0)$. In case (2.7) holds, then we have

$$\begin{aligned}
f_1(m_1(t_0), m_2(t_0)) &\geq m_1'(t_0) = u_1'(t_0) \\
&= f_1(Pu_1(t_0), Pu_2(t_0)) - \frac{u_1(t_0) - Pu_1(t_0)}{1+u_1^2(t_0)} \\
&= f_1(m_1(t_0), u_2(t_0)) + \frac{m_1(t_0) - u_1(t_0)}{1+u_1^2(t_0)} \\
&> f_1(m_1(t_0), m_2(t_0)).
\end{aligned}$$

This leads to a contradiction. Further, in case (2.8) holds, then we have

$$\begin{aligned}
f_1(m_1(t_0), m_2(t_0)) &\geq m_1'(t_0) = u_1'(t_0) \\
&= f_1(Pu_1(t_0), Pu_2(t_0)) - \frac{u_1(t_0) - Pu_1(t_0)}{1+u_1^2(t_0)} \\
&= f_1(m_1(t_0), m_2(t_0)) + \frac{m_1(t_0) - u_1(t_0)}{1+u_1^2(t_0)} \\
&> f_1(m_1(t_0), m_2(t_0)),
\end{aligned}$$

which is again a contradiction.

If $t_0=1$, then $m_1'(1) \geq u_1'(1)$. In this case it is enough to consider $m_2(1) \leq u_2(1) = r_{0,2}$ and we get a contradiction. This proves $m_1(t) \leq u_1(t)$, $m_2(t) \leq u_2(t)$ for $t \in I$ and so $u(t) = (u_1(t), u_2(t))$ is actually a solution of the BVP (2.5). It therefore follows from the definition of maximal solution that $m_1(t) \leq r_1(t)$, $m_2(t) \leq r_2(t)$ on I and hence $m(t) \leq r(t)$ on I and this proves the theorem.

Remark 2 We note that in the special case when f_1 and f_2 in Theorem 2.1 are linear functions of the form $f_1(u_1, u_2) = a_1 u_1 + a_2 u_2$ and $f_2(u_1, u_2) = b_1 u_1 + b_2 u_2$, then the BVP

$$r_1'(t) = a_1 r_1(t) + a_2 r_2(t), \quad r_1(0) = 0,$$

$$-r_2'(t) = b_1 r_1(t) + b_2 r_2(t) \quad , \quad r_2(1) = 0,$$

has a unique trivial solution provided $a_i, b_i \geq 0 (i=1,2)$ are small enough.

In this case Theorem 2.1 implies $m_1(t) \leq 0, m_2(t) \leq 0$ on I .

3. MONOTONE METHOD

In order to develop monotone iterative method for BVP (2.1), we need to consider the following linear systems of initial value problems (IVP)

$$(3.1) \quad x' = F(t, x) \quad , \quad x(0) = x_0,$$

$$(3.2) \quad -y' = G(t, y) \quad , \quad y(1) = y_1,$$

where $F(t, x) = f(t, \eta, \mu) - M(x - \eta)$, $G(t, y) = g(t, \eta, \mu) - M(y - \mu)$ and $\eta, \mu \in C[I, E]$ such that $v_{0,1} \leq \eta \leq w_{0,1}$ and $v_{0,2} \leq \mu \leq w_{0,2}$ for some $M > 0$.

It is easy to observe that there exist unique solutions of the IVP (3.1) and (3.2). For each $\eta, \mu \in C[I, E]$ such that $v_{0,1} \leq \eta \leq w_{0,1}$ and $v_{0,2} \leq \mu \leq w_{0,2}$ on I , we define the mapping A by

$$A(\eta, \mu) = (x, y),$$

where x and y are unique solutions of IVP (3.1) and (3.2) respectively on I . This mapping will be used to define the sequences that converge to the minimal and the maximal solutions of the BVP (2.1)

We first prove the following result concerning the mapping A .

Lemma 3.1 Suppose that the assumptions $(A_1) - (A_4)$ hold. Then

$$(i) \quad (v_{0,1}, v_{0,2}) \leq A(v_{0,1}, v_{0,2}) \quad \text{and} \quad (w_{0,1}, w_{0,2}) \geq A(w_{0,1}, w_{0,2});$$

$$(ii) \quad A \text{ is monotone on } [v_{0,i}, w_{0,i}], (i=1,2), \text{ that is, if } \eta_1, \eta_2 \in [v_{0,1}, w_{0,1}]$$

$$\text{and } \mu_1, \mu_2 \in [v_{0,2}, w_{0,2}] \text{ such that } \eta_1 \leq \eta_2 \text{ and } \mu_1 \leq \mu_2, \text{ then}$$

$$A(\eta_1, \mu_1) \leq A(\eta_2, \mu_2).$$

Proof. Let $A(v_{0,1}, v_{0,2}) = (v_{1,1}, v_{1,2})$, where $v_{1,1}$ and $v_{1,2}$ are the unique solutions of IVP (3.1) and (3.2) corresponding to $v_{0,1}$ and $v_{0,2}$ respectively. Setting $p_1(t) = \phi(v_{0,1}(t) - v_{1,1}(t))$ and $p_2(t) = \phi(v_{0,2}(t) - v_{1,2}(t))$ for some $\phi \in K^*$, we have $p_1(0) \leq 0, p_2(1) \leq 0$ and

$$\begin{aligned} p_1' &= \phi(v_{0,1}' - v_{1,1}') \\ &\leq \phi(f(t, v_{0,1}, v_{0,2}) - f(t, v_{0,1}, v_{0,2}) + M(v_{1,1} - v_{0,1})) \\ &= -Mp_1, \end{aligned}$$

$$\begin{aligned} p_2' &= \phi(v_{0,2}' - v_{1,2}') \\ &\geq \phi(-g(t, v_{0,1}, v_{0,2}) + g(t, v_{0,1}, v_{0,2}) - M(v_{1,2} - v_{0,2})) \\ &= Mp_2. \end{aligned}$$

As a result we have $p_1(t) \leq p_1(0)e^{-Mt} \leq 0$ and $p_2(t) \leq p_2(1)e^{-M(1-t)} \leq 0$.

Since $\phi \in K^*$ is arbitrary, this implies $(v_{0,1}, v_{0,2}) \leq A(v_{0,1}, v_{0,2})$.

Similarly, we can show that $(w_{0,1}, w_{0,2}) \geq A(w_{0,1}, w_{0,2})$.

To prove (ii), let $\eta_1, \eta_2, \mu_1, \mu_2 \in C[I, E]$ such that $v_{0,1} \leq \eta_1 \leq \eta_2 \leq w_{0,1}$ and $v_{0,2} \leq \mu_1 \leq \mu_2 \leq w_{0,2}$. Let $A(\eta_1, \mu_1) = (x_1, y_1)$ and $A(\eta_2, \mu_2) = (x_2, y_2)$ where (x_1, y_1) and (x_2, y_2) are the unique solutions of (3.1)-(3.2) corresponding to (η_1, μ_1) and (η_2, μ_2) respectively. Then setting $p(t) = \phi(x_1(t) - x_2(t))$ for some $\phi \in K^*$, we have $p(0) = 0$ and

$$\begin{aligned}
p' &= \phi(x_1' - x_2') \\
&= \phi(f(t, \eta_1, \mu_1) - M(x_1 - \eta_1) - f(t, \eta_2, \mu_2) + M(x_2 - \eta_2)) \\
&\leq \phi(f(t, \eta_1, \mu_2) - M(x_1 - \eta_1) - f(t, \eta_2, \mu_2) + M(x_2 - \eta_2)) \\
&\leq \phi(M(\eta_2 - \eta_1) - M(x_1 - \eta_1) + M(x_2 - \eta_2)) \\
&= -Mp,
\end{aligned}$$

which yields $p(t) \leq p(0)e^{-Mt} = 0$ on I . This proves $x_1(t) \leq x_2(t)$ on I . Similarly, we can prove that $y_1(t) \leq y_2(t)$ for $t \in I$. This proves that the mapping A is monotone.

In view of Lemma 3.1, we can define sequences $\{v_{n,1}\}, \{v_{n,2}\}, \{w_{n,1}\}, \{w_{n,2}\}$ as follows

$$(v_{n,1}, v_{n,2}) = A(v_{n-1,1}, v_{n-1,2}), \quad (w_{n,1}, w_{n,2}) = A(w_{n-1,1}, w_{n-1,2}).$$

It is easy to see that $\{v_{n,1}\}, \{v_{n,2}\}, \{w_{n,1}\}, \{w_{n,2}\}$ are monotone sequences such that $v_{n,1} \leq w_{n,1}$ and $v_{n,2} \leq w_{n,2}$ and $v_{n,1}, w_{n,1} \in [v_{0,1}, w_{0,1}]$ and $v_{n,2}, w_{n,2} \in [v_{0,2}, w_{0,2}]$.

Next we shall show that there exist subsequences of $\{v_{n,1}\}, \{v_{n,2}\}, \{w_{n,1}\}, \{w_{n,2}\}$ which converge uniformly on I .

Lemma 3.2 Let the assumptions of Lemma 3.1 hold. Further let L and M be small enough, then the sequences $\{v_{n,1}\}, \{v_{n,2}\}, \{w_{n,1}\}, \{w_{n,2}\}$ are uniformly bounded, equicontinuous and relatively compact on I .

Proof. Since the cone K is assumed to be normal, it follows from

$v_{n,1}, w_{n,1} \in [v_{0,1}, w_{0,1}]$ and $v_{n,2}, w_{n,2} \in [v_{0,2}, w_{0,2}]$ that $\{v_{n,1}\}, \{w_{n,1}\}$ and $\{v_{n,2}\}, \{w_{n,2}\}$ are uniformly bounded. This implies the equicontinuity of the sequences by using standard estimates and the fact that f, g maps

bounded sets into bounded sets which is a consequence of (A_1) . Define $m_1(t) = \alpha(\{v_{n,1}(t)\}_{n=1}^{\infty})$ for $i = 1, 2$, then using the standard arguments, we get

$$\begin{aligned} D^+ m_1(t) &\leq (L+2M)m_1(t) + Lm_2(t), \\ -D^+ m_2(t) &\leq Lm_1(t) + Lm_2(t). \end{aligned}$$

Since $m_1(0) = 0$ and $m_2(1) = 0$ and the constants L and M are small enough, then it is immediate from Remark 2.1 that $m_1(t) \equiv 0$ and $m_2(t) \equiv 0$ on I . This implies the relative compactness of the sequences $\{v_{n,1}\}$ and $\{v_{n,2}\}$. Similarly, we can show that the sequences $\{w_{n,1}\}$ and $\{w_{n,2}\}$ are relatively compact on I .

We now apply Ascoli's theorem to the sequences $\{v_{n,i}\}, \{w_{n,i}\}$ ($i=1,2$) to obtain subsequences $\{v_{n_k,i}\}, \{w_{n_k,i}\}$ which converge uniformly on I . Since the sequences are monotone, the whole sequences converge uniformly and monotonically to continuous functions, that is $\lim_{n \rightarrow \infty} v_{n,i}(t) = \rho_i(t)$ and $\lim_{n \rightarrow \infty} w_{n,i}(t) = r_i(t)$ on I for $i = 1, 2$. It then follows easily from (3.1), (3.2) that $(\rho_1(t), \rho_2(t))$ and $(r_1(t), r_2(t))$ are solutions of the BVP (2.1) on I .

Finally, we show that $\rho(t) = (\rho_1(t), \rho_2(t))$ and $r(t) = (r_1(t), r_2(t))$ are minimal and maximal solutions of BVP (2.1). Let (x, y) be any solution of BVP (2.1) on I such that $x \in [v_{0,1}, w_{0,1}]$ and $y \in [v_{0,2}, w_{0,2}]$. Assume that for some integer $j > 0$, $v_{j,1} \leq x \leq w_{j,1}$ and $v_{j,2} \leq y \leq w_{j,2}$. Setting $p_1(t) = \phi(x(t) - v_{j+1,1}(t))$ and $p_2(t) = \phi(y(t) - v_{j+1,2}(t))$ for some $\phi \in K^*$, we have $p_1(0) = 0$, $p_2(1) = 0$ and

$$\begin{aligned}
p_1' &= \phi(x' - v_{j+1,1}') \\
&= \phi(f(t, x, y) - f(t, v_{j,1}, v_{j,2}) + M(v_{j+1,1} - v_{j,1})) \\
&\geq \phi(f(t, x, v_{j,2}) - f(t, v_{j,1}, v_{j,2}) + M(v_{j+1,1} - v_{j,1})) \\
&\geq \phi(-M(x - v_{j,1}) + m(v_{j+1,1} - v_{j,1})) \\
&= -Mp_1,
\end{aligned}$$

$$\begin{aligned}
p_2' &= \phi(y' - v_{j+1,2}') \\
&= \phi(-g(t, x, y) + g(t, v_{j,1}, v_{j,2}) - M(v_{j+1,2} - v_{j,2})) \\
&\leq \phi(-g(t, x, y) + g(t, x, v_{j,2}) - M(v_{j+1,2} - v_{j,2})) \\
&\leq \phi(M(y - v_{j,2}) - M(v_{j+1,2} - v_{j,2})) \\
&= Mp_2,
\end{aligned}$$

which implies $v_{j+1,1} \leq x$ and $v_{j+1,2} \leq y$ on I . Similarly, we can show that $x \leq w_{j+1,1}$ and $y \leq w_{j+1,2}$. Since $x \in [v_{0,1}, w_{0,1}]$ and $y \in [v_{0,2}, w_{0,2}]$ we have by induction $v_{n,1} \leq x \leq w_{n,1}$ and $v_{n,2} \leq y \leq w_{n,2}$ on I for all n . Thus, by taking limit as $n \rightarrow \infty$ we obtain $p_1(t) \leq x(t) \leq r_1(t)$ and $p_2(t) \leq y(t) \leq r_2(t)$ on I , proving $(p_1(t), p_2(t))$ and $(r_1(t), r_2(t))$ are minimal and maximal solutions of BVP (2.1) respectively on I . We have proved the following main result.

Theorem 3.1 Let the cone K be normal and that the assumptions $(A_1) - (A_4)$ hold. Also let the constants L and M be small enough. Then there exist monotone sequences $\{(v_{n,1}, v_{n,2})\}$ and $\{(w_{n,1}, w_{n,2})\}$ which converge uniformly and monotonically to minimal and maximal solutions (p_1, p_2) and (r_1, r_2) .

respectively of the BVP (2.1) on $[v_{0,i}, w_{0,i}]$ for $i = 1, 2$. That is, if (x, y) is any solution of BVP (2.1) such that $x \in [v_{0,1}, w_{0,1}]$ and $y \in [v_{0,2}, w_{0,2}]$, then

$$v_{0,1} \leq v_{1,1} \leq \dots \leq v_{n,1} \leq \dots \leq \rho_1 \leq x \leq r_1 \leq \dots \leq w_{n,1} \leq \dots \leq w_{0,1},$$

and

$$v_{0,2} \leq v_{1,2} \leq \dots \leq v_{n,2} \leq \dots \leq \rho_2 \leq y \leq r_2 \leq \dots \leq w_{n,2} \leq \dots \leq w_{0,2}$$

on I .

Corollary 3.1 If the solutions of the BVP (2.1) are unique, then the assumptions of Theorem 3.1 imply that $\rho_1(t) = x(t) = r_1(t)$ and $\rho_2(t) = y(t) = r_2(t)$ on I . That is $\rho(t) = u(t) = r(t)$ on I , where $u = (x, y)$, $\rho = (\rho_1, \rho_2)$ and $r = (r_1, r_2)$.

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LIST OF SYMBOLS

$\ \cdot \ $	Norm
λ	Lambda
$\cap$	Intersection
$\in$	belongs to
$\subset$	contained in (Subset of)
$-$	Closure
ϕ	phi
α	alpha
η	eta
μ	mu
ρ	rho
ϵ	epsilon