FIXED POINT THEOREMS FOR EXPANDING MAPS

by

A. A. Gillespie and B. B. Williams
Department of Mathematics
The University of Texas at Arlington
Arlington, Texas 76019

Technical Report No. 154

March, 1981

FIXED POINT THEOREMS FOR EXPANDING MAPS

A. A. Gillespie and B. B. Williams

Suppose (X, d) is a metric space, $h > 0$ and $T: X \rightarrow X$. We shall use the notation $T \in E(h)$ to mean $d(T(x), T(y)) \geq h d(x, y)$ for each $x, y \in X$. If $h > 1$, then T will be called an expanding map. Clearly $T \in E(h)$ implies T is a 1-1 function and $d(T^{-1}(x), T^{-1}(y)) \leq \frac{1}{h} d(x, y)$ for each $x, y \in T(X)$.

In this paper some conditions are found to insure that an expanding map will have a fixed point. It is shown that each finite dimensional Banach space X has the following property: each continuous and expanding map from X into X has a fixed point. It is also shown that not all infinite dimensional Banach spaces have the above property.

Theorem 1. Suppose that (X, d) is a complete metric space, T is an expanding map from X into X and $T(X)$ is a closed subset of X . The following are equivalents:

(i) T has a unique fixed point.

(ii) $\bigcap_{n=1}^{\infty} T^n(X) \neq \emptyset$.

(iii) There exists $\{x_n\}$ a sequence in X , such that $d(x_n, T(x_n)) \rightarrow 0$.

Proof. It is clear that (i) implies (ii) and that if T has a fixed point, then it is unique. To see that (ii) implies (iii), suppose $p \in \bigcap_{n=1}^{\infty} T^n(X)$. Let $F = T^{-1}$ and for each $n \in \mathbb{Z}^+$ $x_n = F^n(p)$. Now $d(x_n, T(x_n)) = d(F^n(p), F^{n-1}(p)) \leq (\frac{1}{h})^{n-1} d(F(p), p) \rightarrow 0$. To see that

(iii) implies (i), let $\{x_n\}$ denote a sequence in X such that $d(x_n, T(x_n)) \rightarrow 0$. Now $T \in E(h)$, where $h > 1$, and hence $d(T(x), T(y))(1 - \frac{1}{h}) \leq d(T(x), x) + d(T(y), y)$. It follows that $\{T(x_n)\}$ is a Cauchy sequence in the complete metric space $T(X)$. Thus, there exists $T(p)$ such that $T(x_n) \rightarrow T(p)$. Now T^{-1} is continuous, hence $x_n \rightarrow p$, and since $d(T(p), p) \leq d(T(p), T(x_n)) + d(T(x_n), x_n) + d(x_n, p)$, it follows that $T(p) = p$.

Lemma 1. Suppose (X, d) is a complete metric space, T a function from X into X and $n \in \mathbb{Z}^+$.

(i) $T \in E(h)$ implies $T^n \in E(h^n)$.

(ii) If T is an expanding map, then $T^n(p) = p$ if and only if $T(p) = p$.

(iii) If $T \in E(h)$ and T is continuous, then $T(X)$ is complete.

Proof. The truth of (i) is clear. For (ii), suppose $h > 1$ and $T^n(p) = p$. Now $T(p) = T(T^n(p)) = T^n(T(p))$, hence $d(T(p), p) = d(T^n(T(p)), T^n(p)) \geq h^n d(T(p), p)$ which implies $T(p) = p$. The truth of (iii) follows since T^{-1} is uniformly continuous.

It follows from Lemma 1 (iii), that in Theorem 1, the requirement that $T(X)$ is a closed subset of X could have been replaced by T is a continuous map. The following example shows that Theorem 1 no longer holds if $h = 1$.

Example 1. Define $T: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ by $T(x) = x + e^x$.

The function T in Example 1 is continuous, both (ii) and (iii) of Theorem 1 hold and $T \in E(1)$, yet T has no fixed point.

Below we will show that if $T: \mathbb{R}^1 \rightarrow \mathbb{R}^1$, where T is a continuous and expanding map, then T must have a fixed point. However, there are expanding

maps from R^1 into R^1 with no fixed point.

Example 2. Define $T: R^1 \rightarrow R^1$ by $T(x) = 3x + e^x$ if $x \geq 0$ and $T(x) = 3x - e^x$ if $x < 0$.

The function in Example 2 is an expanding map with no fixed point.

Definition. Let $\mathcal{E}(h)$ denote the class of all complete metric spaces X such that if $T: X \rightarrow X$ where $T \in \mathcal{E}(h)$ and continuous, then T has a fixed point.

Theorem 2.

(i) If $1 < h_1 < h_2$, then $\mathcal{E}(h_1) = \mathcal{E}(h_2)$ (Hence for $h > 1$, we will denote $\mathcal{E}(h)$ by $\mathcal{E}$).

(ii) If X_1 and X_2 are isomorphic Banach spaces, then $X_1 \in \mathcal{E}$ implies $X_2 \in \mathcal{E}$.

(iii) If X and Y are isometric metric spaces, then $X \in \mathcal{E}$ implies $Y \in \mathcal{E}$.

Proof. To prove (i), suppose $X \in \mathcal{E}(h_1)$, $T: X \rightarrow X$, where T is a continuous and $T \in \mathcal{E}(h_2)$. Now $h_1 < h_2$ implies $T \in \mathcal{E}(h_1)$ thus T has a fixed point and hence $X \in \mathcal{E}(h_2)$. Suppose $X \in \mathcal{E}(h_2)$, $T: X \rightarrow X$, where T is continuous and $T \in \mathcal{E}(h_1)$. Thus, by Lemma 1 (i), there exists $n \in \mathbb{Z}^+$ such that $h_1^n > h_2$ and hence $T^n \in \mathcal{E}(h_2)$. Since T^n has a fixed point, it follows by Lemma 1 (ii) that T has a fixed point. Hence, $X \in \mathcal{E}(h_1)$ therefore, $\mathcal{E}(h_1) = \mathcal{E}(h_2)$.

To see that (ii) is true, suppose that F is a linear homeomorphism from the Banach space $(X_1, \|\cdot\|_1)$ onto the Banach space $(X_2, \|\cdot\|_2)$, $X_1 \in \mathcal{E}$

and that T is a continuous and expanding map from X_2 into X_2 . For some $h > 1$, $T \in E(h)$, thus there exists $n \in \mathbb{Z}^+$ such that $1 > \|F^{-1}\| \frac{1}{h^n} \|F\| = \frac{1}{k}$. Define $G: X_1 \rightarrow X_1$ by $G = F^{-1} \circ U \circ F$ where $U = T^n \in E(h^n)$. Clearly G is continuous. To see that G is an expanding map, suppose $x, y \in X_1$ and let $a = G(x)$ and $b = G(y)$. Now $\|x - y\|_1 = \|G^{-1}(a) - G^{-1}(b)\|_1 = \|F^{-1}(U^{-1}(F(a))) - F^{-1}(U^{-1}(F(b)))\|_1 \leq \|F^{-1}\| \|U^{-1}(F(a)) - U^{-1}(F(b))\|_2 \leq \|F^{-1}\| \frac{1}{h^n} \|F(a) - F(b)\|_2 \leq \|F^{-1}\| \frac{1}{h^n} \|F\| \|a - b\|_1 = \frac{1}{k} \|G(x) - G(y)\|_1$. Thus, $G \in E(k)$ where $k > 1$, hence there exists $p \in X_1$ such that $p = G(p) = F^{-1}(U(F(p)))$. Now $T^n(F(p)) = U(F(p)) = F(p)$, thus by Lemma 1, T has a fixed point. Therefore, $X_2 \in \mathcal{E}$.

To see that (iii) is true, suppose that F is an isometry from the metric space (X, d) onto the metric space (Y, p) , $X \in \mathcal{E}$ and that T is a continuous expanding map from Y into Y . Now Y is complete, and $G: X \rightarrow X$ defined by $G = F^{-1} \circ T \circ F$ is a continuous and expanding map. Since G has a fixed point, it follows that T has a fixed point. Therefore, $Y \in \mathcal{E}$.

Theorem 3. If X is a finite dimensional Banach space, then $X \in \mathcal{E}$.

Proof. First suppose $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ where T is a continuous and expanding map. T is a homeomorphism and from Browder's Domain Invariance Theorem [1], $T(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$. From Lemma 1 we have that $T(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^n$, hence $T(\mathbb{R}^n) = \mathbb{R}^n$ and thus $\mathbb{R}^n \in \mathcal{E}$. $\mathbb{C}^n$ is isometric to $\mathbb{R}^{2n}$ therefore, $\mathbb{C}^n \in \mathcal{E}$. Since each finite dimensional Banach space is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ for some n , thus each finite dimensional Banach space is an element of $\mathcal{E}$.

To find a class of Banach spaces that has no elements in $\mathcal{E}$, we can

use the following lemma.

Lemma 2. If X is a Banach space, then the following statements are equivalent:

(i) $X \in \mathcal{E}$.

(ii) If $T: X \rightarrow X$ where T is a continuous and expanding map, then $0 \in T(X)$.

(iii) If $T: X \rightarrow X$ where T is a continuous and expanding map, then $T(X) = X$.

Proof. It is clear that (iii) implies (i). To see that (i) implies (ii), suppose $X \in \mathcal{E}$, $F: X \rightarrow X$ where F is a continuous and expanding map. There exists $h > 1$ such that $F \in E(h)$. Define $T: X \rightarrow X$ by $T(x) = F^n(x) + x$, where $n \in \mathbb{Z}^+$ such that $h^n > 2$. Since $\|T(x) - T(y)\| \geq h^n \|x - y\| - \|x - y\|$, we have that T is a continuous and expanding map. Thus, there exists $p \in X$ such that $p = T(p) = F^n(p) + p$ and hence $0 \in F(X)$.

To see that (ii) implies (iii), suppose $F: X \rightarrow X$, F is a continuous and expanding map and $x_0 \in X$. Define $T: X \rightarrow X$ by $T(x) = F(x) - x_0$. T is a continuous and expanding map, thus there exists $p \in X$ such that $0 = F(p) - x_0$ and hence $F(X) = X$.

Theorem 4. Let X denote a Banach space. If there exists $T: X \rightarrow X$ where T is a linear homeomorphism and $T(X) \neq X$, then $X \notin \mathcal{E}$.

Proof. Suppose T is a linear homeomorphism that maps X into X where $T(X) \neq X$. Define $F: X \rightarrow X$ by $F(x) = 2\|T^{-1}\|T(x)$. F is a continuous and expanding map and $F(X) = T(X) \neq X$, thus by Lemma 2, $X \notin \mathcal{E}$.

REFERENCE

1. James Dugundji, Topology, Allyn and Bacon, Inc., 1966.