**MINIMUM PATH PROBLEMS IN NORMED
SPACES, REFLECTION AND REFRACTION**

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ABSTRACT

The main minimum (or extremum) path problem in this paper deals with the "law of refraction" at a curve separating the plane into two parts with different norms. Analytic and geometric characterization for the point at which refraction takes place and formulas for the angles that this incident and refracted rays make with a fixed axis or with the normal to the curve are established. The case where the unit circle of the two norms are Euclidean circles with different radii leads to the traditional Snell's Law. The other problem deals with the "law of reflection" from a curve in the normed plane, which in the case of Euclidean norm asserts the equality of the angles of incidence and reflection. The 3-dimensional case, where the separating curve is replaced by a surface, is also considered. Finally it is shown that minimization of path length with respect to non-Euclidean norms is not a special case of Fermat's principle of minimizing the line integral $\int n ds$ for a suitable refraction index n .

1. Introduction. The principle of shortest optical path or of least light time was first pronounced by Fermat and is usually referred to as Fermat's principle. According to it any medium in 3-space is characterized by a function $n(x, y, z)$, the index of refraction, and the path which a light ray takes between two points O_1, O_2 minimizes the line integral $\int_{O_1}^{O_2} n ds$ (actually, it renders the integral an extremum, i.e. $\delta \int_{O_1}^{O_2} n ds = 0$ in the sense of the calculus of variations. As Caratheodory [2] showed, this is a genuine minimum if O_1, O_2 are sufficiently close). Much of the results of geometric optics can be derived from Fermat's principle, the best known are the laws of reflection and refraction. These were first established as empirical laws, the former goes back at least to Euclid, the latter to Willebrord Snell, a Dutch astronomer and mathematician (1591-1626). Later on these laws were derived from various physical theories of light. Snell's law was deduced from an early corpuscular theory of light by René Descartes (1596-1650) and is also known as Descartes' law. Christian Huygens (1629-45) derived these laws from his geometrical "wave" theory of light. Eventually they became byproducts of the all-embracing electromagnetic field theory of James Clark Maxwell (1831-79). For more of the history and physics of these laws see the monograph by Morris Kline and Irwin W. Kay [7].

With no reference to physics the laws of reflection and refraction can be deduced mathematically from the variational principle of shortest light time. For the former this is done by the well-known geometric construction ascribed to Heron in Greek antiquity. For the more difficult case of refraction a geometric proof was found by Huygens [6] and another, based on Ptolemy's theorem, by Pedoe [8]. The laws can also be derived by the use of elementary inequalities; this was done for the law of reflection by Beckenbach and Bellman [1] and for the law of refraction by Golomb [5]. Of course, the general methodology of the calculus of variations can be used for this purpose, see e.g. Caratheodory [2] or Gelfand-Fomin [4, p.60]. Polya [9] has a mechanical interpretation of Snell's law. Chakerian-Ghandehari [3] treat a related problem, that of minimizing the sum of distances of a point from the three vertices of a triangle in a normed space.

The equations of reflection and refraction established in this paper are not physical laws, what they have in common with the above optical laws is that they are solutions of extremal path problems, where arc length is not defined by a refraction index, but by the way distance is measured in the space. The traditional method of the calculus of variations (Euler equation) do not apply here since path length cannot be expressed by a line integral.

2. The Refraction Problem. Let X be a 2-dimensional vector space over the reals with points $x, y, \dots$. Let there be given a simple smooth curve $C : x = x_c(t), x_c \in C^1(-\infty, \infty), x'_c(t) \neq 0$ for all t , which divides X into two regions X_1, X_2 with common boundary C . Let these regions be normed by $\|\cdot\|_1, \|\cdot\|_2$, respectively. The unit circle

$$U_1 = \{x \in X_1 : \|x\|_1 = 1\}, U_2 = \{x \in X_2 : \|x\|_2 = 1\}$$

are assumed to be smooth and strictly convex, *i.e.* they contain no straight line segment (for the norms this means: $\|x + y\|_i = \|x\|_i + \|y\|_i$ implies $x = 0$ or $y = tx$ for some $t \geq 0; i = 1, 2$).

We define the length of a (continuous) curve γ in X to be the least upper bound of the lengths of all the polygonal lines inscribed in γ . Here the length of a polygonal line is the sum of the lengths of its segments, and the length of a segment with terminals x, y is $\|x - y\|_i$ if both x and y are in $X_i (i = 1, 2)$, and it is $\|x - z\|_2 + \|y - z\|_1$ if $x \in X_1, y \in X_2$ and z is the point of intersection of the segment with C . From the triangle inequality property of norms it follows that the length of any curve with terminals x_1, x_2 is at least equal to the length of the line segment λ joining x_1 and x_2 . The strict convexity that we postulated for the norms implies that the length of γ is actually greater than the length of λ unless γ coincides with λ .

Now suppose points $x_1 \in X_1$, $x_2 \in X_2$ are given. We look for the curve γ of the shortest length joining x_1 and x_2 . γ intersects C in at least one point, say x_* . The arc of γ with terminals x_i, x_* ($i = 1, 2$) has length greater than $\|x_i - x_*\|_i$. Unless the arc coincides with the line segment joining x_i and x_* . Thus, in searching for the curve of shortest length joining x_1 and x_2 , we can restrict the search to finding some $x_* \in C$ such that $\|x_1 - x_*\|_1 + \|x_* - x_2\|_2$ is minimal. It is not hard to see that the function

$$s(t) = \|x_1 - x_c(t)\|_1 + \|x_2 - x_c(t)\|_2$$

attains an absolute minimum at some t_* (not necessarily unique) for the point $x_* = x_c(t_*)$. Because of the assumptions made about the norms and the curve C , $s \in C^1(-\infty, \infty)$, hence

$$(*) \quad s'(t_*) = 0.$$

The first problem is to characterize the point x_* and the vectors $x_1 - x_*$, $x_2 - x_*$ by some analytic and geometric condition. The other problems concern the angles involved. Let n_* be the Euclidean normal to C at x_* , α_1 the angle between the vectors $x_1 - x_*$ and n ("angle of incidence"), α_2 the angle between the vectors $x_2 - x_*$ and n ("angle of refraction"). We establish the relation between α_1 and α_2 that corresponds to Snell's law in optics. Also the relation between θ_1^* and θ_2^* , which are the angles that the vectors $x_1 - x_*$ and $x_2 - x_*$ make with a base line. The analogous problems are solved for the case of reflection from the curve C , where only one norm $\|\cdot\|$ is involved, and a condition corresponding to the optical law of reflection is established.

3. Determination of x_* . Let $|\cdot|$ denote the Euclidean norm in X . Since U_i ($i = 1, 2$) is a simple closed curve and the interior of U_i is convex, U_i can be represented in Euclidean polar coordinates ρ, θ ; $\rho = |x|$, $\theta = \arg(x)$, by

$$U_i = \{x \in X_i : \rho = u_i(\theta), u_i \in C^1\} \quad i = 1, 2$$

$\{\theta = 0\}$ is some base ray with terminal at the center of U_i , $\{\theta = \frac{\pi}{2}\}$ is the ray turned counterclockwise by a right angle. Then

$$(1) \quad ||x||_1 = \frac{\rho}{u_1(\theta)} = \frac{|x|}{u_1(\arg x)}, \quad ||x||_2 = \frac{\rho}{u_2(\theta)} = \frac{|x|}{u_2(\arg x)}$$

Set

$$r_i = r_i(t) = x_i - x_c(t), \quad \rho_i = \rho_i(t) = |x_i(t)|, \quad \theta_i = \theta_i(t) = \arg(r_i(t)), \quad i = 1, 2$$

Then by (1), the minimizing condition (*) becomes

$$(2) \quad \left. \frac{d}{dt} \frac{\rho_i(t)}{u_1 \circ \theta_1(t)} \right|_{t=t_*} = - \left. \frac{d}{dt} \frac{\rho_2(t)}{u_2 \circ \theta_2(t)} \right|_{t=t_*}.$$

Carrying out the differentiations in (2), we obtain

$$(3) \quad \frac{\rho'_1}{u_1 \circ \theta_1} - \rho_1 \frac{(u'_1 \circ \theta_1) \theta'_1}{(u_1 \circ \theta_1)^2} = \frac{-\rho'_2}{u_2 \circ \theta_2} + \rho_2 \frac{(u'_2 \circ \theta_2) \theta'_2}{(u_2 \circ \theta_2)^2}$$

The argument in (3) is t_* throughout.

Besides the polar coordinates we will also use Cartesian coordinates (ξ, η) for vectors: $\xi = \rho \cos \theta$, $\eta = \rho \sin \theta$. In particular, we set $\xi_i = \xi_i(t) = \rho_i(t) \cos \theta_i(t)$, $\eta_i = \eta_i(t) = \rho_i(t) \sin \theta_i(t)$, also $X_c = (\xi_c, \eta_c)$. Then

$$\rho'_i = \frac{\xi_i \xi'_i + \eta_i \eta'_i}{\rho_i} = -(\xi'_c \cos \theta_i + \eta'_c \sin \theta_i)$$

$$(4) \quad \theta'_i = -\frac{1}{\rho_i} (\eta'_c \cos \theta_i - \xi'_c \sin \theta_i) \quad \begin{matrix} i = 1, 2 \\ t = t_* \end{matrix}$$

We now use for the variable t the Euclidean arc length of C . Then $\xi'_c = \cos \tau$, $\eta'_c = \sin \tau$, where τ is the angle that the target vector to C at x_* makes with the line $\theta = 0$. Using τ in

(4), we have

$$(5) \quad \rho'_i = -\cos(\tau - \theta_i), \quad \theta'_i = -\frac{1}{\rho_i} \sin(\tau - \theta_i) \quad i = 1, 2; \quad t = t_*$$

In place of τ we use the angles α_i , where α_1 is the angle that the vector $x_1 - x_*$ makes with the normal vector to C at x_* pointing toward x_1 (the "angle of incidence") and α_2 is the angle that the vector $x_2 - x_*$ makes with the normal vector to C at x_* pointing toward x_2 (the "angle of refraction"). Then (5) becomes

$$(6) \quad \rho'_1 = -\sin \alpha_1, \theta'_1 = -\frac{1}{\rho_1} \cos \alpha_1, \quad t = t_*$$

$$\rho'_2 = \sin \alpha_2, \theta'_2 = \frac{1}{\rho_2} \cos \alpha_2.$$

Using (6) in (3), we end up with

$$(7) \quad \frac{\sin \alpha_1}{u_1(\theta_1)} - \frac{u'_1(\theta_1)}{u_1^2(\theta_1)} \cos \alpha_1 = \frac{\sin \alpha_2}{u_2(\theta_2)} - \frac{u'_2(\theta_2)}{u_2^2(\theta_2)} \cos \alpha_2.$$

This is an explicit form of the minimizing condition (*), involving the (not independent) variables α_i, θ_i .

We now give a geometric interpretation of (7). We make x_* the center of both U_1 , and U_2 and also the origins of the $\xi\eta$ -coordinate system. The intersection of the line joining x_i and x_* with U_i is the point P_i with Cartesian coordinates $(u_i(\theta_i) \cos \theta_i, u_i(\theta_i) \sin \theta_i)$. A simple calculation gives the slope m_i of U_i at P_i :

$$(8) \quad m_i = \frac{u_i(\theta_i) \cos \theta_i + u'_i(\theta_i) \sin \theta_i}{-u_i(\theta_i) \sin \theta_i + u'_i(\theta_i) \cos \theta_i}, \quad i = 1, 2$$

Thus, the equation of the tangent line T_i to U_i at P_i is

$$(9) \quad \frac{\eta - u_i(\theta_i) \sin \theta_i}{\xi - u_i(\theta_i) \cos \theta_i} = m_i, \quad i = 1, 2$$

The equation of the tangent line T_* to C at x_* is $\eta/\xi = \eta'_c/\xi'_c = \tan \tau$. So the point of

intersection Q_i of T_i and T_* has the coordinates (ξ_{Q_i}, η_{Q_i}) , which are found by another calculation.

$$(10) \quad (\xi_{Q_1}, \eta_{Q_1}) = \frac{-1}{\frac{\sin \alpha_1}{u_1(\theta_1)} - \frac{u'_1(\theta_1)}{u_1^2(\theta_1)} \cos \alpha_1} (\cos \tau, \sin \tau)$$

$$(\xi_{Q_2}, \eta_{Q_2}) = \frac{-1}{\frac{\sin \alpha_2}{u_2(\theta_2)} - \frac{u'_2(\theta_2)}{u_2^2(\theta_2)} \cos \alpha_2} (\cos \tau, \sin \tau)$$

Comparing (10) with (7), we conclude

$$(11) \quad (\xi_{Q_1}, \eta_{Q_1}) + (\xi_{Q_2}, \eta_{Q_2}) = 0$$

We also see by (7) that the denominator of one of the fractions in (10) is zero if and only if the denominator of the other fraction is zero. This is the case if and only if $m_i = \tan \tau$, i.e. the tangent lines T_i and T_* either coincide or are parallel. For the geometric proposition to be stated we exclude this possibility. Equation (11) says that the point $x_* = (0, 0)$ is the midpoint of the segment $\overline{Q_1 Q_2}$.

Equations (10) were derived independently of (7). Therefore, conditions (7) and (11) are equivalent. We have proved the following:

Theorem 1. Let the line from the given point $O_i (i = 1, 2)$ to a point O_* on the curve C intersect the norm unit circle U_i with center at O_* at P_i , and let Q_i be the intersection of the tangent to U_i at P_i with the tangent to C at O_* . Then the minimizing condition(*) is equivalent to: O_* is the midpoint of the segment $\overline{Q_1 Q_2}$. See Figure 1.

4. Reformulation of the Minimizing Condition. Condition (7) involves besides the angles α_1, α_2 also the angles θ_1, θ_2 . Clearly the variables are not independent, so we will establish conditions which involve only the θ_i or the α_i .

To get the relation between $\theta_1^* = \theta_1(t_*)$ and $\theta_2^* = \theta_2(t_*)$, we assume that the curve C is star-shaped both with respect to x_1 and x_2 . Then the parameter t in condition (*) may be replaced by either $\theta_1 = \arg r_1(t)$ or $\theta_2 = \arg t_2(t)$. C is now represented by

both $x = g_1(\theta_1)$ and $x = g_2(\theta_2)$. Using ρ_i again for $|r_i|$, we put $\rho_i = \rho_i(\theta_i) = |x_i - g_i(\theta_i)|$. Then (2) becomes, using θ_1 for t :

$$(12) \quad \left. \frac{d}{d\theta_1} \frac{\rho_1(\theta_1)}{u_1(\theta_1)} \right|_{\theta_1 = \theta_1^*} = - \left. \frac{d}{d\theta_2} \frac{\rho_2(\theta_2)}{u_2(\theta_2)} \cdot \frac{d\theta_2}{d\theta_1} \right|_{\theta_1 = \theta_1^*}$$

Because of our assumption on C , both θ_1 and θ_2 are strictly monotone, θ_1 is increasing, θ_2 decreasing with respect to t . Since $g_1(\theta_1) = g_2(\theta_2)$, we have

$$g'_1(\theta_1) = g'_2(\theta_2) \frac{d\theta_2}{d\theta_1}, |g'_1(\theta_1)| = - |g'_2(\theta_2)| \frac{d\theta_2}{d\theta_1}.$$

Hence (12) becomes

$$(13) \quad \left. \frac{1}{|g'_1(\theta_1)|} \frac{d}{d\theta_1} \frac{|x_1 - g_1(\theta_1)|}{u_1(\theta_1)} \right|_{\theta_1 = \theta_1^*} = \left. \frac{1}{|g'_2(\theta_2)|} \frac{d}{d\theta_2} \frac{|x_2 - g_2(\theta_2)|}{u_2(\theta_2)} \right|_{\theta_2 = \theta_2^*}$$

This is the explicit relation between θ_1^* and θ_2^* .

Now take α_1, α_2 as the basic variables. Suppose as above that C is star-shaped both with respect to x_1 and x_2 , then $\rho_i = |x - x_i| = |x_i - g_i(\theta_i)|$ is a function of θ_i : $\rho_i = \rho_i(\theta_i)$, and by (6)

$$(14) \quad \tan \alpha_i = \frac{1}{\rho_i} \frac{d\rho_i}{d\theta_i} = \frac{d}{d\theta_2} \log \rho_i(\theta_i) \quad i = 1, 2$$

This equation expresses α_i as a (monotone) function of θ_i . If we now assume that $\theta_i \rightarrow \log \rho_i$ has no inflection point then the inversion of (15) gives θ_i as a (monotone) function of α_i , $\theta_i = \theta_i(\alpha_i)$, and also $x_c = g_i(\theta_i) = f_i(\alpha_i)$ (two representations), $u_i = u_i(\alpha_i)$. Then proceeding as above, we obtain

$$(15) \quad \left. \frac{1}{|f'_1(\alpha_1)|} \frac{d}{d\alpha_1} \frac{|x_1 - f_1(\alpha_1)|}{u_1(\alpha_1)} \right|_{\alpha_1 = \alpha_1^*} = \left. \frac{1}{|f'_2(\alpha_2)|} \frac{d}{d\alpha_2} \frac{|x_2 - f_2(\alpha_2)|}{u_2(\alpha_2)} \right|_{\alpha_2 = \alpha_2^*}$$

This is the explicit relation between the angle of incidence and the angle of refraction, the analogue of Snell's Law.

If X_1 and X_2 are Euclidean spaces, then $u_1(\alpha_1)$ and $u_2(\alpha_2)$ are constant, so (15) become

$$(16) \quad \frac{|f'_2(\alpha_2)|}{|f'_1(\alpha_1)|} \cdot \frac{|x_1 - f_1(\alpha_1)|'}{|x_2 - f_2(\alpha_2)|'} = \frac{u_1}{u_2}$$

But

$$\begin{aligned} |x_i - f_i(\alpha_i)|' &= (x_i - f_i(\alpha_i)) \cdot (x_i - f_i(\alpha_i))' / |x_i - f_i(\alpha_i)| \\ &= |f'_i(\alpha_i)| \cdot \cos \angle (x_i - f_i(\alpha_i), -f'_i(\alpha_i)) = |f'_i(\alpha_i)| \sin \alpha_i. \end{aligned}$$

So (16) becomes

$$(17) \quad \frac{\sin \alpha_1^*}{\sin \alpha_2^*} = \frac{u_1}{u_2},$$

which is the traditional Snell's Law. In the optical interpretation u_1, u_2 are the indices of refraction of the two media X_1, X_2 .

5. The Reflection Problems. In the case of reflection there is only one normed space $X_1 = X_2 = X$, $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|$, and the points x_1, x_2 are on the same side of curve C . There is only one unit circle, $U_1 = U_2 = U$: $U = \{x \in X : \rho = u(\theta), u \in C^1\}$. The minimizing condition (7) becomes:

$$(18) \quad \frac{\sin \alpha_1}{u(\theta_1)} - \frac{u'(\theta_1)}{u^2(\theta_1)} \cos \alpha_1 = \frac{\sin \alpha_2}{u(\theta_2)} - \frac{u'(\theta_2)}{u^2(\theta_2)} \cos \alpha_2.$$

Theorem 1 of Sec. 2 applies here too:

Theorem 2. Let the lines joining the points O_1, O_2 to a point O_* on the curve C intersect the norm unit circle U with center at O_* at the points P_1, P_2 resp. Let the tangent lines to U at the points P_1, P_2 , intersect the tangent line to E at the points Q_1, Q_2 resp. Then

the sum of the distance from O_1 and O_2 to $O = O_*$ is an extremum for all points O on C if and only if O_* is the midpoint of the segment $\overline{Q_1 Q_2}$. See Figure 2.

The special case of parallel tangent lines occurs here if and only if P_1 and P_2 coincide, hence there is only one tangent line to U at $P_1 = P_2$ parallel to the tangent to C at O_* , and in this case the incident and reflected rays are collinear.

For Euclidean space X , (18) becomes $\alpha_1^* = 2\pi - \alpha_2^*$, the angles of incidence and reflection are equal, which is the traditional law of reflection.

6. Addenda. We consider briefly the 3-dimensional case. Assume that X is a 3-dimensional vector space over the reals and there is a surface S that divides X into two regions X_1, X_2 , which are normed by $\|\cdot\|_1, \|\cdot\|_2$. For simplicity we assume that S is the graph of a function in $C^1(\mathbb{R}_2)$. For given points $x_1 \in X_1, x_2 \in X_2$ we seek a curve with terminals x_1, x_2 which makes its length (as derived from the norms, see Sec. 2) an extremum. As in the 2-dimensional case, we may restrict the search to finding a point $x_* \in S$ for which $\|x_1 - x_*\|_1 + \|x_* - x_2\|_2$ is an extremum. There must be such a point, in particular a point x_* for which this sum is a minimum. Let the plane determined by the points x_1, x_2, x_* (if these points are collinear choose any plane containing them) intersect the surface S in the curve C . We are now back in the 2-dimensional case and Theorem 1 may be used to determine the point x_* .

Theorem 3. Let the plane P_* , determined by the given points O_1, O_2 and a chosen point O_* on the surface S , intersect S in the curve C . Let $P_i (i = 1, 2)$ be the point where the line joining O_i and O_* intersects the norm unit ball U_i with center at O_* , and let Q_i be the intersection of the tangent to U_i at P_i in the plane P_* with the tangent to C at O_* . Then the path which is the union of the line segments $\overline{O_1 O_*}$ and $\overline{O_* O_2}$ has extremal length among all the curves that join O_1 to O_2 .

We forego stating the analogous propositions for the case of reflection from the surface S .

Fermat's principle postulates that for every optical medium M in Euclidean space there is a function n with domain M such that light passing from a point O_1 to a point O_2 follows a path that makes the line integral $\int_{O_1}^{O_2} n \cdot ds$ an extremum. The question arises whether the extremal path problem treated in this paper can be formulated as special cases of this principle. Let us consider the straight line path given by $x = tu$, $0 \leq t \leq s$, u a Euclidean unit vector, $|u| = 1$, in the space with norm $\|\cdot\|$. If a function n as postulated above exists, then we must have $\int_s^0 n(tu) dt = \|su - 0\| = s\|u\|$ for every $s > 0$. Differentiation with respect to s gives $n(su) = \|u\|$, or if we set $su = x$, we find $n(x) = \|x\|/|x|$ for every $x \neq 0$. Now consider the path given by $x = x_0 + tu$, $0 \leq t \leq s$ for some x_0 . Then we should have

$$\int_s^0 n(x_0 + tu) dt = \int_s^0 \frac{\|x_0 + tu\|}{\|x_0 + tu\|} dt = \|x_0 + su - x_0\| = s\|u\|,$$

hence $\|x_0 + su\| = \|u\| \|x_0 + su\|$, in particular $\|x_0\| = \|u\| \|x_0\|$ for every x_0 . This is true only if the norm $\|\cdot\|$ differs from the Euclidean norm $|\cdot|$ but by a constant factor, in which case $n = \text{const}$. Thus, the variational problems for the functional $\int n ds$ cannot be replaced by the variational problems for a line integral

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FIGURE 2

FIGURE 1