NUMERICAL SIMULATION OF MISCIBLE AND
IMMISCIBLE FLUID FLOWS IN POROUS MEDIA

by

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ABSTRACT

Fast, economical, and accurate finite difference numerical methods are developed for approximating continuum models of both miscible and immiscible fluid flows in porous media. The methods apply to very general systems of equations and boundary configurations. Computer examples are described and discussed.

1. Introduction. A variety of ecological problems require the study of porous flows which involve two different fluids. Typical examples are the extraction of oil from a porous reservoir by means of pressure from an injected liquid or gas, and contamination of fresh water wells near a salt water body (see, e.g., refs. [1], [7] - [9], [15] - [17], [19], [21], [23], [24], [26]-[28], [32] and the references contained therein). Such time dependent, two-fluid porous flow problems often fall into one of the following two categories: (a) the miscible case, in which the fluids are allowed to mix, and (b) the immiscible case, in which the fluids are not allowed to mix. In both cases, the determination of such quantities as velocity, pressure, concentration, and, in the immiscible case, the fluid interface are of physical interest and must be determined numerically. Computer oriented techniques developed for these and for related purposes include finite elements, finite differences, general variational methods, Galerkin methods, particle methods, and the method of characteristics (see, e.g., [7], [8], [10], [12] - [14], [16], [18], [20], [22], [23], [28] - [30]).

In this paper we will develop two related finite difference techniques, one for the miscible problems and the second for the immiscible ones. The power of the methods is derived from their hybrid structure, which incorporates some of the best aspects of the MAC method, the Courant, Isaacson and Rees method, and several special techniques developed by Bulgarelli and the present writers [2], [4], [5], [11], [25], [31]. The resulting techniques are fast, economical, and yield numerical solutions which are in complete agreement with

physical expectations, as will be seen from the computer examples which are described and discussed in the last section.

2. Governing Equations. Consider first the motion of two miscible fluids in a porous medium. Considering this motion as that of a single, nonhomogeneous fluid, the governing equations follow directly from the conservation of momentum, volume, and mass, and, in two dimensions, are, respectively,

$$(1) \quad \begin{cases} \frac{\rho}{\eta} \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} - \frac{\mu}{K} u \\ \frac{\rho}{\eta} \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial y} - \frac{\mu}{K} v - \rho g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ \eta \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\partial}{\partial x} \left(\tau \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left(\tau \frac{\partial c}{\partial y} \right) \end{cases},$$

where $\rho = \rho(c)$ is the density of the mixture, $\eta = \eta(x, y)$ is the porosity of the medium, u and v are the components of fluid velocity, p is the pressure, $\mu = \mu(c)$ is the viscosity, $K = K(x, y)$ is the permeability of the medium, g is the acceleration due to gravity, c is the concentration of one fluid in the mixture, and $\tau = \tau(x, y)$ is the mass diffusivity. The first two equations of system (1) will be called Darcy's equations, the third is called the incompressibility condition, and the last is called the transport equation. If there exists a free surface which can be represented as a single valued function $y = h(x, t)$, the change in surface elevation is determined by the kinematic condition

$$(2) \quad \eta_s \frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} = v_s ,$$

where η_s is the porosity of the medium at the free surface, and u_s and v_s are the seepage velocity components at the fluid surface.

In applying (1) and (2), the following initial conditions are employed:

$$(3) \quad \begin{cases} u(x, y, t_0) = u_0(x, y) \\ v(x, y, t_0) = v_0(x, y) \\ c(x, y, t_0) = c_0(x, y) \\ h(x, t_0) = h_0(x) \end{cases}$$

The boundary conditions for (1) are given as follows. Either the pressure p_s or the normal velocity u_n is given as a function of time on the boundary. In particular, at the free surface, the pressure is zero at all times, while, at the impenetrable boundaries, the normal velocity is always zero. Also, either the concentration or its normal derivative is given on the boundary. In particular, at the impenetrable boundaries the normal derivative is always zero. The free surface equation (2) needs one boundary condition at the point of intersection with the fixed boundary only if the fixed boundary is an inflow boundary at such a point. In this case the surface height $h(x, t)$ must be specified at this point as a function of time.

Now consider the motion of two immiscible fluids. The conservation principles in this case imply

$$(4) \begin{cases} \frac{\rho_r}{n} \frac{\partial u_r}{\partial t} = - \frac{\partial p_r}{\partial x} - \frac{\mu_r}{KK_r} u_r & , r=1,2 \\ \frac{\rho_r}{n} \frac{\partial v_r}{\partial t} = - \frac{\partial p_r}{\partial y} - \frac{\mu_r}{KK_r} v_r - \rho_r g & , r=1,2 \\ n \frac{\partial S_r}{\partial t} + \frac{\partial u_r}{\partial x} + \frac{\partial v_r}{\partial y} = 0 & , r=1,2 \end{cases}$$

and

$$(5) \quad S_1 + S_2 = 1 ,$$

where, in the model, ρ_r and μ_r are constant, S_r is the saturation, and K_r is the relative permeability. The difference $p_2 - p_1$ is called the capillary pressure p_c , so that

$$(6) \quad p_2 - p_1 = p_c .$$

We assume that p_c , K_1 and K_2 are known functions of the saturation.

If there exists a free surface, the change in elevation is determined, again, by (2), where u_s and v_s are the total seepage velocity components at the fluid surface, that is, $u_s = u_{1,s} + u_{2,s}$ and $v_s = v_{1,s} + v_{2,s}$.

We assume now that, throughout, S denotes the saturation S_1 of the first fluid and p denotes the pressure p_1 of the first fluid.

The initial conditions for model (2), (4), (5) are

$$(7) \begin{cases} u_r(x, y, t_0) = [u_r(x, y)]_0 & , r=1,2 \\ v_r(x, y, t_0) = [v_r(x, y)]_0 & , r=1,2 \\ S(x, y, t_0) = S_0(x, y) \\ h(x, t_0) = h_0(x) . \end{cases}$$

The boundary conditions for (4) are given by assigning either the pressure p_g or the normal components $u_{1,l}$ and $u_{2,l}$ of the

velocities of the two fluids. In addition, at the inflow boundaries only, if the pressure p_s is specified, then the saturation must and will be specified also.

3. Difference Approximations for the Miscible Model. Our finite difference approximations for the system (1) are based on the Los Alamos MAC (Marker-and-Cell) method [25],[31]. Specifically, the finite difference mesh consists of rectangular cells of width Δx and height Δy . On each cell, the field variables u , v , p , and c are defined at the locations shown in Figure 1: the u component of velocity is defined at the center of each vertical side, the v component at the center of each horizontal side, and the pressure p and concentration c at each cell center.

Fig. 1. Positions of the field variables

The implicit finite difference equations corresponding to the first two equations of system (1), that is, to Darcy's equations, are

$$(8) \begin{cases} u_{i+\frac{1}{2},j}^{n+1} = \lambda_{i+\frac{1}{2},j}^{n+1} \left[\frac{\rho_{i+\frac{1}{2},j}^{n+1}}{\eta_{i+\frac{1}{2},j}} u_{i+\frac{1}{2},j}^n - \Delta t \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{\Delta x} \right] \\ v_{i,j+\frac{1}{2}}^{n+1} = \lambda_{i,j+\frac{1}{2}}^{n+1} \left[\frac{\rho_{i,j+\frac{1}{2}}^{n+1}}{\eta_{i,j+\frac{1}{2}}} v_{i,j+\frac{1}{2}}^n - \Delta t \left(\frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta y} + \rho_{i,j+\frac{1}{2}}^{n+1} \varepsilon \right) \right], \end{cases}$$

where the λ coefficients are given by

$$\lambda_{i+\frac{1}{2},j}^{n+1} = (\eta_{i+\frac{1}{2},j} K_{i+\frac{1}{2},j}) / (\rho_{i+\frac{1}{2},j}^{n+1} K_{i+\frac{1}{2},j} + \Delta t \mu_{i+\frac{1}{2},j}^{n+1} \eta_{i+\frac{1}{2},j})$$

$$\lambda_{i,j+\frac{1}{2}}^{n+1} = (\eta_{i,j+\frac{1}{2}} K_{i,j+\frac{1}{2}}) / (\rho_{i,j+\frac{1}{2}}^{n+1} K_{i,j+\frac{1}{2}} + \Delta t \mu_{i,j+\frac{1}{2}}^{n+1} \eta_{i,j+\frac{1}{2}}).$$

Of course, implementation of (8) requires first the determination of the λ and ρ coefficients and also the pressure. Since ρ and μ are known functions of the concentration c , determination of the λ 's and ρ 's requires discretization of the fourth equation of system (1), that is, the transport equation. As in Casulli [4], the finite difference equation chosen, which remains stable even in the absence of diffusivity, is given by

$$(9) \quad \eta \bar{c}_t + u \bar{c}_x + v \bar{c}_y = (\overline{\tau \bar{c}_x})_x + (\overline{\tau \bar{c}_y})_y,$$

where

$$\eta \bar{c}_t = \eta_{i,j} \frac{c_{i,j}^{n+1} - c_{i,j}^n}{\Delta t}$$

$$u \bar{c}_x = u_{i,j}^n \frac{c_{i,j}^n - c_{i-1,j}^n}{\Delta x}, \quad \text{if } u_{i,j}^n \geq 0$$

$$u \bar{c}_x = u_{i,j}^n \frac{c_{i+1,j}^n - c_{i,j}^n}{\Delta x}, \quad \text{if } u_{i,j}^n < 0$$

$$\begin{aligned} \bar{v}c_y &= v_{1,j}^n \frac{c_{1,j}^n - c_{1,j-1}^n}{\Delta y}, & \text{if } v_{1,j}^n \geq 0 \\ \bar{v}c_y &= v_{1,j}^n \frac{c_{1,j+1}^n - c_{1,j}^n}{\Delta y}, & \text{if } v_{1,j}^n < 0 \end{aligned}$$

$$(\bar{\tau}c_x)_x = \frac{\tau_{1+\frac{1}{2},j} \frac{c_{1+\frac{1}{2},j}^n - c_{1,j}^n}{\Delta x} - \tau_{1-\frac{1}{2},j} \frac{c_{1,j}^n - c_{1-\frac{1}{2},j}^n}{\Delta x}}{\Delta x}$$

$$(\bar{\tau}c_y)_y = \frac{\tau_{1,j+\frac{1}{2}} \frac{c_{1,j+\frac{1}{2}}^n - c_{1,j}^n}{\Delta y} - \tau_{1,j-\frac{1}{2}} \frac{c_{1,j}^n - c_{1,j-\frac{1}{2}}^n}{\Delta y}}{\Delta y},$$

and $u_{1,j}^n$ and $v_{1,j}^n$ are simple averages from the closest scalar grid points at which they are defined.

Next, the determination of the pressure field $p_{1,j}^{n+1}$ is accomplished as follows. The finite difference equation corresponding to the third equation of system (1), that is, to the incompressibility condition, in each cell is given by

$$(10) \quad \frac{u_{1+\frac{1}{2},j}^{n+1} - u_{1-\frac{1}{2},j}^{n+1}}{\Delta x} + \frac{v_{1,j+\frac{1}{2}}^{n+1} - v_{1,j-\frac{1}{2}}^{n+1}}{\Delta y} = 0.$$

As in Casulli and Greenspan [5], the pressure field $p_{1,j}^{n+1}$ at the center of each cell must be computed so that the discrete incompressibility condition (10) is valid throughout. In developing a finite difference equation for the pressure, we require that the seepage velocity components computed with (8) also satisfy (10). We first define

$$F_{i+\frac{1}{2},j}^{n+1} = \lambda_{i+\frac{1}{2},j}^{n+1} \frac{\rho_{i+\frac{1}{2},j}^{n+1}}{\tau_{i+\frac{1}{2},j}} u_{i+\frac{1}{2},j}^n$$

$$G_{i,j+\frac{1}{2}}^{n+1} = \lambda_{i,j+\frac{1}{2}}^{n+1} \left[\frac{\rho_{i,j+\frac{1}{2}}^{n+1}}{\tau_{i,j+\frac{1}{2}}} v_{i,j+\frac{1}{2}}^n - (\Delta t) g \rho_{i,j+\frac{1}{2}}^{n+1} \right],$$

so that (8) is equivalent to

$$(11) \quad \begin{cases} u_{i+\frac{1}{2},j}^{n+1} = F_{i+\frac{1}{2},j}^{n+1} - (\Delta t) \lambda_{i+\frac{1}{2},j}^{n+1} \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{\Delta x} \\ v_{i,j+\frac{1}{2}}^{n+1} = G_{i,j+\frac{1}{2}}^{n+1} - (\Delta t) \lambda_{i,j+\frac{1}{2}}^{n+1} \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta y} \end{cases}$$

Substitution of (11) into (10) yields the following difference equation for pressure:

$$(12) \quad \lambda_{i+\frac{1}{2},j}^{n+1} \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{(\Delta x)^2} - \lambda_{i-\frac{1}{2},j}^{n+1} \frac{p_{i,j}^{n+1} - p_{i-1,j}^{n+1}}{(\Delta x)^2} \\ + \lambda_{i,j+\frac{1}{2}}^{n+1} \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{(\Delta y)^2} - \lambda_{i,j-\frac{1}{2}}^{n+1} \frac{p_{i,j}^{n+1} - p_{i,j-1}^{n+1}}{(\Delta y)^2} \\ = \frac{p_{i+\frac{1}{2},j}^{n+1} - p_{i-\frac{1}{2},j}^{n+1}}{(\Delta x)(\Delta t)} + \frac{G_{i,j+\frac{1}{2}}^{n+1} - G_{i,j-\frac{1}{2}}^{n+1}}{(\Delta y)(\Delta t)}$$

Equation (12) for the unknowns $p_{i,j}^{n+1}$ is then solved at each time step by an iteration that adjusts the velocities through changes in the pressure field [4].

4. Difference Approximations for the Immiscible Model. The difference approximations for the equations of system (4) are, essentially, the same as those for the miscible model. The finite difference mesh consists of rectangular cells and the field variables u_r , v_r , and p_r , $r=1,2$, are defined at locations shown in Figure 1. In addition, the saturations S_r , $r=1,2$, are defined at each cell center. The implicit difference equations corresponding to the first two equations of system (4), that is, to Darcy's equations, are analogous to (8), but now have a set of equations for each fluid:

$$(13) \begin{cases} (u_r)_{i+\frac{1}{2},j}^{n+1} = (\lambda_r)_{i+\frac{1}{2},j}^{n+1} \left[\frac{\rho_r}{\eta_{r+\frac{1}{2},j}} (u_r)_{i+\frac{1}{2},j}^n - (\Delta t) \frac{(p_r)_{i+\frac{1}{2},j}^{n+1} - (p_r)_{i,j}^{n+1}}{\Delta x} \right] \\ (v_r)_{i,j+\frac{1}{2}}^{n+1} = (\lambda_r)_{i,j+\frac{1}{2}}^{n+1} \left[\frac{\rho_r}{\eta_{i,j+\frac{1}{2}}} (v_r)_{i,j+\frac{1}{2}}^n - (\Delta t) \left(\frac{(p_r)_{i,j+\frac{1}{2}}^{n+1} - (p_r)_{i,j}^{n+1}}{\Delta y} + \rho_r g \right) \right], \end{cases}$$

where $r=1,2$ and where the coefficients λ_r are given by

$$(\lambda_r)_{i+\frac{1}{2},j}^{n+1} = (\eta_{i+\frac{1}{2},j} K_{i+\frac{1}{2},j} K_r) / (\rho_r K_{i+\frac{1}{2},j} K_r + (\Delta t) \mu_r \eta_{i+\frac{1}{2},j})$$

$$(\lambda_r)_{i,j+\frac{1}{2}}^{n+1} = (\eta_{i,j+\frac{1}{2}} K_{i,j+\frac{1}{2}} K_r) / (\rho_r K_{i,j+\frac{1}{2}} K_r + (\Delta t) \mu_r \eta_{i,j+\frac{1}{2}})$$

Here also, the implementation of (13) requires first the determination of the coefficients $(\lambda_r)^{n+1}$ and the pressures $(p_r)^{n+1}$, $r=1,2$. Since K_r and p_c are known functions of S , we need to discretize the third equation of system (4) to evaluate $S_{i,j}^{n+1}$, which results in

$$(14) \quad S_{i,j}^{n+1} = S_{i,j}^n - \frac{\Delta t}{\eta_{i,j}} \left[\frac{(u_1)_{i+\frac{1}{2},j}^n - (u_1)_{i-\frac{1}{2},j}^n}{\Delta x} + \frac{(v_1)_{i,j+\frac{1}{2}}^n - (v_1)_{i,j-\frac{1}{2}}^n}{\Delta y} \right].$$

Equation (14) allows us to determine $K_r(S_{i,j}^{n+1})$ and $(p_c)_{i,j}^{n+1} = p_c(S_{i,j}^{n+1})$.

The determination of the pressure fields $(p_r)_{i,j}^{n+1}$ is accomplished as follows. From (5) and the two equations for the saturations S_1 and S_2 in system (4), we find

$$(15) \quad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = \eta \frac{\partial (S_1 + S_2)}{\partial t} = 0,$$

which is the incompressibility condition for system (4). The corresponding difference approximation to be used is

$$(16) \quad \frac{(u_1)_{i+\frac{1}{2},j}^{n+1} - (u_1)_{i-\frac{1}{2},j}^{n+1}}{\Delta x} + \frac{(v_1)_{i,j+\frac{1}{2}}^{n+1} - (v_1)_{i,j-\frac{1}{2}}^{n+1}}{\Delta y} + \frac{(u_2)_{i+\frac{1}{2},j}^{n+1} - (u_2)_{i-\frac{1}{2},j}^{n+1}}{\Delta x} + \frac{(v_2)_{i,j+\frac{1}{2}}^{n+1} - (v_2)_{i,j-\frac{1}{2}}^{n+1}}{\Delta y} = 0.$$

Again, the pressure fields $(p_r)_{i,j}^{n+1}$, $r=1,2$, at the center of each cell must be computed so that discrete incompressibility condition (16) is valid throughout. To find a finite difference equation for the pressure, we require that the seepage velocity components of the two fluids, computed with (13), satisfy (16). Recall, $p_1 = p$ and define

$$(F_1)_{i+\frac{1}{2},j}^{n+1} (\lambda_1)_{i+\frac{1}{2},j}^{n+1} \frac{\rho_1}{\eta_{i+\frac{1}{2},j}} (u_1)_{i+\frac{1}{2},j}^n$$

$$\begin{aligned}
(G_1)_{i,j+\frac{1}{2}}^{n+1} &= (\lambda_1)_{i,j+\frac{1}{2}}^{n+1} \left[\frac{\rho_1}{\eta_{i,j+\frac{1}{2}}} (v_1)_{i,j+\frac{1}{2}}^n - (\Delta t) g \rho_1 \right] \\
(F_2)_{i+\frac{1}{2},j}^{n+1} &= (\lambda_2)_{i+\frac{1}{2},j}^{n+1} \left[\frac{\rho_2}{\eta_{i+\frac{1}{2},j}} (u_2)_{i+\frac{1}{2},j}^n - (\Delta t) \frac{(p_c)_{i+1,j}^{n+1} - (p_c)_{i,j}^{n+1}}{\Delta x} \right] \\
(G_2)_{i,j+\frac{1}{2}}^{n+1} &= (\lambda_2)_{i,j+\frac{1}{2}}^{n+1} \left[\frac{\rho_2}{\eta_{i,j+\frac{1}{2}}} (v_2)_{i,j+\frac{1}{2}}^n - (\Delta t) \left(\frac{(p_c)_{i,j+1}^{n+1} - (p_c)_{i,j}^{n+1}}{\Delta y} + g \rho_2 \right) \right],
\end{aligned}$$

then equations (13) are equivalent to

$$(17) \quad \begin{cases} (u_r)_{i+\frac{1}{2},j}^{n+1} = (F_r)_{i+\frac{1}{2},j}^{n+1} - \Delta t (\lambda_r)_{i+\frac{1}{2},j}^{n+1} \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{\Delta x} \\ (v_r)_{i,j+\frac{1}{2}}^{n+1} = (G_r)_{i,j+\frac{1}{2}}^{n+1} - \Delta t (\lambda_r)_{i,j+\frac{1}{2}}^{n+1} \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta y} \end{cases}$$

Substitution of (17) into (16) yields the pressure equation

$$\begin{aligned}
(18) \quad & (\lambda_1 + \lambda_2)_{i+\frac{1}{2},j}^{n+1} \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{(\Delta x)^2} - (\lambda_1 + \lambda_2)_{i-\frac{1}{2},j}^{n+1} \frac{p_{i,j}^{n+1} - p_{i-1,j}^{n+1}}{(\Delta x)^2} \\
& + (\lambda_1 + \lambda_2)_{i,j+\frac{1}{2}}^{n+1} \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{(\Delta y)^2} - (\lambda_1 + \lambda_2)_{i,j-\frac{1}{2}}^{n+1} \frac{p_{i,j}^{n+1} - p_{i,j-1}^{n+1}}{(\Delta y)^2} \\
& = \frac{(F_1 + F_2)_{i+\frac{1}{2},j}^{n+1} - (F_1 + F_2)_{i-\frac{1}{2},j}^{n+1}}{(\Delta t)(\Delta x)} + \frac{(G_1 + G_2)_{i,j+\frac{1}{2}}^{n+1} - (G_1 + G_2)_{i,j-\frac{1}{2}}^{n+1}}{(\Delta t)(\Delta y)}
\end{aligned}$$

Equation (18) for the unknowns $p_{i,j}^{n+1} = (p_1)_{i,j}^{n+1}$ is then solved at

each time step by an iteration that adjusts the velocities through changes in the pressure field [4].

In the previous notation, since the saturation S is defined only at the cell center, the discrete variables $(\lambda_r)_{i+\frac{1}{2},j}^{n+1}$ and $(\lambda_r)_{i,j+\frac{1}{2}}^{n+1}$ are not defined. However, since the fluids are immiscible, the saturation S and hence $\lambda_r(S)$ have strong variations near the interface of the two fluids. Therefore, a simple average from the closest grid points would yield too crude an approximation for the saturation. For this reason, we use

$$S_{i+\frac{1}{2},j}^{n+1} \equiv S_{i,j}^{n+1}, \quad \text{if } u_{i+\frac{1}{2},j}^{n+1} > 0$$

$$S_{i+\frac{1}{2},j}^{n+1} \equiv S_{i+1,j}^{n+1}, \quad \text{if } u_{i+\frac{1}{2},j}^{n+1} < 0$$

$$S_{i+\frac{1}{2},j}^{n+1} \equiv \frac{1}{2}(S_{i,j}^{n+1} + S_{i+1,j}^{n+1}), \quad \text{if } u_{i+\frac{1}{2},j}^{n+1} = 0$$

$$S_{i,j+\frac{1}{2}}^{n+1} \equiv S_{i,j}^{n+1}, \quad \text{if } v_{i,j+\frac{1}{2}}^{n+1} > 0$$

$$S_{i,j+\frac{1}{2}}^{n+1} \equiv S_{i,j+1}^{n+1}, \quad \text{if } v_{i,j+\frac{1}{2}}^{n+1} < 0$$

$$S_{i,j+\frac{1}{2}}^{n+1} \equiv \frac{1}{2}(S_{i,j}^{n+1} + S_{i,j+1}^{n+1}), \quad \text{if } v_{i,j+\frac{1}{2}}^{n+1} = 0$$

5. Approximation of Free Surface Motion. When there exists a free surface, which, in the case of porous flow, is a moving boundary between an area in which fluid fills the pores and one in which there is no fluid present, equation (2) is discretized with the Courant, Isaacson, and Rees method, which is consistent with the correct physical boundary conditions [2], [4], [5]. The surface height $h_{i+\frac{1}{2}}^n$ is defined on each vertical grid line and (2) is approximated by

$$(19) \quad h_{i+\frac{1}{2}}^{n+1} = h_{i+\frac{1}{2}}^n + \frac{\Delta t}{(\eta_s)_{i+\frac{1}{2}}} \left[v_s^n - u_s^n \frac{\Delta h_{i+\frac{1}{2}}^n}{\Delta x} \right],$$

where

$$\begin{aligned} \Delta h_{i+\frac{1}{2}}^n &= h_{i+\frac{1}{2}}^n - h_{i-\frac{1}{2}}^n, & \text{if } u_s^n \geq 0 \\ \Delta h_{i+\frac{1}{2}}^n &= h_{i+(3/2)}^n - h_{i+\frac{1}{2}}^n, & \text{if } u_s^n < 0 \end{aligned}$$

The coefficients u_s^n and v_s^n in (19) are obtained as weighted averages of the nearest cell velocities.

6. Boundary Conditions. In this section, we allude primarily to those applied problems in which the investigator knows that the boundary of a basin or a reservoir is not approximated well by a polygon with relatively few sides which are all either horizontal or vertical.

In order to calculate the seepage velocities with (8) for the miscible case, or with (13) for the immiscible case, the pressure field $p_{i,j}^{n+1}$ is required at each cell center, including those cells which are crossed by boundary lines, and are called boundary cells. When the pressure p_s^{n+1} is specified as a boundary condition, the pressure $p_{i,j}^{n+1}$ at the boundary cell center is chosen such that a linear interpolation between it and the pressure in the nearest cell yields the pressure p_s^{n+1} . As an example, the pressure $p_{i,j}^{n+1}$ for the configuration in Figure 2 is given by

$$(20) \quad p_{i,j}^{n+1} = q p_s^{n+1} + (1-q) p_{i,j-1}^{n+1}$$

where

$$q = \frac{2 \Delta y}{3 \Delta y - (\xi_{i-\frac{1}{2}} + \xi_{i+\frac{1}{2}})}.$$

Fig. 2. Boundary configuration

Particular care must be taken also when the normal seepage velocity is prescribed on a boundary. Let us consider the miscible case first. If we rewrite (10) as

$$(21) \quad u_{i+\frac{1}{2},j}^{n+1}(\Delta y) - u_{i-\frac{1}{2},j}^{n+1}(\Delta y) + v_{i,j+\frac{1}{2}}^{n+1}(\Delta x) - v_{i,j-\frac{1}{2}}^{n+1}(\Delta x) = 0, \quad ,$$

then (21) has the following physical meaning: the volume of fluid entering cell (i,j) must balance the volume of fluid leaving the cell. It is this principle which is applied to a boundary cell. With reference to Figure 3, where u_1 denotes the normal velocity at which the fluid enters cell (i,j) through the boundary, the analogue of (21) for such a cell is given by

$$(22) \quad u_{i+\frac{1}{2},j}^{n+1} \xi_{i+\frac{1}{2}} - u_{i-\frac{1}{2},j}^{n+1} \xi_{i-\frac{1}{2}} + v_{i,j+\frac{1}{2}}^{n+1} - u_1^{n+1} ((\Delta x)^2 + (\xi_{i-\frac{1}{2}} - \xi_{i+\frac{1}{2}})^2)^{\frac{1}{2}} = 0.$$

Fig. 3. Boundary configuration

The difference equation for pressure in a boundary cell is obtained by substituting (11) into (22):

$$\begin{aligned}
 (23) \quad & \lambda_{i+1/2,j} \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{\Delta x} (\xi_{i+1/2}) - \lambda_{i-1/2,j} \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta x} (\xi_{i-1/2}) \\
 & + \lambda_{i,j+1/2} \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta y} (\Delta x) = \frac{F_{i+1/2,j}^{n+1} \xi_{i+1/2} - F_{i-1/2,j}^{n+1} \xi_{i-1/2} + G_{i,j+1/2}^{n+1} (\Delta x)}{\Delta t} \\
 & + u_{\perp}^{n+1} \frac{((\Delta x)^2 + (\xi_{i-1/2} - \xi_{i+1/2})^2)^{1/2}}{\Delta t} .
 \end{aligned}$$

The system (12) and (20) or (23) form a linear algebraic system for the unknowns $p_{i,j}^{n+1}$.

Let us consider next the immiscible case. The approach is similar to that described above. For the pressure in a boundary cell where the normal seepage velocities are prescribed for the two immiscible fluids, the analogue of the discrete incompressibility condition (22) is, with reference to Figure 3, given by

$$(24) \quad (u_1 + u_2)_{i+\frac{1}{2},j}^{n+1} \xi_{i+\frac{1}{2}} - (u_1 + u_2)_{i-\frac{1}{2},j}^{n+1} \xi_{i-\frac{1}{2}} + (v_1 + v_2)_{i,j+\frac{1}{2}}^{n+1} (\Delta x) - (u_{1,\perp} + u_{2,\perp})^{n+1} ((\Delta x)^2 + (\xi_{i-\frac{1}{2}} - \xi_{i+\frac{1}{2}})^2)^{\frac{1}{2}} = 0.$$

The difference equation for pressure in a boundary cell is obtained by substitution of (17) into (24) and a nonlinear algebraic system results this time. This is due to the fact that the coefficients $\lambda(S)$ depend on the directions of the velocities which are functions of the pressure.

The boundary conditions for the concentration c in the transport equation of model (1) are treated numerically in the usual way for diffusion equations (see, e.g., Caponi [3] and any standard finite difference text). Of course, model (4) has no diffusion equation. With reference to Figure 3, difference equation (14) for saturation in a boundary cell becomes

$$(25) \quad A_{i,j} s_{i,j}^{n+1} = A_{i,j} s_{i,j}^n - \frac{\Delta t}{\tau_{i,j}} \left\{ (u_1)_{i+\frac{1}{2},j}^n \xi_{i+\frac{1}{2}} - (u_1)_{i-\frac{1}{2},j}^n \xi_{i-\frac{1}{2}} + (v_1)_{i,j+\frac{1}{2}}^n (\Delta x) - (u_{1,\perp})^n ((\Delta x)^2 + (\xi_{i-\frac{1}{2}} - \xi_{i+\frac{1}{2}})^2)^{\frac{1}{2}} \right\},$$

where $A_{i,j}$ is the fluid volume in the boundary cell, i.e.,

$$A_{i,j} = \frac{1}{2} (\Delta x) (\xi_{i+\frac{1}{2}} + \xi_{i-\frac{1}{2}})$$

The requirement of the specification of the saturation S_b at the inflow boundary where the pressure p_g is specified, which is physically correct, is properly utilized numerically in that equation (13) requires the knowledge of the coefficients $\lambda_r(S_b)$ on these boundaries.

7. Stability and Accuracy Considerations. The seepage velocity equations (8) and (13) in both models are first order accurate in time and second order in space. Difference equations (12) and (18) for pressure are second order accurate in space for both models. Free surface difference equation (19) is first order accurate in space and time. In the model for miscible fluids, the upstream difference approximations used for the transport equation make difference equation (9) first order accurate in space and in time [28]. In the model for immiscible fluids, the difference approximation (14) for the saturation is first order accurate in time and second order in space.

Note that since incompressibility condition (10) or (16) is satisfied, the numerical schemes conserve fluid volume up to an error ϵ , which is chosen for the iterative procedure for pressure evaluation.

Once the space steps Δx and Δy are chosen, the choice of time step Δt necessary for stability is governed by a condition for the transport equation (9) or for the saturation equation (14), and if there exists a free surface, by a surface wave Courant condition. The condition for the transport equation yields the following restriction on Δt [28]:

$$(26) \quad \Delta t < \min_{u,v,r,\eta} \left\{ \left[\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + 2r \left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right) \right]^{-1} \eta \right\}.$$

When stability condition (26) is satisfied, the numerical values for $c_{i,j}^n$ are always bounded by the minimum and the maximum values for the concentration assigned as initial data or as boundary conditions.

The stability condition for saturation equation (14) is given by

$$(27) \quad \begin{cases} \Delta t \leq \min_{i,j} \left[\frac{S_{i,j}^n \eta_{i,j}}{(D_1)_{i,j}} \right] & , \text{ if } (D_1)_{i,j} > 0 \\ \Delta t \leq \min_{i,j} \left[\frac{(S_{i,j}^n - 1) \eta_{i,j}}{(D_1)_{i,j}} \right] & , \text{ if } (D_1)_{i,j} < 0 \end{cases}$$

where

$$(D_1)_{i,j} = \frac{(u_1)_{i+\frac{1}{2},j}^n - (u_1)_{i-\frac{1}{2},j}^n}{\Delta x} + \frac{(v_1)_{i,j+\frac{1}{2}}^n - (v_1)_{i,j-\frac{1}{2}}^n}{\Delta y}.$$

When stability condition (27) is satisfied throughout, the numerical values for $S_{i,j}^n$ are always bounded by zero and unity, which is physically meaningful.

If there exists a free surface, the surface wave Courant condition [4] appears as:

$$(28) \quad \Delta t < (\Delta x)/(gh_{\max})^{\frac{1}{2}}.$$

Note finally that equations of motion (8) and (13) are derived from implicit discretizations and are always stable.

8. Summary of the Computer Algorithms. The algorithms for proceeding from time step t_n to time step t_{n+1} for both models are now summarized as follows.

For the miscible model:

Step 1. Determine the values for concentration by an explicit calculation with (9) which uses the previous time concentration and velocity.

Step 2. Determine the pressure field so that (10) or (22) is valid in each cell. This is done iteratively by solving the linear algebraic system (12), (20) and/or (23).

Step 3. Determine the velocity field by an explicit calculation with (8) which uses the previous time velocities, the concentrations calculated in Step 1, and the pressure field calculated in Step 2.

Step 4. Determine the new free surface position by an explicit calculation with (19), using the previous time velocities and surface position.

Next, for the immiscible model:

Step 1. Determine the saturation values explicitly with (14), using the previous time saturations and velocities.

Step 2. Determine the pressure field so that (16) or (24) is valid in each cell. This is done iteratively by solving the nonlinear system (18), (20), and or an equation corresponding to (23) for each boundary cell.

Step 3. Determine the velocity field by an explicit calculation with (13), using the previous time velocities, the saturation computed in Step 1, and the pressure field computed in Step 2.

Step 4. Determine the new free surface position explicitly from (19) using previous time velocities and surface position.

With regard to the above algorithms, Δt for the first model has to satisfy stability conditions (26) and (28), while for the second model it has to satisfy (27) and (28). Note also that the diffusivity coefficient τ can be arbitrarily small and difference equation (9) will always remain stable if (26) is valid.

9. Computer Examples. The two numerical schemes described have been tested on a variety of examples. For clarity, we will confine our discussion here to problems which do not have a free surface, since free surface problems have been treated extensively elsewhere [2],[4].

Let us consider a confined square region whose sides are parallel to the axes and which contains a homogeneous porous material saturated with a fluid having density $\rho_2 = 0.8$ and viscosity $\mu_2 = 1.3$. The length L of the side of the square is taken as $L = 2$ and the porous material contained in the interior is assumed to have porosity $\alpha = 0.1$ and permeability $K = 1$. The gravity acceleration constant is normalized to $g = 1$. At the initial time $t_0 = 0$, another fluid with density $\rho_1 = 1.0$ and viscosity $\mu_1 = 0.7$ is injected at the lower left hand corner of the square with a pressure p_b which is thereafter constant. The boundary is assumed to be open also at the upper right hand corner where a pressure $p = 0$ is assumed at all times. The initial velocities are zero. The problem is to describe the flow within the square as the fluid being injected at the lower left hand corner

continually forces fluid out the upper right hand corner.

The flow through the porous medium has been simulated for different values of p_b using both models. In the miscible case, we have assumed a constant diffusivity $\bar{T} = 0.001$. In the immiscible case we have assumed the relative permeabilities K_1 and K_2 shown in Figure 4 and the capillary pressure shown in Figure 5. As shown in Figure 6, a grid of 10×10 cells with $\Delta x = \Delta y = 0.2$ was used.

In the first example using the miscible model, the applied pressure is $p_b = 20$ and the time step is $\Delta t = 0.001$. The density ρ and the viscosity μ are defined as functions of the concentration and are then determined by linear interpolation from $\rho_1, \rho_2, \mu_1, \mu_2$. Figures 7-9 show the isoconcentration lines at $t = 0.3, 0.6, 0.9$, respectively. Tables 1-3 in the Appendix of [6] give the corresponding numerical values. As expected, the two fluids mix and flow to the exit point. Figure 10 shows the outflow discharge with time of the two fluids of the mixture.

A second run was made with all the parameters the same except $p_b = 2$ and $\Delta t = 0.01$. Since the difference in pressures at the entry and exit points is now much smaller, the flow is slow and the effect of gravity is pronounced. The inflow fluid, being heavier, now tends to remain in the lower part of the region. Figures 11-13 show the isoconcentration lines at $t = 3, 6, 9$, respectively. Tables 4-6 in the Appendix of [6] give the corresponding numerical values.

The problems used in Examples 1 and 2, above, were next reconsidered under the assumption that the fluids were immiscible. The relative permeabilities and the capillary pressure were again those shown in Figures 4 and 5, respectively.

Fig. 4. Relative permeability curves

Fig. 6. Computational region

Fig. 7. Numerical results at $t=0.3$

Fig. 8. Numerical results at $t=0.6$

Fig. 9. Numerical results at $t=0.9$

Fig. 10. Outflow discharge. Crosses are fluid 1 and dots fluid 2.

Fig. 11. Numerical results at $t=3$

Fig. 12. Numerical results at $t=6$

Fig. 13. Numerical results at $t=9$

In the third example, then, with $p_b = 20$ and $\Delta t = 0.001$, Figure 14 shows the saturation front of the two fluids at $t = 0.3, 0.6, 0.9$. Tables 7-9 in the Appendix of [6] give the corresponding values for the saturation S . Figure 15 shows the outflow discharge rate of the two fluids. It is interesting to compare Figures 10 and 15 to note the different behaviors of miscible and immiscible fluids.

For the fourth example, the third example was rerun with all parameters the same except $p_b = 2$ and $\Delta t = 0.01$. As in Example 2, the flow is slow and more sensitive to gravity. Figure 16 shows the saturation front at $t = 3, 6, 9$. Tables 10-12 in the Appendix of [6] give the numerical values for S at $t = 3, 6, 9$.

Figures 17 and 18 show the outflow discharge obtained numerically from the examples 2 and 4, respectively. Here we can compare how the phenomena evolve. For an applied constant pressure $p_b = 2$, the flow develops from zero initial velocities and these increase slowly. We note the greater velocities in Example 2, shown in Figure 17, than in Example 4, shown in Figure 18. This is correct because in addition to the absolute permeability, the immiscible model also has the relative permeabilities given in Figure 4. Note also from the figures that at $t = 8$, the outflow discharge of fluid 1 becomes greater than that of fluid 2. This phenomenon occurs in both examples at approximately $t = 8$. The difference is that in Example 4 the discharge of fluid 2 goes rapidly to zero as the discharge of fluid 1 increases, in accord with the physics of phenomena for immiscible fluids. Further analysis of Figure 18 reveals that the total discharge of the steady state is not as great as the outflow discharge was for fluid 2. This is due to the fact that $\rho_1 > \rho_2$ and the effect of gravity is accentuated by the relatively small value $p_b = 2$. The

discontinuity at the breakthrough time $t = 8$, which appears clearly in Example 3, almost disappears in the slow flow of Example 4.

Note also that since the two methods are conservative, the total volume is conserved in the sense that for all the examples the difference between inflow and discharge can be made as small as we wish for fixed time and space steps.

Finally, note that phenomena like those in Examples 1 and 3 occur in the process of oil recovery from a porous medium by injection of water, while phenomena like those of Examples 2 and 4 are typical of seawater intrusion in aquifers.

Fig. 14. Saturation fronts

Fig. 15. Outflow discharge. Crosses for fluid 1, dots for fluid 2

Fig. 16. Saturation fronts

Fig. 17. Outflow discharge. Crosses are fluid 1, dots fluid 2.

Fig. 18. Outflow discharge. Crosses are fluid 1, dots fluid 2.

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APPENDIX

Table 1

ITER = 3		TIME = 3.000E-01				CYCLE = 300			
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.019	0.008	0.003	0.001	0.000	0.000	0.000	0.000	0.000	0.000
0.098	0.049	0.017	0.004	0.001	0.000	0.000	0.000	0.000	0.000
0.329	0.190	0.075	0.021	0.004	0.001	0.000	0.000	0.000	0.000
0.692	0.480	0.231	0.075	0.017	0.003	0.000	0.000	0.000	0.000
0.946	0.796	0.481	0.191	0.049	0.008	0.001	0.000	0.000	0.000
0.999	0.946	0.693	0.330	0.099	0.019	0.002	0.000	0.000	0.000

Table 3

ITER= 3		TIME= 9.000E-01				CYCLE= 900			
0.091	0.087	0.082	0.077	0.076	0.080	0.092	0.113	0.138	0.131
0.423	0.408	0.387	0.367	0.353	0.349	0.355	0.359	0.319	0.139
0.763	0.746	0.720	0.689	0.656	0.623	0.583	0.511	0.359	0.114
0.938	0.930	0.913	0.888	0.852	0.802	0.722	0.584	0.357	0.093
0.990	0.988	0.982	0.979	0.946	0.898	0.803	0.626	0.353	0.081
0.999	0.999	0.998	0.994	0.982	0.947	0.855	0.661	0.359	0.078
1.000	1.000	1.000	0.999	0.994	0.971	0.891	0.695	0.374	0.080
1.000	1.000	1.000	1.000	0.998	0.983	0.917	0.728	0.396	0.085
1.000	1.000	1.000	1.000	0.999	0.989	0.934	0.756	0.419	0.091
1.000	1.000	1.000	1.000	0.999	0.991	0.943	0.774	0.437	0.096

Table 4

ITER = 3		TIME = 3.000E+00				CYCLE = 300			
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.004	0.003	0.002	0.001	0.001	0.000	0.000	0.000	0.000	0.000
0.026	0.021	0.014	0.009	0.005	0.002	0.001	0.000	0.000	0.000
0.100	0.081	0.057	0.035	0.019	0.009	0.004	0.001	0.000	0.000
0.270	0.226	0.163	0.102	0.055	0.025	0.009	0.003	0.001	0.000
0.532	0.464	0.355	0.234	0.130	0.059	0.021	0.006	0.001	0.000
0.795	0.728	0.600	0.428	0.254	0.120	0.044	0.012	0.002	0.000
0.949	0.911	0.814	0.639	0.414	0.210	0.079	0.021	0.004	0.000
0.994	0.982	0.933	0.801	0.572	0.315	0.126	0.035	0.006	0.001
1.000	0.997	0.974	0.883	0.679	0.401	0.169	0.048	0.008	0.001

Table 5

ITER = 3		TIME = 6.000E+00				CYCLE = 600			
0.079	0.078	0.078	0.079	0.082	0.088	0.099	0.116	0.133	0.136
0.244	0.243	0.241	0.242	0.247	0.257	0.272	0.282	0.260	0.154
0.503	0.497	0.489	0.480	0.473	0.466	0.452	0.412	0.312	0.149
0.749	0.741	0.726	0.706	0.681	0.649	0.596	0.502	0.340	0.149
0.908	0.901	0.886	0.864	0.833	0.786	0.708	0.576	0.373	0.159
0.977	0.974	0.965	0.951	0.925	0.880	0.797	0.649	0.419	0.179
0.997	0.995	0.993	0.986	0.972	0.939	0.867	0.721	0.477	0.207
1.000	1.000	0.999	0.997	0.991	0.971	0.917	0.789	0.543	0.242
1.000	1.000	1.000	0.999	0.997	0.987	0.950	0.844	0.610	0.281
1.000	1.000	1.000	1.000	0.999	0.993	0.967	0.883	0.667	0.317

Table 6

ITER = 3		TIME = 0.999E+00				CYCLE = 900			
0.365	0.368	0.373	0.383	0.399	0.423	0.459	0.511	0.574	0.666
0.627	0.629	0.636	0.648	0.667	0.697	0.739	0.787	0.821	0.767
0.848	0.849	0.851	0.856	0.865	0.880	0.898	0.910	0.885	0.773
0.955	0.955	0.954	0.954	0.955	0.958	0.960	0.951	0.901	0.777
0.990	0.990	0.989	0.988	0.987	0.986	0.982	0.967	0.911	0.789
0.999	0.998	0.998	0.998	0.997	0.995	0.992	0.976	0.924	0.806
1.000	1.000	1.000	1.000	0.999	0.999	0.996	0.984	0.938	0.826
1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.989	0.951	0.846
1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.993	0.963	0.864
1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.995	0.972	0.879

Table 7

ITER= 26		TIME= 3.000E-01				CYCLE= 390			
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.029	0.002	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.612	0.280	0.004	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.629	0.629	0.434	0.004	0.0	0.0	0.0	0.0	0.0	0.0
0.629	0.633	0.630	0.284	0.002	0.0	0.0	0.0	0.0	0.0
0.630	0.629	0.628	0.622	0.033	0.0	0.0	0.0	0.0	0.0

Table 8

ITER = 21		TIME = 6.000E-01				CYCLE = 600			
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.201	0.124	0.012	0.000	0.0	0.0	0.0	0.0	0.0	0.0
0.627	0.629	0.626	0.206	0.004	0.0	0.0	0.0	0.0	0.0
0.628	0.630	0.631	0.629	0.549	0.004	0.0	0.0	0.0	0.0
0.628	0.631	0.635	0.633	0.632	0.559	0.005	0.0	0.0	0.0
0.628	0.629	0.633	0.633	0.631	0.633	0.218	-0.000	0.0	0.0
0.629	0.630	0.636	0.629	0.634	0.634	0.629	0.022	0.0	0.0
0.629	0.633	0.630	0.630	0.630	0.630	0.629	0.154	0.0	0.0
0.630	0.629	0.629	0.628	0.628	0.627	0.628	0.258	0.001	0.0

Table 9

ITER = 7		TIME = 9.000E-01					CYCLE = 900		
0.554	0.486	0.424	0.342	0.374	0.047	0.244	-0.006	0.001	0.000
0.629	0.631	0.631	0.632	0.643	0.633	0.649	0.653	0.506	-0.000
0.629	0.631	0.630	0.634	0.632	0.638	0.630	0.637	0.644	0.117
0.626	0.630	0.630	0.631	0.609	0.639	0.609	0.640	0.618	0.248
0.629	0.630	0.630	0.630	0.638	0.632	0.638	0.631	0.649	0.061
0.627	0.631	0.635	0.633	0.632	0.633	0.630	0.612	0.646	0.205
0.627	0.630	0.633	0.633	0.632	0.633	0.634	0.635	0.632	0.470
0.629	0.630	0.636	0.629	0.634	0.634	0.631	0.631	0.631	0.470
0.629	0.633	0.630	0.630	0.630	0.630	0.630	0.629	0.631	0.528
0.630	0.629	0.629	0.627	0.629	0.625	0.627	0.630	0.627	0.624

Table 10

ITEP= 1		TIME= 3.000E+00				CYCLE= 300			
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.170	0.066	-0.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.610	0.613	0.424	0.104	-0.000	0.0	0.0	0.0	0.0	0.0
0.616	0.616	0.617	0.616	0.282	0.001	0.0	0.0	0.0	0.0
0.621	0.620	0.619	0.618	0.616	0.239	-0.000	0.0	0.0	0.0
0.624	0.623	0.621	0.619	0.616	0.617	0.086	0.0	0.0	0.0
0.627	0.626	0.623	0.621	0.619	0.616	0.381	0.007	0.0	0.0

Table 11

ITER= 12		TIME= 6.000E+00				CYCLE= 600			
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.063	0.039	0.008	-0.000	-0.000	-0.000	0.0	0.0	0.0	0.0
0.597	0.600	0.591	0.489	0.340	0.216	0.077	-0.000	0.0	0.0
0.611	0.611	0.612	0.611	0.612	0.612	0.614	0.454	0.058	0.0
0.616	0.616	0.616	0.615	0.614	0.613	0.613	0.613	0.617	0.107
0.620	0.619	0.619	0.618	0.616	0.615	0.613	0.613	0.613	0.622
0.623	0.622	0.620	0.620	0.618	0.617	0.615	0.614	0.613	0.621
0.625	0.624	0.623	0.621	0.620	0.618	0.617	0.615	0.613	0.619
0.627	0.626	0.624	0.623	0.621	0.620	0.619	0.617	0.613	0.615
0.628	0.627	0.627	0.625	0.624	0.623	0.622	0.620	0.617	0.615

Table 12

HEF = 2

TIME = 8.999E+00

CYCLE = 900

0.244	0.317	0.338	0.364	0.405	0.403	0.360	0.407	0.469	0.542
0.607	0.607	0.607	0.608	0.608	0.609	0.609	0.609	0.611	0.616
0.614	0.614	0.614	0.614	0.614	0.614	0.614	0.615	0.616	0.622
0.617	0.617	0.617	0.617	0.616	0.615	0.614	0.614	0.614	0.622
0.620	0.619	0.619	0.619	0.617	0.616	0.615	0.614	0.613	0.622
0.622	0.621	0.621	0.620	0.619	0.618	0.617	0.615	0.613	0.621
0.624	0.624	0.622	0.621	0.620	0.619	0.618	0.617	0.614	0.619
0.626	0.625	0.624	0.623	0.622	0.621	0.620	0.619	0.616	0.617
0.627	0.627	0.625	0.625	0.623	0.623	0.622	0.621	0.619	0.619
0.629	0.627	0.627	0.626	0.626	0.625	0.625	0.624	0.623	0.622