PARABOLIC DIFFERENTIAL INEQUALITIES IN CONES

by

V. Lakshmikantham and Randy Vaughn

Department of Mathematics
The University of Texas at Arlington
Arlington, Texas 76019

Technical Report No. 78

March 1978

PARABOLIC DIFFERENTIAL INEQUALITIES IN CONES

by

V. Lakshmikantham and Randy Vaughn

In this paper we investigate the theory of parabolic differential inequalities in arbitrary cones. After discussing the fundamental results concerning parabolic inequalities in cones, we prove a result on flow-invariance which is then used to obtain a comparison theorem. This comparison result is useful in deriving upper and lower bounds on solutions of parabolic differential equations in terms of the solutions of ordinary differential equations. We treat the Dirichlet problem in this paper since its theory follows the general pattern of ordinary differential equations and requires less restrictive assumptions. The treatment of Neumann problem, on the other hand, demands stronger smoothness assumptions and depends heavily on strong maximum principle. The study of the corresponding results relative to Neumann problem is discussed elsewhere.

I. A PRELIMINARY RESULT

Let Ω be a bounded domain in R^n and let $H = (t_0, \infty) \times \Omega$, $t_0 \geq 0$. Let $\bar{H}$ denote the closure of H and ∂H , the boundary of H .

A proper subset K of R^N is called a solid cone if (i) $\lambda K \subset K$, $\lambda \geq 0$, (ii) $K + K \subset K$, (iii) $K = \bar{K}$, (iv) $K \cap (-K) = \{0\}$ and

* Research supported by U.S. Army Research Grant DGAG29-77-G0062.

(v) K^0 is nonempty, where K^0 is the interior of K .

The cone K induces the ordering relation on R^N defined by

$$u \leq v \text{ iff } v - u \in K \text{ and } u < v \text{ iff } v - u \in K^0.$$

The set K^* defined by $K^* = [\phi \in R^N: \phi(u) \geq 0 \text{ for all } u \in K]$

satisfies the properties (i) to (v) and is called the adjoint cone.

We note that $K = (K^*)^*$, $u \in K^0$ iff $\phi(u) > 0$ for all $\phi \in K_0^*$ and $u \in \partial K$ iff $\phi(u) = 0$ for some $\phi \in K_0^*$, where $K_0 = K - \{0\}$.

The following lemma is basic in our discussions.

Lemma 1.1. Assume that

(i) $m \in C[\bar{H}, R^N]$ and the partial derivatives m_t, m_x, m_{xx} exist and are continuous in H ;

(ii) $m(t, x) < 0$ on ∂H ;

(iii) if $(t_1, x_1) \in H$ and $\phi \in K_0^*$ such that $m(t_1, x_1) \leq 0$,

$\phi(m(t_1, x_1)) = 0$, $\phi(m_{x_i}(t_1, x_1)) = 0$, $i = 1, 2, \dots, n$ and the quadratic form

$$\sum_{i,j=1}^n \lambda_i \lambda_j \phi(m_{x_i x_j}(t_1, x_1)) \leq 0$$

for arbitrary $\lambda \in R^n$, then $\phi(m_t(t_1, x_1)) < 0$.

Then $m(t, x) < 0$ on $\bar{H}$.

Proof. If the conclusion of the lemma is false then the set $M = [\phi \in K_0^*: \phi(m(t, x)) \geq 0 \text{ for some } (t, x) \in \bar{H}]$ is nonempty. Then the set

$S = [(t, x) \in \bar{H}: d(-m(t, x), \partial K) > 0]$ is a nonempty open set containing

$\{t_0\} \times \bar{\Omega}$. Since $\bar{\Omega}$ is compact, there exists a $t^* > t_0$ such that $[t_0, t^*) \times \bar{\Omega} \subseteq S$. Let $\phi \in M$ and $Z_\phi = \{(t, x) : \phi(m(t, x)) \geq 0\}$. Let $Z_{\phi, t}$ be the projection of Z_ϕ on the t -axis and set $t_\phi = \inf_t Z_{\phi, t}$. Note that $t_\phi \geq t^*$. Thus for each $\phi \in M$, $[t_0, t^*] \subset [t_0, t_\phi]$ and consequently $[t_0, t^*] \subseteq \bigcap_{\phi \in M} [t_0, t_\phi] = [t_0, t_1]$. Clearly $\phi(m(t, x)) \leq 0$ on $[t_0, t_1] \times \bar{\Omega}$ and $\phi(m(t, x)) < 0$ on $[t_0, t_1) \times \bar{\Omega}$, for each $\phi \in K_0^*$.

We claim that there exists an $x_1 \in \bar{\Omega}$ such that $-m(t_1, x_1) \in \partial K$. For, if not, $-m(t_1, x_1) \in K^0$ for each $x_1 \in \bar{\Omega}$ and therefore $\phi(m(t_1, x_1)) < 0$ for every $x_1 \in \bar{\Omega}$ which contradicts the definition of t_1 . By assumption (ii), $x_1 \notin \partial\Omega$ and hence $-m(t_1, x_1) \in \partial K$ for some $x_1 \in \Omega$. By Mazur's theorem there exists a $\phi \in K_0^*$ such that $\phi(m(t_1, x_1)) = 0$ and necessarily $\phi \in M$. We have then that $\phi(m(t_1, x))$ attains an interior maximum at (t_1, x_1) . Thus $\phi(m_t(t_1, x_1)) \geq 0$. We also have that $\phi(m_{x_i}(t_1, x_1)) = 0$, $i = 1, 2, \dots, n$ and the quadratic form

$$\sum_{i,j=1}^n \lambda_i \lambda_j \phi(m_{x_i x_j}(t_1, x_1)) < 0$$

for arbitrary $\lambda \in R^N$. Hence by assumption $\phi(m_t(t_1, x_1)) < 0$. This contradiction shows that M is empty and the proof is complete.

II. MAIN RESULTS

In order to discuss the fundamental parabolic differential inequalities in cones, we need the following definitions relative to K . Let $f \in C[\bar{H} \times R^N \times R^{Nn} \times R^{Nn^2}, R^N]$. Then f is said to be elliptic at a

point (t, x) relative to K , if for any u, P, Q, R and $\phi \in K_0^*$, the quadratic form

$$\sum_{i,j=1}^n \lambda_i \lambda_j \phi(Q_{ij} - R_{ij}) \leq 0$$

for arbitrary $\lambda \in R^N$ implies

$$\phi(f(t, x, u, P, Q)) \leq \phi(f(t, x, u, P, R)).$$

If this property holds for each $(t, x) \in H$ then f is said to be elliptic relative to K in H .

A function f is said to be quasimonotone nondecreasing in u relative to K , if for any u, v, u_x, v_x, Q and $\phi \in K_0^*$ such that $u \leq v$, $\phi(u) = \phi(v)$, $\phi(u_{x_i}) = \phi(v_{x_i})$, $i = 1, 2, \dots, n$, we have

$$\phi(f(t, x, u, u_x, Q)) \leq \phi(f(t, x, v, v_x, Q)).$$

Theorem 2.1. Suppose that

- (i) $u, v \in C[\bar{H}, R^N]$ and the partial derivatives $u_t, u_x, u_{xx}, v_t, v_x, v_{xx}$ exist and are continuous in H ;
 (ii) $f \in C[\bar{H} \times R^N \times R^{Nn} \times R^{Nn^2}, R^N]$, f is elliptic and quasimonotone nondecreasing in u relative to K , and

$$\begin{aligned} u_t &\leq f(t, x, u, u_x, u_{xx}), \\ v_t &\geq f(t, x, v, v_x, v_{xx}) \quad \text{on } H; \end{aligned}$$

- (iii) $u(t, x) < v(t, x)$ on ∂H .

Then $u(t, x) < v(t, x)$ on $\bar{H}$ if one of the inequalities in (ii) are strict.

Proof. Suppose that one of the inequalities in (ii) are strict. Then we show that $m(t, x) = u(t, x) - v(t, x)$ satisfies the conditions of Lemma 1.1. Clearly the conditions (i) and (ii) of Lemma 1.1 are satisfied because of (i) and (iii) of Theorem 2.1. We therefore need to verify (iii) of Lemma 1.1. Let $(t_1, x_1) \in (t_0, \infty) \times \Omega$ and suppose that $\phi \in K_0^*$ so that $\phi(m(t_1, x_1)) = 0$, $m(t_1, x_1) \leq 0$, $\phi(m_{x_i}(t_1, x_1)) = 0$, $i = 1, 2, \dots, n$ and the quadratic form $\sum_{i,j=1}^n \lambda_i \lambda_j \phi(m_{x_i x_j}(t_1, x_1)) \leq 0$ for arbitrary $\lambda \in R^n$. This implies that $u(t_1, x_1) \leq v(t_1, x_1)$, $\phi(u(t_1, x_1)) = \phi(v(t_1, x_1))$,

$$\phi(u_{x_i}(t_1, x_1)) = \phi(v_{x_i}(t_1, x_1)), \quad i = 1, 2, \dots, n, \quad \text{and}$$

$$\sum_{i,j=1}^n \lambda_i \lambda_j \phi(u_{x_i x_j}(t_1, x_1) - v_{x_i x_j}(t_1, x_1)) \leq 0.$$

Since f is elliptic relative to K , we have

$$\begin{aligned} & \phi(f(t_1, x_1, u(t_1, x_1), u_x(t_1, x_1), u_{xx}(t_1, x_1))) \\ & \leq \phi(f(t_1, x_1, u(t_1, x_1), u_x(t_1, x_1), v_{xx}(t_1, x_1))). \end{aligned}$$

The quasimonotonicity of f shows that

$$\begin{aligned} & \phi(f(t_1, x_1, u(t_1, x_1), u_x(t_1, x_1), v_{xx}(t_1, x_1))) \\ & \leq \phi(f(t_1, x_1, v(t_1, x_1), v_x(t_1, x_1), v_{xx}(t_1, x_1))). \end{aligned}$$

$$\begin{aligned} \text{Thus } \phi(m_t(t_1, x_1)) & \leq \phi(u_t(t_1, x_1) - v_t(t_1, x_1)) \\ & < \phi(f(t_1, x_1, u(t_1, x_1), u_x(t_1, x_1), u_{xx}(t_1, x_1))) \\ & \quad - f(t_1, x_1, v(t_1, x_1), v_x(t_1, x_1), v_{xx}(t_1, x_1))) \\ & \leq 0. \end{aligned}$$

Hence Lemma 1.1 implies that $m(t, x) < 0$ on $\bar{H}$ and this proves the theorem.

We can dispense with the strict inequalities needed in Theorem 2.1 if f satisfies a one sided uniqueness condition. Since such a condition is also required later, we shall list it separately.

$$(H_1) \quad f(t, x, s_1, P, R) - f(t, x, s_2, P, R) \leq g(t, s_1 - s_2), \quad s_1 \geq s_2,$$

where $g \in C[[t, \infty) \times \mathbb{R}^N, \mathbb{R}^N]$, $g(t, 0) \equiv 0$, $g(t, u)$ is quasimonotone nondecreasing in u relative to K and the maximal solution of

$$r' = g(t, r), \quad r(t_0) = 0,$$

is identically zero on $[t_0, \infty)$.

Theorem 2.2. Assume that the conditions (i) and (ii) of Theorem 2.1 hold. Suppose that (H_1) is satisfied. Then $u(t, x) \leq v(t, x)$ on ∂H implies $u(t, x) \leq v(t, x)$ on $\bar{H}$.

Proof. Consider the interval $[t_0, T] \subset [0, \infty)$. Let $r(t, \varepsilon)$ be a solution of

$$r' = g(t, r) + \varepsilon, \quad r(t_0) = \varepsilon,$$

where $\varepsilon \in K^0$ and $||\varepsilon||$ is sufficiently small. The conditions on g imply that $r(t, \varepsilon) \in K^0$ on the interval $[t_0, T]$ and $\lim_{\varepsilon \rightarrow 0} r(t, \varepsilon) \equiv 0$ on $[t_0, T]$, see [1,3]. Hence by defining $z(t, x, \varepsilon) = v(t, x) + r(t, \varepsilon)$, we have $u(t, x) < z(t, x, \varepsilon)$ on $[t_0, T] \times \partial\Omega$. Furthermore, in view of (H_1) , we see that

$$\begin{aligned}
z_t(t, x, \varepsilon) &\geq f(t, x, v(t, x), v_x(t, x), v_{xx}(t, x)) + g(t, r(t, \varepsilon)) + \varepsilon \\
&> f(t, x, v(t, x) + r(t, \varepsilon), v_x(t, x), v_{xx}(t, x)) \\
&= f(t, x, z(t, x, \varepsilon), z_x(t, x, \varepsilon), z_{xx}(t, x, \varepsilon)) \quad \text{on } [t_0, T] \times \Omega.
\end{aligned}$$

An application of Theorem 2.1 shows that $u(t, x) < z(t, x, \varepsilon)$ on $[t_0, T] \times \bar{\Omega}$ and consequently $u(t, x) \leq v(t, x)$ on $[t_0, T] \times \bar{\Omega}$. Since T is arbitrary, the proof is complete.

Let us now consider the Dirichlet problem

$$(2.1) \quad u_t = f(t, x, u, u_x, u_{xx}),$$

$$(2.2) \quad u(t, x) = u_0(t, x) \quad \text{on } \partial H,$$

where $f \in C[H \times R^N \times R^{Nn} \times R^{Nn^2}, R^N]$, $u_x = \left(\frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_1}{\partial x_n}, \dots, \frac{\partial u_N}{\partial x_1}, \dots, \frac{\partial u_N}{\partial x_n} \right)$
and $u_{xx} = \left(\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 u}{\partial x_{n^2}}, \dots, \frac{\partial^2 u}{\partial x_1^2}, \dots, \frac{\partial^2 u}{\partial x_{n^2}} \right)$

Given an initial function $u_0(t, x)$ which is continuous on ∂H , a solution of (2.1), (2.2) is any function $u(t, x)$ satisfying the following properties:

- (i) $u(t, x)$ is continuous on $\bar{H}$;
- (ii) $u(t, x)$ possesses continuous partial derivatives u_t, u_x, u_{xx} in H and satisfies (2.1) in H ;
- (iii) $u(t, x) = u_0(t, x)$ on ∂H .

We shall assume that f is elliptic relative to K in H so that the system (2.1) is parabolic relative to K . We shall also assume that the problem (2.1), (2.2) has solutions.

A closed set $F \subset R^N$ is said to be flow-invariant relative to the system (2.1), (2.2) if for every solution $u(t, x)$ of (2.1), (2.2) we have

$$u_0(t, x) \in F \text{ on } \partial H \text{ implies } u(t, x) \in F \text{ on } \bar{H}.$$

To consider the invariance properties of solutions of (2.1), (2.2), we need, in addition to (H_1) , the following condition:

$$(H_2) \quad \text{for any } \phi \in K_0^*, \quad (t, x) \in \bar{H}, \quad \phi(f(t, x, u, u_x, 0)) \leq 0$$

provided $u \in \bar{Q}$, $\phi(u) = 0$ and $\phi(u_{x_i}) = 0$, $i = 1, 2, \dots, n$, where $Q = \{u \in R^N: u < 0\}$ and $\bar{Q}$ is the closure of Q .

Theorem 2.3. If the assumptions (H_1) and (H_2) hold, then the closed set $\bar{Q}$ is flow-invariant relative to system (2.1), (2.2).

Proof. Let $u(t, x)$ be any solution of (2.1), (2.2) such that $u_0(t, x) \in \bar{Q}$ on ∂H . Consider the interval $[t_0, T] \subset [t, \infty)$. Let $r(t, \epsilon)$ be a solution of $r' = g(t, r) + \epsilon$, $r(t_0) = \epsilon$, on $[t_0, T]$, where $\epsilon \in K^0$ such that $||\epsilon||$ is sufficiently small. Define $m(t, x) = u(t, x) - r(t, \epsilon)$ on $[t_0, T] \times \bar{\Omega}$. We wish to apply Lemma 1.1. Clearly (i) of Lemma 1.1 holds. Since the assumptions on g imply $r(t, \epsilon) \in K^0$ on $[t_0, T]$, $m(t, x) < u(t, x) \leq 0$ on $[t_0, T] \times \partial\Omega$, verifying (ii). Let $(t_1, x_1) \in (t_0, T] \times \Omega$ and $\phi \in K_0^*$ with $m(t_1, x_1) \leq 0$, $\phi(m(t_1, x_1)) = 0$, $\phi(m_{x_i}(t_1, x_1)) = 0$, $i = 1, 2, \dots, n$ and the quadratic form

$$\sum_{i,j=1}^n \lambda_i \lambda_j \phi(m_{x_i x_j}(t_1, x_1)) \leq 0,$$

for arbitrary $\lambda \in R^n$. Using successively, (H_1) , the ellipticity of f and (H_2) , we obtain

$$\begin{aligned} \phi(m_t(t_1, x_1)) &= \phi(f(t_1, x_1, u(t_1, x_1), u_x(t_1, x_1), u_{xx}(t_1, x_1)) - g(t, r(t, \epsilon))) \\ &\quad + \phi(-\epsilon) \\ &\leq \phi(f(t_1, x_1, m(t_1, x_1), m_x(t_1, x_1), m_{xx}(t_1, x_1))) + \phi(-\epsilon) \\ &\leq \phi(f(t_1, x_1, m(t_1, x_1), m_x(t_1, x_1), 0)) + \phi(-\epsilon) \\ &\leq \phi(-\epsilon) < 0, \end{aligned}$$

which checks the condition (iii). Hence Lemma 1.1 yields $m(t, x) < 0$ on $[t_0, T] \times \bar{\Omega}$. This implies that $u(t, x) \in Q$ on $[t_0, T] \times \bar{\Omega}$ because of the fact $\lim_{\epsilon \rightarrow 0} r(t, \epsilon) \equiv 0$ on $[t_0, T]$. Since $[t_0, T]$ is arbitrary the proof is complete.

In certain situations the flow-invariance of the closed set $\bar{W}$ where $W = \{u \in R^N: \alpha < u < b, \alpha, b \in R^N\}$ relative to system (2.1), (2.2) becomes important. This requires a modification of (H_2) , namely

$$\begin{aligned} (H_3) \quad \text{For any } \phi \in K_0^*, \quad (t, x) \in \bar{H}, \quad \phi(f(t, x, u, u_x, 0)) &\leq 0 \text{ if} \\ u \in W, \quad \phi(u) = \phi(b), \quad \phi(u_{x_i}) = 0, \quad i = 1, 2, \dots, n, \quad \text{and } \phi(f(t, x, u, u_x, 0)) & \\ \geq 0 \text{ if } u \in \bar{W}, \quad \phi(u) = \phi(\alpha), \quad \phi(u_{x_i}) = 0, \quad i = 1, 2, \dots, n. & \end{aligned}$$

The following Corollary can be proved by applying Theorem 2.3 to $u - b$ and $\alpha - u$ respectively relative to Q .

Corollary 2.1. If (H_1) and (H_3) hold, then the closed set $\bar{W}$ is flow-invariant relative to system (2.1), (2.2).

We shall next prove a comparison theorem for the solutions of (2.1), (2.2).

Theorem 2.4. Assume that

(i) $g_1, g_2 \in C[\bar{H} \times R^N, R^N]$, $g_1(t, r)$, $g_2(t, r)$ are quasimonotone nondecreasing in r relative to K and for $(t, x, u) \in H \times R^N$, $\phi \in K_0^*$,

$$\phi(g_2(t, u)) \leq \phi(f(t, x, u, u_x, 0)) \leq \phi(g_1(t, u)),$$

whenever $\phi(u_{x_i}) = 0$, $i = 1, 2, \dots, n$;

(ii) the condition (H_1) holds;

(iii) $r(t, t_0, r_{10})$, $\rho(t, t_0, \rho_{20})$ are the maximal and minimal solutions of $r' = g_1(t, r)$, $r(t_0) = r_{10}$, $\rho' = g_2(t, \rho)$, $\rho(t_0) = \rho_{20}$ respectively existing on $[t_0, \infty)$.

Then if $u(t, x)$ is any solution of (2.1), (2.2) such that

$$\rho(t, t_0, \rho_{20}) \leq u(t, x) \leq r(t, t_0, r_{10}) \quad \text{on } \partial H$$

we have

$$\rho(t, t_0, \rho_{20}) \leq u(t, x) \leq r(t, t_0, r_{10}) \quad \text{on } \bar{H}.$$

Proof. Consider the function $z(t, x) = u(t, x) - r(t, t_0, r_{10})$ so that

$$z_t(t, x) = F(t, x, z(t, x), z_x(t, x), z_{xx}(t, x))$$

where $F(t, x, z, z_x, z_{xx}) = f(t, x, z+r, u_x, u_{xx}) - g_1(t, r)$. We note that

$z(t, x) \leq 0$ on ∂H . We wish to apply Theorem 2.3. In view of (H_1) , we

get, if $z_1 \geq z_2$,

$$F(t, x, z_1, P, Q) - F(t, x, z_2, P, Q) \leq g(t, z_1 - z_2).$$

Moreover, for $\phi \in K_0^*$ if $z \leq 0$, $\phi(z) = 0$, and $\phi(z_{x_i}) = 0$, $i = 1, 2, \dots, n$, we obtain, using the quasimonotonicity of $g_1(t, r)$ and (H_2) ,

$$\begin{aligned} \phi(F(t, x, z, P, 0)) &= \phi(f(t, x, z+r, P, 0) - g_1(t, r)) \\ &\leq \phi(f(t, x, z+r, P, 0) - g_1(t, z+r)) \\ &= 0. \end{aligned}$$

Thus the function F satisfies (H_1) and (H_2) and consequently by Theorem 2.3, we have $z(t, x) \in \bar{Q}$ on $\bar{H}$. A similar argument holds for $z(t, x) = \rho(t, t_0, \rho_{20}) - u(t, x)$. The proof is therefore complete.

Corollary 2.2. If $\bar{W}$ is flow-invariant relative to system (2.1), (2.2), then there exist functions g_1, g_2 satisfying the assumption (1) of Theorem 2.4.

Proof. We can construct g_1, g_2 when $\bar{W}$ is flow-invariant as follows: for $\phi \in K_0^*$, we set

$$\begin{aligned} \phi(g_1(t, u)) &= \sup[\phi(f(t, x, u, u_x, 0)): x \in \bar{\Omega}, a \leq v \leq u \text{ with} \\ &\quad \phi(u) = \phi(v) \text{ and } \phi(u_{x_i}) = 0, i = 1, 2, \dots, n], \end{aligned}$$

and

$$\begin{aligned} \phi(g_2(t, u)) &= \inf[\phi(f(t, x, u, u_x, 0)): x \in \bar{\Omega}, u \leq v \leq b \text{ with} \\ &\quad \phi(u) = \phi(v) \text{ and } \phi(u_{x_i}) = 0, i = 1, 2, \dots, n]. \end{aligned}$$

It is clear that the functions $g_1(t, r), g_2(t, r)$ are quasimonotone

nondecreasing in u relative to K .

The idea of constructing a function with quasimonotone property when a given function does not have that property is described in [2] in a different situation. Also, the advantage of employing an arbitrary cone is suggested in [2] and is exploited in [4]. In this paper, these ideas are developed for parabolic systems.

III. APPLICATIONS

Let us consider a weakly coupled system of parabolic differential equations

$$(3.1) \quad u_t = A\Delta u + F(t, u),$$

in H with the Dirichlet data

$$(3.2) \quad u(t, x) = u_0(t, x) \quad \text{on } \partial H.$$

In (3.1), Δ denotes the Laplace operator in $x \in R^n$, $u, F, u_0 \in R^N$ and A is a diagonal matrix. The weak coupling of (3.1) suggests the choice of cone $K = R_+^N$. Thus the inequality $u \leq v$ implies the component-wise inequalities $u_i \leq v_i$, $i = 1, 2, \dots, N$.

Concerning the problem (3.1), (3.2), we can derive the following result.

Theorem 3.1. Assume that $F \in C[R_+ \times R^N, R^N]$ and $F(t, u)$ is Lipschitzian in u . Suppose that $A \geq 0$ and $u(t, x)$ is any solution of (3.1), (3.2). Then

- (a) if $u_i = 0$, $u_j \geq 0$, $i \neq j$, $j = 1, 2, \dots, N$ implies $F_i(t, u) \geq 0$, then $u(t, x) \geq 0$ on $\bar{H}$ provided $u(t, x) \geq 0$ on ∂H .
- (b) if $F(t, u)$ is quasimonotone nondecreasing in u , that is, for each i , $F_i(t, u)$ is nondecreasing in u_j , $j \neq i$, $j = 1, 2, \dots, N$ and if the solutions $r(t, t_0, r_{10})$, $\rho(t, t_0, \rho_{20})$ of $r' = F(t, r)$ with $r(t_0) = r_{10}$, $\rho(t_0) = \rho_{20}$ exist on $[t_0, \infty)$, then

$$\rho(t, t_0, \rho_{20}) \leq u(t, x) \leq r(t, t_0, r_{10}), \text{ on } \bar{H},$$

provided $\rho(t, t_0, \rho_{20}) \leq u_0(t, x) \leq r(t, t_0, r_{10})$ on ∂H ;

- (c) if, in addition, $F(t, 0) \equiv 0$ in (b), then

$0 \leq u(t, x) \leq r(t, t_0, r_{10})$ on H provided $0 \leq u_0(t, x) \leq r(t, t_0, r_{10})$ on ∂H ;

In particular, $0 < u_0(t, x) \leq r(t, t_0, r_{10})$ on ∂H implies ,

$$0 < u(t, x) \leq r(t, t_0, r_{10}) \text{ on } \bar{H}.$$

Proof. The assertion (a) follows from Theorem 2.3 with $\bar{Q} = R_+^N$. Theorem 2.4 yields (b) with $g_1 = g_2 = F$. Uniqueness of solutions of $r' = F(t, r)$ and the fact $F(t, 0) \equiv 0$ implies (c).

Suppose that $F(t, u)$ is not quasimonotone nondecreasing in u . If the closed set $\bar{W}$ (see Corollary 2.2) is flow-invariant for (3.1), (3.2), then there exist functions g_1, g_2 satisfying the assumption (i) of Theorem 2.4. In fact, for each i ,

$$g_{1i}(t, u) = \sup[F_i(t, u): a \leq u \leq v \text{ with } u_i = v_i],$$

$$g_{2i}(t, u) = \inf[F_i(t, u): u \leq v \leq b \text{ with } u_i = v_i].$$

Consequently, we can obtain upper and lower bounds as in Theorem 3.1 on the solutions of (3.1), (3.2) in terms of solutions of ordinary differential equations.

We also see from (c) that the stability properties of the trivial solution of $r' = g_1(t, r)$ imply the corresponding stability properties of the trivial solution of (3.1), (3.2).

The advantage of employing an appropriate cone K other than R_+^N in the study of parabolic differential inequalities is demonstrated in [5] where the Neumann problem is considered.

REFERENCES

- [1] Elsner, L., *Quasimonotonic und ungleichungen in halbgeordneten Raumen*, Linear Algebra Appl. 8 (1974), 249-261.
- [2] Lakshmikantham, V., *Vector Lyapunov functions*, Proce. Twelfth Annual Allerton Conf. Circuit and System Theory, (1974), 71-77.
- [3] Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol. I and II, Academic Press, New York, 1969.
- [4] _____, *Cone valued Lyapunov functions*, Nonlinear Analysis, Theory, Methods and Appl. 1 (1977), 215-2-2.
- [5] Lakshmikantham, V. and Vaughn, R., *Reaction-Diffusion Inequalities in Cones*, (to appear).