SYSTEM IDENTIFICATION OF MODELS
EXHIBITING EXPONENTIAL, HARMONIC
AND RESONANT MODES

by

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I. INTRODUCTION:

A classical problem arising in compartmental analysis is the so called identification problem or the inverse problem. One is presented with the linear time invariant compartmental model

$$\dot{z}(t) = Az(t) + Bu(t) \quad t > 0, \quad z(0) = 0 \quad (1.1)$$

$$y(t) = Cz(t) \quad (1.2)$$

Where A is a square matrix and the dimensions of B and C are consistent with that of A . The problem is to estimate a certain subset of the matrix elements $\{a_{ij}\}$, $\{b_{ij}\}$ and $\{c_{ij}\}$ from discrete observations of the input vector $u(t)$ and the output vector $y(t)$. When instantaneous mixing is assumed the analogous equations (1.1) are replaced by

$$\dot{z}(t) = Az(t) \quad t > 0, \quad z(0) = z_0 \quad (1.1)'$$

An intermediate problem is that of identifying the respective impulse response matrix i. e the matrix

$$D(t) = C \exp(tA)B \quad (1.3)$$

In the case of the model (1.1) the corresponding problem is of identifying matrix

$$\hat{D}(t) = C \exp(tA) \quad (1.3)$$

In either case each entry $x(t) = d_{ij}(t)$ or $x(t) = \hat{d}_{ij}(t)$ of the matrix satisfies a differential equation

$$x^{(N)}(t) + c_{N-1} x^{(N-1)}(t) + \dots + c_0 x(t) = 0 \quad (1.4)$$

with constant coefficients c_i $i = 0, 1, 2, \dots, N-1$ where $N \leq$ the degree of the minimal polynomial of A . Thus $x(t)$ has the form:

$$x(t) = \sum_{j=1}^n \sum_{i=1}^{n_j} a_{ij} \frac{t^{i-1}}{t} e^{\lambda_j t} + \sum_{p=1}^m \sum_{u=1}^{m_p} \frac{t^{u-1}}{t} e^{\gamma_p t} \left(\eta_{up} \sin(\beta_p t) + \psi_{up} \cos(\beta_p t) \right) \quad (1.5)$$

Where λ_j $j = 1, 2, \dots, n$ are distinct real numbers.

$\gamma_p = \alpha_p + i\beta_p$ $p = 1, 2, \dots, m$ are distinct complex numbers such

that $\beta_p \neq 0$ $p = 1, 2, \dots, m$, and

$$N = \sum_{j=1}^n n_j + 2 \sum_{p=1}^m m_p \quad (1.6)$$

This paper deals with the identification of all parameters and integers occurring in equation (1.5). One of the most challenging of the computational problems is to identify the number of components N as defined in (1.6)

The identification procedure to be presented is based on the method of moments. We call this method LAM (Linear algebraic method). The method requires the estimation of "generalized moments (which we define in Section 2) denoted by s_k $k = 1, 2, \dots, 2L$.

$$s_k = \int_a^b x(t) w_k(t) dt \quad k=1, 2, \dots, 2L \quad (1.7)$$

Where $w_k(t)$ represents respective weighting functions, interval $[a, b]$ represents the interval on which observations are available and $L \leq N$ is known. If $x(t)$ is not known directly, however data is collected for $y(t)$ and $u(t)$, then one may estimate the integrals s_k by a deconvolution method proposed in [2].

Hence we restate our problem as follows: One is presented with the discrete (noisy) observations $x(ti)$ $i = 1, 2, \dots, q$ and the general model (1.5). The problem is to identify all parameters and integers occurring in (1.5).

The identification process may be separated into two stages. The first deals with mapping discrete data into a "generalized moment sequence". Second stage, discussed in Section 4, shows how such a mapping can be accomplished by means of a suitably chosen "weighting function." The identification algorithm is presented in section 5. Sections 2 and 3 deal with the theory of moment sequences. Numerical examples are presented in Section 6.

This work is an extension of authors' previous paper [1] where $x(t)$

has the more special form

$$x(t) = \sum_{j=1}^n \alpha_j \exp(\lambda_j t) \quad (1.8)$$

Where λ_j 's are distinct and the α_j 's are positive. The success of the algorithm and proofs of the theorems will be reported in [9]

2. Generalized discrete moment sequence (g.d.m.s)

We consider first the classical discrete moment problem [7].

Given a sequence $s_1, s_2, \dots$ of real numbers, it is required to determine real numbers α_j and λ_j $j = 1, 2, \dots, N$ such that the following equations hold:

$$s_k = \sum_{j=1}^n \alpha_j \lambda_j^{k-1} \quad k = 1, 2, \dots \quad (2.1)$$

$$\alpha_j > 0 \quad j = 1, 2, \dots \quad (2.2)$$

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_N \quad (2.3)$$

The sequence of real numbers $\{s_k\}$ $k = 1, 2, \dots$ satisfying (2.1) - (2.3) is called a discrete moment sequence.

This discussion can be extended to a more general situation. We define a generalized discrete moment sequence as follows:

Given a sequence $s_1, s_2, \dots, s_{2L}$ of real numbers, it is required to determine integers N, n, n_j , $j = 1, 2, \dots, n$; m, m_j , $j = 1, 2, \dots, m$; real numbers λ_j , $j = 1, 2, \dots, n$; α_{ij} $i = 1, 2, \dots, n_j$, $j = 1, 2, \dots, n$; and complex numbers w_{ij} $i = 1, 2, \dots, m_j$, $j = 1, 2, \dots, m$;

and $z_j, j = 1, 2, \dots, m$, such that the following relations hold:

$$s_k = \left[\sum_{i=1}^k \sum_{j=1}^n \binom{k-1}{i-1} \alpha_{ij} \lambda_j^{k-i} \right] + \left[\sum_{i=1}^k \sum_{j=1}^m \binom{k-1}{i-1} \left(w_{ij} z_j^{k-i} + \bar{w}_{ij} \bar{z}_j^{k-i} \right) \right] \quad (2.4)$$

$$k = 1, 2, \dots, 2L.$$

Here:

$z_j, j = 1, 2, \dots, m$ are distinct complex numbers such that

$$z_j = \gamma_j + i\beta_j, \beta_j \neq 0, j = 1, 2, \dots, m, \quad (2.5)$$

$$\lambda_j, j = 1, 2, \dots, n \text{ are distinct real numbers,} \quad (2.6)$$

$$\alpha_{n_j j} > 0, j = 1, 2, \dots, n, \quad (2.7)$$

$$\alpha_{ij} = 0 \quad \text{for all } i > n_j, j = 1, 2, \dots, n, \quad (2.8)$$

$$w_{ij} = 0 \quad \text{for all } i > m_j, j = 1, 2, \dots, m, \quad (2.9)$$

$$N = \sum_{j=1}^n n_j + 2 \sum_{j=1}^m m_j, \quad (2.10)$$

$$L \geq N \quad \text{where } L \text{ is known.} \quad (2.11)$$

The sequence of real numbers $s_k, k = 1, 2, \dots, 2L$ satisfying (2.4) - (2.11) is called a generalized discrete moment sequence (g.d.m.s).

Let us define operations:

$$D_j s_k = s_{k+1} - \lambda_j s_k \quad \begin{matrix} k = 1, 2, 3, \dots, 2L-1 \\ j = 1, 2, \dots, n, \end{matrix} \quad (2.12)$$

$$E_j s_k = (s_{k+2} - z_j s_{k+1}) - \bar{z}_j (s_{k-1} - \bar{z}_j s_k), \quad (2.13)$$

$$i.e., E_j s_k = s_{k+2} - 2\gamma_j s_{k+1} + (\gamma_j^2 + \beta_j^2) s_k,$$

$$k = 1, 2, 3, \dots, 2L - 2$$

$$j = 1, 2, \dots, m.$$

Remark:

Let $\{s_k\}_{k=1}^{2L}$ be a g.d.m.s then

$$\prod_{i=1}^n D_i^{n_i} \prod_{j=1}^m E_j^{m_j} s_k = 0$$

$$k = 1, 2, \dots, n, \dots, 2L - N. \quad (2.14)$$

3. Hankel matrices of generalized discrete moment sequences:

Let $\{s_k\}_{k=1}^{2L}$ be a g.d.m.s. Define the Hankel matrices

$$G_j^{(k)} = \begin{pmatrix} s_k & s_{k+1} & \dots & s_{k+j-1} \\ s_{k+j-1} & s_{k+j} & \dots & s_{2j+k-2} \end{pmatrix}$$

$$j = 1, 2, 3, \dots$$

$$k = 1, 2, \dots, 2(L-j+1).$$
(3.1)

One may obtain following results:

Theorem 1. Let $\{s_k\}_{k=1}^{2L}$ be a g.d.m.s. and let the integer Q be defined by

$$Q = (1/2) \left(\sum_{j=1}^n n_j^2 - N \right) \quad (3.2)$$

then

$$\det G_j^{(k)} = 0 \quad \text{for} \quad j = N + 1, N + 2, \dots, L$$

$$k = 1, 2, \dots, 2(L - j + 1), \quad (3.3)$$

$$\det G_N^{(1)} \neq 0, \quad \text{and}$$

$$\det G_N^{(1)} > 0 \quad \text{if } Q \text{ is even,} \quad (3.4)$$

$$\det G_N^{(1)} < 0 \quad \text{if } Q \text{ is odd.} \quad (3.5)$$

Theorem 2. Let $\{s_k\}_{k=1}^{2L}$ be a g.d.m.s. and $P_N(\mu) = \frac{q_N(\mu)}{\det G_N^{(1)}}$ where

$$q_N(\mu) = \begin{vmatrix} 1 & \mu & \mu^2 & \dots & \mu^N \\ s_1 & s_2 & s_3 & \dots & s_{N+1} \\ & & & & \\ s_N & s_{N+1} & s_{N+2} & \dots & s_{2N} \end{vmatrix} \quad (3.6)$$

then:

- (i) n = number of distinct real roots of $P_N(\mu)$,
- (ii) λ_j $j = 1, 2, \dots, n$ are the distinct real roots of $P_N(\mu)$ with respective multiplicities n_j $j = 1, 2, \dots, n$,
- (iii) m = number of distinct complex roots of $P_N(\mu)$,
- (iv) z_j , $j = 1, 2, \dots, m$ are the distinct complex roots of $P_N(\mu)$ with respective multiplicities m_j , $j = 1, 2, \dots, m$.

Remark: Hankel matrices $G_j^{(k)}$ (3.1) can be decomposed as

$$G_j^{(k)} = P_j Q_j^{(k)} P_j^T \quad \begin{matrix} j = 1, 2, \dots, 2L \\ k = 1, 2, \dots, 2(L-j+1) \end{matrix} \quad (3.7)$$

where: P_j is of dimension $j \times N$,
 $Q_N^{(k)}$ is of dimension $N \times N$,

and: P_j^T = Transpose of P_j .

In particular, for $j = N$, $G_N^{(1)}$ is nonsingular as in Theorem 1 and hence P_N and $Q_N^{(1)}$ both are nonsingular.

Theorem: 3. Let $\{s_k\}_{k=1}^{2L}$ be a g.d.m.s. then

$$\bar{\alpha} = (P_N^{-1}) \bar{s} \quad (3.8)$$

where $\bar{\alpha} = [\alpha_{11} \ \alpha_{21} \ \dots \ \alpha_{nn}, w_{11}, w_{21}, \dots, w_{nn}, w_{11}, w_{21}, \dots, w_{nn}]^T$, $\bar{s} = [s_1, s_2, \dots, s_N]^T$ and P_N is a nonsingular square matrix defined in (3.7).

4. Generation of g.d.m.s.

Define set X as follows:

$$X = \left\{ x(t) \mid \begin{array}{l} a \leq t \leq b, x(t) \text{ is of the form (1.5)} \\ \lambda_j, j = 1, 2, \dots, n \text{ be distinct real numbers} \end{array} \right\}$$

$$z_j = \gamma_j + \beta_j, \beta_j \neq 0 \quad j = 1, 2, \dots, m \text{ be distinct complex numbers}$$

(4.1)

Notice that with this definition of X and discussion in section 1, X is a vector space. X is finite dimensional and dimension of X is N where N is as defined in (2.6)

$$W_L = \left\{ w(t) \mid \begin{array}{l} w(t) > 0 \quad t \in (a, b), w(t) \in C^{2L}[a, b] \\ w^{(i-1)}(a) = w^{(i-1)}(b) = 0 \quad i = 1, 2, \dots, 2L - 1 \end{array} \right\}. \quad (4.2)$$

We call W_L , a set of weighting functions.

Define a sequence of linear functionals on X as follows:

$$w_p: X \rightarrow \mathbb{R}$$

$$\langle x(t), w_k(t) \rangle = (-1)^{k-1} \int_a^b x(t) w^{(k-1)}(t) dt \quad (4.3)$$

$$k = 1, 2, \dots, 2L$$

$$w(t) \in W_L \text{ such that}$$

$$L \geq N + 1.$$

$$\text{Denote } s_k = \langle x(t), w_k(t) \rangle \quad k = 1, 2, \dots, 2L. \quad (4.4)$$

Using above definitions one may prove following:

Theorem 4. A sequence of real numbers generated in (4.4) is a g.d.m.s

if $\alpha_{n_j}, j > 0 \quad j = 1, 2, \dots, n$ for $x(t) \in X$.

5. Identification procedure:

We are given noisy data $x(t_i)$ $i = 0, 1, 2, \dots, q$. We assume that $L \geq N + 1$ is known and $\alpha_{n_j}, j > 0$ for $j = 1, 2, \dots, n$. Our identification procedure is discussed in following 3 steps. Step 1. Choose $w(t) \in W_L$

as defined in (4-2) $w(t) = \sin^{(2L-1)} \left[\frac{\pi(t-a)}{(b-a)} \right]$ or $w(t) = (t-a)^{2L-1} (b-t)^{2L-1}$.
(5.1)

Estimate s_k $k = 1, 2, \dots, 2L$ using definitions (4.3) - (4.4) and any appropriate quadrature formula for integration. compute $\det G_j^{(1)}, j = 1, 2, \dots, L$. Using Theorem 1 and Theorem 4

$$\hat{N} = \max_{1 \leq j \leq L} \left\{ j \mid \det G_j^{(1)} \neq 0 \right\} \quad (5.2)$$

In practical situations, because of experimental and computational errors, one does not get $\det G_j^{(1)} = 0$ for all $j = N + 1, N + 2, \dots$ but one finds a sharp drop in the value of $\det G_j^{(1)}$ to a number ϵ relatively small in absolute value. Hence instead of (5.2) we consider

$$\hat{N} = \max_{1 \leq j \leq L} \left\{ j \mid \left| \det G_j^{(1)} \right| > \epsilon, 1 \leq j \leq L \right\}. \quad (5.3)$$

Step 2. Once N is identified, using Theorem 2 we can theoretically identify $n, n_j, j = 1, 2, \dots, n, m, m_j, j = 1, 2, \dots, m, \lambda_j, j = 1, 2, \dots, n$, and $z_j, j = 1, 2, \dots, m$. Since the roots of the polynomial may be extremely sensitive to small changes in coefficients,

the errors involved in computing coefficients of the monic polynomial $p_N(\mu)$ (3.6) may lead us to completely different roots and hence we may end up with completely different structure of $x(t)$. To handle this situation, along with Theorem 2 we use (3.4), (3.5) and (2.14), to identify the aforesaid parameters. Numerical examples for the case $N \leq 3$ are presented in the following section.

Step 3. Once Step 1 and step 2 are finished, the problem of identifying α_{ij} and w_{ij} is of lower order of difficulty. We use Theorem 3 to identify these parameters. One may also use linear least squares method to identify these parameters, as the respective model (1.5) is linear in these parameters.

6. Numerical examples:

In this section we present numerical examples using computer simulated data. All computations have been performed on the IBM 370-155 computer in FORTRAN, using single precision arithmetic. The noise which has a uniform distribution is also added to $x(t)$. For these examples we assumed $L = 4$ and $N \leq 3$ and we chose the weighting function $w(t) \in W_4$ as

$$w(t) = \sin^7 \left[\frac{\pi(t-a)}{(b-a)} \right] \quad (6.1)$$

For convenience we chose equally spaced data points $x(j)$, $j = 0, 1, 2, \dots, 70$. The g.d.m.s. $\{s_k\}$ $k = 1, 2, \dots, 8$ is computed using the composite Simpson's rule of integration.

In these examples we consider all possible structures.

I. $N = 1$ We have only one possible case $x(t) = \alpha_{11} e^{\lambda_1 t}$, examples for this case have been presented in [1]

II. $N = 2$ We have three possible structures of $x(t)$:

(i) $x(t) = \alpha_{11} e^{\lambda_1 t} + \alpha_{12} e^{\lambda_2 t}$ (examples for this case have been presented in [1]).

$$(ii) \quad x(t) = \alpha_{11} e^{\lambda_1 t} + \alpha_{21} t e^{\lambda_1 t},$$

$$(iii) \quad x(t) = e^{\lambda_1 t} \left[\eta_{11} \sin \beta_1 t + \psi_{11} \cos \beta_1 t \right]$$

For cases (ii) and (iii) we consider the following two examples.

Example 1.

$$x(t) = 5.0 e^{-.01t} + 0.1 t e^{-.01t}$$

Expected parameters: $N = 2$, $n = 1$, $n_1 = 2$, $m = 0$, $\lambda_1 = -0.01$,
 $\alpha_{11} = 5.0$, $\alpha_{21} = 0.1$

Step 1. From the computed determinants, $\det G_i^{(1)}$ $i = 1, \dots, 4$
 were respectively $0.12103900E = 03$, $-0.20730250E + 01$, -0.62276790
 $E-07$, $0.47945110E-14$.

We estimated $\hat{N} = 2$

Step 2. Let $p_2(\mu) = \mu^2 + c_1 \mu + c_0$ We estimated

$$\hat{c}_1 = 0.20039256E-01 \quad \text{and} \quad \hat{c}_0 = 0.99821260E-04, \quad \hat{c}_1 - 4\hat{c}_0 = 0.22867402E-05.$$

Since $\det G_2^{(1)} < 0.$, $n \neq 2$. Averaging the two roots of $p_2(\mu)$ we got

$$\hat{\lambda}_1 = -0.10005300E-.01 \quad \text{thus} \quad \hat{n} = 1, \quad \hat{n}_1 = 2, \quad \hat{m} = 0 \quad \text{and} \quad \hat{x}(t) = \hat{\alpha}_{11} e^{\hat{\lambda}_1 t} + \hat{\alpha}_{21} t e^{\hat{\lambda}_1 t}.$$

Check: For this structure (2.14) implies $s_3 - 2\lambda_1 s_2 + \lambda_1^2 s_1 = 0$.

Now $\hat{s}_3 - 2\hat{\lambda}_1 \hat{s}_2 + \hat{\lambda}_1^2 \hat{s}_1 = 0.28690843E-05$ which is small.

Step 3. Solving (3.8) we got $\hat{\alpha}_{11} = 0.49977800E-.01$

$$\text{and} \quad \hat{\alpha}_{21} = 0.10019210E00.$$

Example 2. $x(t) = e^{-.05t} \left(\sin(.05t) + 0.5 \cos(.05t) \right)$

Determinants: $0.34962540E+01, -0.39188754E-01, 0.25200293E-08,$
 $0.90490722E-17$

	N	n	m	m_1	γ_1	β_1	n_{11}	ψ_{11}
Expected	2	0	1	1	-0.50000000E-01	0.50000000E-01	0.10000000E+01	0.50000000E+00
Estimated	2	0	1	1	-0.49998611E-01	0.4999814E-01	0.99956610E+00	0.50000972E00

III. $N = 3$ We have four possible structures of $x(t)$

Example 3. $x(t) = (1.0 + 10.0t + 2.0t^2) e^{-.05t}$

Determinants: $0.94841328E+04, -0.13220200E+06, -0.35761895E+04, 0.40516555E-02$

	N	n	m_1	m	λ_1	α_{11}	α_{21}	α_{31}
Expected	3	1	3	0	-0.50000000E-01	0.10000000E+02	0.10000000E+02	0.20000000E+01
Estimated	3	1	3	0	-0.50000627E-01	0.10328026E+01	0.99949007E+01	0.19999180E+01

Example 4. $x(t) = (2.0 + 0.01t) e^{.03t} + 0.50e^{-.05t}$

Determinants: 0.14366412E+03, 0.15601616E+01, -0.28224720E-04, -0.90954327E-12

	N	n	n ₁	n ₂	m	λ_1	λ_2	α_{11}	α_{21}	α_{12}
Expected	3	2	2	1	0	0.30000000E-01	0.20000000E-01	0.20000000E+01	0.50000000E+00	0.50000000E+00
Estimated	3	2	2	1	0	0.31871039E-01	0.50268579E-01	0.20348787E+01	0.88596896E-02	0.46314700E+00

Example: 5. $x(t) = (2.0e^{-.05t}) + e^{-.03t} \left[3.0 \sin(0.02t) + 0.01 \cos(0.02t) \right]$

Determinants: 0.21071804E+02, -0.84263563E-01, -0.92183501E-06, -0.26345270E-14

	N	n	n ₁	m	m ₁	λ_1	γ_1	β_1	α_{11}	η_{11}	ψ_{11}
Expected	3	1	1	1	1	-.50000000E-01	-.30000000E-01	.20000000E-01	0.20000000E+01	0.30000000E+01	0.10000000E-01
Estimated	3	1	1	1	1	-.50671406E-01	-.29773440E-01	.19862771E-01	0.34186697E-01	0.29853060E+01	0.10763079E-01

Example 6. $x(t) = 2.0e^{-.05t} + 3.0e^{-.03t} + .01e^{-.02t}$

Determinants: $0.30050079E+2$, $0.10564530E+00$, $0.33736160E-06$, $0.23740237E-14$

	N	n	m_1	m_2	m_3	λ_1	λ_2	λ_3
Expected	3	3	1	1	1	$-.50000000E-01$	$-.30000000E-01$	$0.20000000E-01$
Estimated	3	3	1	1	1	$-.50520600E-01$	$-.30386200E-01$	$0.17941800E-01$

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