**AN ASYMPTOTIC RESULT IN FORCED OSCILLATIONS
OF PENDULUM-TYPE EQUATIONS**

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1. Introduction

The forced pendulum-type equation is given by

$$x'' + g(x(t)) = p(t) \quad (1.1)$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and T -periodic and $p(t)$ is 2π -periodic. When $g(x) = a \sin x$, $a > 0$, we obtain the classical pendulum equation. The question of existence of 2π -periodic solution of (1.1) for a given $p(t)$ has been studied recently in [1], [3], [5] (cf. [4] for an extensive bibliography).

Throughout this paper we denote by p_0 the average of $p(t)$ i.e., $p_0 = (1/2\pi) \int_0^{2\pi} p(t) dt$. Some of the existence literature obtains sufficient conditions on the magnitudes of p_0 and $p - p_0$ in order that (1.1) have 2π -periodic solutions. A second category of results in the literature involves studying the problem (1.1) as characterizing the $p(t)$ that are in the range of the operator $\frac{d^2}{dt^2} + g(\cdot)$ acting on 2π -periodic functions.

In Section 2 of this paper we will show (cf. also [4]) that for a given $p_1(t)$, with $\int_0^{2\pi} p_1 = 0$, there exist two numbers $d(p_1)$ and $D(p_1)$ such that (1.1), given by

$$\begin{aligned} x'' + g(x) &= p_0 + p_1(t) \\ x(0) &= x(2\pi), x'(0) = x'(2\pi) \end{aligned} \quad (1.2)$$

has 2π -periodic solutions only for all $p_0 \in \mathbb{R}$ satisfying

$$d(p_1) \leq p_0 \leq D(p_1)$$

This paper deals itself with studying the asymptotic behavior of the functionals d and D . More precisely we shall show that,

$$\lim_{|\lambda| \rightarrow \infty} d(\lambda p_1) = \lim_{|\lambda| \rightarrow \infty} D(\lambda p_1) = \bar{g} = \frac{1}{T} \int_0^T g(x) dx$$

in a generic sense i.e., the set of all $p_1(t)$ with $\int_0^{2\pi} p_1(t) dt = 0$ satisfying the above property contains an open and dense in the class of all 2π -periodic functions with average zero. Thus, in the case of the classical pendulum equation where $\bar{g} = 0$ our result proves that as $p_1(t)$ becomes "large" in magnitude, p_0 gets "small" in (1.2) in order that there exist solutions.

In [1], [2], [4] it was proved that (1.2) always has a 2π -periodic solution if $p_0 = \bar{g}$. Thus in the case of the classical pendulum it is known that (1.2) has a 2π -periodic solution for all p_1 if $p_0 = 0$. In view of the asymptotic result of this paper it now follows that for the classical pendulum as $|\lambda| \rightarrow \infty$, $p_0 \rightarrow 0$ for "most" p_1 in order that

$$x'' + a \sin x = p_0 + \lambda p_1(t)$$

have 2π -periodic solutions.

We also present an example here to illustrate that the generic nature of the result cannot be strengthened very much. Finally we note that the results obtained on the range of $\frac{d^2}{dt^2} + g(\cdot)$ acting on 2π -periodic functions indicate

the contrast in behaviour with respect to operators $\frac{d^2}{dt^2} + g_1(\cdot)$ where g_1 satisfies Landesman-Lazer type hypotheses.

2. Let X be the Banach space of continuous 2π -periodic functions $p(t)$ with norm $\|p\|_X = \|p\|_\infty = \max_{t \in [0, 2\pi]} |p(t)|$.

We denote by X_0 the constant functions and by X_1 the continuous 2π -periodic functions $p_1(t)$ satisfying $\int_0^{2\pi} p_1(t) dt = 0$. Let H be the linear compact operator defined by $H: X_1 \rightarrow X_1$, $p_1 \rightarrow Hp_1 = x_1$ where $x_1 \in X_1$ is the unique solution of

$$x'' = p_1(t).$$

We now obtain the following theorem on existence of 2π -periodic solutions of (1.2).

Theorem 1.

There exist functionals $d, D: X_1 \rightarrow \mathbb{R}$ such that (1.2) has 2π -periodic solutions if and only if

$$d(p_1) < p_0 < D(p_1) \quad (2.1)$$

In addition we have:

- i) $\min_{x \in \mathbb{R}} g(x) < d(p_1) < \bar{g} = (1/T) \int_0^T g(x) dx < D(p_1) < \max_{x \in \mathbb{R}} g(x)$;
- ii) $d(0) = \min_{x \in \mathbb{R}} g(x)$, $D(0) = \max_{x \in \mathbb{R}} g(x)$;
- iii) given $\{p_{1n}\}$ and p_1 in X_1 , such that $Hp_{1n} \rightarrow Hp_1$ uniformly we have

$$d(p_{1n}) \rightarrow d(p_1) \quad \text{and} \quad D(p_{1n}) \rightarrow D(p_1)$$

Proof. We denote by

$$R(p_1) = \{p_0 \in X_0 : (1.2) \text{ has a } 2\pi\text{-periodic solution}\}$$

Then an integration of (1.2) over a period results in

$$\min_{x \in R} g(x) \leq \inf R(p_1) \leq \sup R(p_1) \leq \max_{x \in R} g(x).$$

As in [4], (1.2) has a 2π -periodic solution if $p_0 = \bar{g}$.

Thus

$$d(p_1) = \inf R(p_1) \leq \bar{g} \leq \sup R(p_1) = D(p_1).$$

If $\theta = x - Hp_1$ then (1.2) is equivalent to

$$\theta'' + g(Hp_1(t) + \theta) = p_0 \quad (2.2)$$

Then if $p_{on} \in R(p_1)$ it follows from the periodicity of g and (2.2) that there exist 2π -periodic solutions θ_n of (2.2) such that $\|\theta_n\|_\infty, \|\theta_n''\|_\infty$ are uniformly bounded. By standard compactness arguments it follows that if $p_{on} \rightarrow d(p_1)$ then $d(p_1) \in R(p_1)$. Similarly $D(p_1) \in R(p_1)$.

Since the solutions corresponding to $d(p_1)$ and $D(p_1)$ act as lower and upper solutions for (1.2) where $p_0 \in [d(p_1), D(p_1)]$ it follows that

$$p_0 \in [d(p_1), D(p_1)] \Rightarrow p_0 \in R(p_1).$$

It follows from above that

$$D(p_1) = \max_{\theta \in X \cap C^2(R)} \left[\min_{t \in R} (\theta'' + g(Hp_1 + \theta)) \right] \quad (2.3)$$

and

$$d(p_1) = \min_{\theta \in X \cap C^2(R)} \left[\max_{t \in R} (\theta'' + g(Hp_1 + \theta)) \right] \quad (2.4)$$

The quantities inside the square brackets of (2.3) and (2.4) are continuous with respect to $\|Hp_1\|_\infty$ implying thereby that

D is l.s.c. and d is u.s.c. with respect to the same norm.

Similar ideas may be used to show that if p_{1n}, p_1 belonging to X_1 are such that $Hp_{1n} \rightarrow Hp_1$ uniformly then from

$$\theta_n'' + g(Hp_{1n} + \theta_n) = d(p_n)$$

with $\theta_n(0) \in [0, T]$ (without loss of generality), it follows that there exists a limit point of θ_n , say θ such that

$$\theta'' + g(Hp_1 + \theta) = \lim d(p_n)$$

i.e. $d(p_1) \leq \liminf d(p_{1n})$.

Remarks

- a) The above results hold true for weak solutions (i.e., $p_1 \in L^1(0, 2\pi)$).
- b) Note that the assumption $Hp_{1n} \rightarrow Hp_1$ uniformly is rather weak. Thus, for example, $p_{1n} \rightarrow p_1$ in $L^2(0, 2\pi)$ or $p_{1n} \rightarrow p_1$ in $L^1(0, 2\pi)$ implies $Hp_{1n} \rightarrow Hp_1$ uniformly.
- c) In [1] similar results were obtained under the additional assumption that $(g(u) - g(v))/(u - v) < 1$ for all $u, v \in \mathbb{R}, u \neq v$. Also the continuity of d and D was established in the weak topology of $L^2(0, 2\pi)$.
- d) We also refer to [2] where a similar result was stated and the continuity was in the strong topology of $L^1(0, 2\pi)$.
- e) Part of Theorem 2.1 may also be seen in [4].

3. Asymptotic behaviour of d and D .

Let us denote by

$$L = \left\{ p_1 \in X_1 : \lim_{|\lambda| \rightarrow \infty} d(\lambda p_1) = \lim_{|\lambda| \rightarrow \infty} D(\lambda p_1) = \bar{g} \right\}$$

Theorem 2. L is generic in X_1 i.e., L contains an open and dense subset of X_1 .

The proof requires the following lemmas.

Lemma 1. For $\alpha \in L^1(a, b)$, $-\infty < a < b < \infty$, we have

$$\int_a^b g(\lambda t) \alpha(t) dt \rightarrow \bar{g} \int_a^b \alpha(t) dt \quad \text{as } |\lambda| \rightarrow \infty.$$

(For a proof see [8]).

Lemma 2. Let $\varphi \in C^1[a, b]$ with a finite number of critical points and $\{\theta_\lambda\}_{\lambda \in \mathbb{R}}$ be functions in $C^1[a, b]$ satisfying

$$\|\theta_\lambda\|_\infty + \|\theta'_\lambda\|_\infty \leq M, \text{ a constant independent of } \lambda. \text{ Then}$$

$$\frac{1}{b-a} \int_a^b g[\lambda \varphi(t) + \theta_\lambda(t)] dt \rightarrow \bar{g} \quad \text{as } |\lambda| \rightarrow \infty.$$

(A similar result when $g(x) = e^{ix}$ may be seen in [2]).

Proof Since g is bounded, the integral over small interval is small and thus it suffices to prove the lemma assuming φ' does not vanish. Let λ_0 be such that $\lambda_0 |\varphi'(t)| \geq M$ for all $t \in [a, b]$. For λ satisfying $|\lambda| \geq \lambda_0$ let $\varphi_\lambda = \varphi + \theta_\lambda$. Then $\varphi'_\lambda \neq 0$ over $[a, b]$ and

$$\int_a^b g[\lambda \varphi(t) + \theta_\lambda(t)] dt = \int_a^b g(\lambda \varphi_\lambda(t)) dt$$

$$= \int_{\varphi_\lambda([a, b])} g(\lambda s) |\varphi'_\lambda(\varphi_\lambda^{-1}(s))|^{-1} ds$$

By Lemma 1, $\int_{\varphi([a,b])} g(\lambda s) |\varphi'(\varphi^{-1}(s))| ds \rightarrow \bar{g}(b-a)$ and hence

Lemma 2 follows.

Lemma 3. The set $M = \{p_1 \in X_1 : Hp_1 \text{ is a Morse function}\}$ is dense and open in X_1 .

Proof It is easy to see M is open. In order to prove that M is dense we note that for $p_1 \in X_1$ if we identify Hp_1 as a C^2 -function on the 1-sphere then we can find [6] 2π -periodic Morse functions φ_n such that $\varphi_n \rightarrow Hp_1$ in the C^2 -uniform topology. Then if $p_{1n} = \varphi_n''$ it follows that $p_{1n} \rightarrow p_1$ in X and $Hp_{1n} = \varphi_n - (1/2\pi) \int_0^{2\pi} \varphi_n dx$ is also a Morse function and $p_{1n} \in M$.

Proof of Theorem 2. From Lemma 3 it suffices to prove that $M \subset L$. For any $p_1 \in M$, Hp_1 can have only a finite number of critical points in $[0, 2\pi]$.

Also for a given λ let θ_λ be a 2π -periodic function such that

$$\theta_\lambda'' + g(\lambda Hp_1(t) + \theta_\lambda) = d(\lambda p_1)$$

with $\theta_\lambda(0) \in [0, T]$.

Then $\|\theta_\lambda''\|_\infty + |\theta_\lambda(0)|$ is uniformly bounded and hence so is $\|\theta_\lambda''\|_\infty + \|\theta_\lambda\|_\infty$. From Lemma 2,

$$d(\lambda p_1) = (1/2\pi) \int_0^{2\pi} g(\lambda Hp_1 + \theta_\lambda) dt \rightarrow \bar{g}$$

as $|\lambda| \rightarrow \infty$. Similarly $D(\lambda p) \rightarrow \bar{g}$ and hence the Theorem follows.

4. An example showing L is not open

$$\text{Let } p_{1\alpha}(t) = \begin{cases} t(t-\alpha)[\alpha^2 + 5(t-\alpha)t], & 0 \leq t \leq \alpha \\ 0, & \alpha \leq t \leq \pi. \end{cases}$$

Then we set $p_{1\alpha}(2\pi-t) = -p_{1\alpha}(t)$ and extend $p_{1\alpha}$ by periodicity to all of $\mathbb{R}$.

Then it can be seen that $p_{1\alpha} \in X$ and $Hp_{1\alpha}$ is given by

$$Hp_{1\alpha}(t) = \begin{cases} \frac{1}{6}(t-\alpha)^3 t^3, & 0 \leq t \leq \alpha \\ 0, & t \leq \alpha \leq \pi \end{cases}$$

$$\text{and } Hp_{1\alpha}(2\pi-t) = -Hp_{1\alpha}(t), \quad 0 \leq t \leq \pi.$$

Let $X_\pi = \{p \in X : p(\pi) = 0\}$ and let $H_\pi : X_1 \rightarrow X_\pi$ be defined to be the unique solution of $x'' = p$, $x \in X_\pi$.

Then for each λ there exists a 2π -periodic solution of

$$\begin{aligned} \theta_{\lambda\alpha}'' + g(\lambda Hp_{1\alpha} + \theta_{\lambda\alpha}) &= (1/2\pi) \int_0^{2\pi} g(\lambda Hp_{1\alpha} + \theta_{\lambda\alpha}) dt \\ \theta_{\lambda\alpha}(\pi) &= \xi_0, \text{ where } g(\xi_0) = \max_{x \in \mathbb{R}} g(x), \end{aligned} \quad (4.1)$$

This existence follows from the Schauder fixed point theorem applied to the operator $T : X_1 \rightarrow X_\pi$ defined by

$$x \mapsto -H_\pi \left[g(\lambda Hp_{1\alpha} + x + \xi_0) - (1/2\pi) \int_0^{2\pi} g(\lambda Hp_{1\alpha} + x + \xi_0) dt \right]$$

and setting $\theta_{\lambda\alpha} = x + \xi_0$.

From now on we assume that $g \not\equiv 0$ and let $\delta > 0$ be such that, for a given $\eta > 0$,

$$g(\xi) \geq \bar{g} + \eta \quad \text{if} \quad |\xi - \xi_0| \leq \delta. \quad (4.2)$$

Then, if $|g(x)| \leq M$ for $x \in \mathbb{R}$, it follows from (4.1) that $\|\theta_{\lambda\alpha}\|_\infty \leq 2M$.

Thus $\|\theta_{\lambda\alpha}'\|_{\infty} \leq 4\pi M$ and

$$|\theta_{\lambda\alpha}(t) - \theta_0| \leq 4\pi M(\pi - \alpha) \text{ if } \alpha \leq t \leq 2\pi - \alpha.$$

Let α be chosen close to π such that

$$4\pi M(\pi - \alpha) \leq \delta.$$

Then from (4.1) we have

$$\begin{aligned} D(\lambda p_{1\alpha}) &\geq (1/2\pi) \int_0^{2\pi} g(\lambda H p_{1\alpha} + \theta_{\lambda\alpha}) dt \\ &= (1/2\pi) \left[\int_0^{\alpha} + \int_{\alpha}^{2\pi-\alpha} + \int_{2\pi-\alpha}^{2\pi} \right]. \end{aligned}$$

By Lemma 2, the first and third integrals tend to $\alpha \bar{g}$ as $|\lambda| \rightarrow \infty$. Also, from (4.2),

$$\int_{\alpha}^{2\pi-\alpha} g(\theta_{\lambda\alpha}) dt \geq 2(\bar{g} + \eta)(\pi - \alpha).$$

Thus

$$\liminf_{|\lambda| \rightarrow \infty} D(\lambda p_{1\alpha}) \geq \bar{g} + (1 - \alpha/\pi)\eta$$

and thus $p_{1\alpha} \notin L$. Setting $\alpha \uparrow \pi$, $p_{1\alpha} \rightarrow p_{1\pi}$ in X_1 , and $p_{1\pi} \in L$ because $H p_{1\pi}$ has only a finite number of critical points in $[0, 2\pi]$.

Remarks

a) Following the Galerkin approximation ideas it follows that if $X_n = \langle 1, \sin t, \cos t, \dots, \sin nt, \cos nt \rangle$ and $g \in C^1(\mathbb{R})$ then

$$d_n(p_1) = \inf_{\theta \in X_n} [\max_{t \in \mathbb{R}} (\theta'' + g(H p_1 + \theta))]$$

satisfies $d_n(p_1) \downarrow d(p_1)$ and may be seen as an approximation to $d(p)$ over a finite dimensional subspace. Similar remarks

are valid for $D(p_1)$.

b) It is well-known that when g_1 is a monotone decreasing function or when g_1 satisfies Landesman-Lazer type hypotheses then

$$R\left(\frac{d^2}{dt^2}\right) + R(g_1) \cong R\left(\frac{d^2}{dt^2} + g_1\right),$$

Thus for example if $g_1(x) = \arctan x$ then the equation

$$\frac{d^2 x}{dt^2} + g_1(x) = p_0 + p_1(t)$$

has a 2π -periodic solution for $p_1(t)$ arbitrary and $p_0 \in (-\pi/2, \pi/2)$. This is in contrast with the result of Theorem 2. For further discussions on pendulum-type equations we refer to [7].

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