SEPARATION AND MONOTONICITY RESULTS FOR
THE ROOTS OF THE MOMENT PROBLEM

by

James Hallmark¹

and

Jerome Eisenfeld^{1,2}

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¹Department of Mathematics, University of Texas at Arlington, Arlington Texas, 76019.

²Department of Medical Computer Science, University of Texas Health Science Center at Dallas, Dallas, Texas 75235.

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I. INTRODUCTION

Consider the system identification problem

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) \quad 0 < t < T,$$

where $u(t)$ and $y(t)$ are given discretely on the interval $[0, T]$ and we wish to determine information about the unknown constant matrices A , B , and C .

Many techniques are available for approaching this problem. Several of these methods lead to what is known as the moment problem, [1]-[6].

This paper pertains to the moment problem which will now be introduced.

II. MOMENT PROBLEM

Assume we are given a moment sequence $\{s_k\}$, $k = 0, 1, 2, \dots$. Several previously mentioned methods used in solving the system identification problem will give us such a sequence. From the analysis involved in these methods, it can be shown that the terms in the moment sequence have the form

$$s_k = \sum_{i=1}^N \alpha_i \lambda_i^k \quad (2.1)$$

where $\alpha_i > 0$, λ_i are real and distinct, and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_N$. Thus the moment problem is: given a moment sequence $\{s_k\}$, where s_k has the form (2.1), determine N , α_i , λ_i , $i = 1, 2, \dots, N$. The values of $\lambda_1, \lambda_2, \dots, \lambda_N$ will be called the eigenvalues of the moment problem. Some useful results will now be stated about the moment problem.

Methods exist for determining N , the number of components, and are discussed in [3]. Assuming now that N is known, then the eigenvalues of the moment problem are the N real distinct roots of the equation

$$\det \left(G_N^{k+1} - \lambda G_N^k \right) = 0 \quad (2.2)$$

where G_N^k is the $N \times N$ matrix

$$G_N^k = \begin{pmatrix} s_k & s_{k+1} & \dots & s_{k+N-1} \\ s_{k+1} & & & \vdots \\ \vdots & & & \vdots \\ s_{k+N-1} & \dots & & s_{k+2N-2} \end{pmatrix}.$$

It may be shown that $\lambda_1, \lambda_2, \dots, \lambda_N$ are also the N roots of the N th degree polynomial

$$\det \begin{pmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^N \\ \theta_k & \theta_{k+1} & \theta_{k+2} & \dots & \theta_{k+n} \\ \vdots & & & & \vdots \\ \theta_{k+N-1} & & & \dots & \theta_{k+2N-1} \end{pmatrix} = 0. \quad (2.3)$$

Changes in k are referred to as *moment displacements*. This terminology is due to Isenberg [4].

Roots of equations (2.2) and (2.3) are independent of k . For a discussion of these results see [2]-[4].

Several problems are encountered when solving the moment problem. Because of insufficient data or numerical errors, the value of N may be difficult to compute. This is certainly the case if N is 'large'. Even if N is known, solutions of equations (2.2) and (2.3) may contain large numerical errors if N is large, say $N \geq 5$. The following question now arises. If we assume only n terms ($n = 1, 2, \dots, N$) in model (2.1) and then solve equation (2.2) or (2.3), how will these n roots (to be called subeigenvalues) relate to the actual eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$? The remainder of this paper will be directed at answering this question.

III. THE SUBEIGENVALUE PROBLEM

Consider the $n \times n$ matrices

$$G_n^k = \begin{pmatrix} s_k & s_{k+1} & \dots & s_{k+n-1} \\ s_{k+1} & & & \vdots \\ \vdots & & & \vdots \\ s_{k+n-1} & \dots & s_{k+2n-2} \end{pmatrix} \quad (3.1)$$

where $n = 1, 2, 3, \dots, N$ and $k = 0, 1, 2, \dots$.

Let $\lambda_{n,i}^k$ ($i = 1, 2, \dots, n$) be the n roots of the equation

$$\det \left(G_n^{k+1} - \lambda G_n^k \right) = 0. \quad (3.2)$$

with $\lambda_{n,1}^k \leq \lambda_{n,2}^k \leq \dots \leq \lambda_{n,n}^k$. These values of $\lambda_{n,i}^k$ will be referred to as the subeigenvalues of the moment problem. As before, an equivalent method to determine these subeigenvalues is to determine the n roots of the polynomial

$$\det \begin{pmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^n \\ s_k & s_{k+1} & s_{k+2} & \dots & s_{k+n} \\ \vdots & & & & \vdots \\ s_{k+n-1} & & & \dots & s_{k+2n-1} \end{pmatrix} = 0. \quad (3.3)$$

These are several ways to represent $\lambda_{n,i}^k$. Two such representations, which will be of help later, are:

$$\lambda_{n,i}^k = \max_{y_j \in R^n} \min_{\substack{(x, y_j)=0 \\ x \neq 0}} R_n^k(x) \quad j = 1, 2, \dots, i-1 \quad (3.4)$$

$$= \min_{y_j \in R^n} \max_{\substack{(x, y_j)=0 \\ x \neq 0}} R_n^k(x) \quad j = 1, 2, \dots, n-i \quad (3.5)$$

where (x, y_j) is the inner product in R^n of x with y_j and

$$R_n^k(x) = \frac{\left(G_n^{k+1} x, x \right)}{\left(G_n^k x, x \right)} = \frac{x^T G_n^{k+1} x}{x^T G_n^k x}.$$

Henceforth, when $i = 1$ equation (3.4) will become

$$\lambda_{n,1}^k = \min_{\substack{x \in R^n \\ x \neq 0}} R_n^k(x)$$

and when $i = n$, equation (3.5) will be replaced by

$$\lambda_{n,n}^k = \max_{\substack{x \in R^n \\ x \neq 0}} R_n^k(x).$$

For verification of the above results, see [5].

Using (3.1) we have

$$\left(G_n^k x, x \right) = \sum_{i,j=1}^n s_{k+i+j-2} x_i x_j \quad (3.6)$$

where $x = (x_1, x_2, \dots, x_n)^T$. Since s_k has the form (2.1), we have

$$\left(G_n^k x, x \right) = \sum_{i=1}^N \alpha_i \lambda_i^k \gamma_i(x) \quad (3.7)$$

where $\gamma_i(x) = \left(\sum_{j=1}^n \lambda_i^{j-1} x_j \right)^2$.

We now have

$$R_n^k(x) = \frac{\left(G_n^{k+1} x, x\right)}{\left(G_n^k x, x\right)} = \frac{\sum_{i=1}^N \alpha_i \lambda_i^k \gamma_i(x) \lambda_i}{\sum_{i=1}^N \alpha_i \lambda_i^k \gamma_i(x)}. \quad (3.8)$$

Note that $\alpha_i \lambda_i^k \gamma_i(x) \geq 0$, so that (3.8) is a weighted center of mass of $\lambda_1, \lambda_2, \dots, \lambda_N$ with λ_i having weight $\alpha_i \lambda_i^k \gamma_i(x)$. Since $\lambda_1 < \lambda_2 < \dots < \lambda_N$, it follows from (3.8) that

$$\lambda_1 \leq R_n^k(x) \leq \lambda_N \quad (3.9)$$

for $n = 1, 2, \dots, N$. Combining (3.4) and (3.9), we have the following result:

Theorem 1. Let $\lambda_{n,i}^k$ be the n roots of (3.3) with $\lambda_{n,1}^k \leq \lambda_{n,2}^k \leq \dots \leq \lambda_{n,n}^k$. Then

$$\lambda_1 \leq \lambda_{n,i}^k \leq \lambda_N$$

for $n = 1, 2, \dots, N$ and $k = 0, 1, 2, \dots$.

We will now prove a separation result about the subeigenvalues.

Theorem 2. As n increases, $n = 2, 3, \dots, N$, the subeigenvalues of the moment problem interlace for a fixed moment displacement k . That is

$$\lambda_{n,i}^k \leq \lambda_{n-1,i}^k \leq \lambda_{n,i+1}^k \quad \text{for } i = 1, 2, \dots, n-1$$

Proof. First it will be shown that

$$\lambda_{n,i}^k \leq \lambda_{n-1,i}^k \quad \text{for fixed } i > 1. \quad (3.10)$$

Let s^n = any set of $i-1$ vectors from R^n

S^n = collection of all sets s^n

$T^n = \{x \in R^n \mid (x, y) = 0 \text{ for all } y \in s^n \text{ and } x \neq 0\}$ for a given s^n

$\hat{T}^n = \{x \in T^n \mid \text{nth component } x = 0\}$

Using this notation and (3.4) we have the following:

$$\lambda_{n,i}^k = \max_{s^n \in S^n} \min_{x \in T^n} R_n^k(x) \quad (3.11)$$

Since $\hat{T}^n \subset T^n$, we have

$$\max_{s^n \in S^n} \min_{x \in T^n} R_n^k(x) \leq \max_{s^n \in S^n} \min_{x \in \hat{T}^n} R_n^k(x).$$

Thus to prove (3.10), it suffices to show that

$$\max_{s^n \in S^n} \min_{x \in \hat{T}^n} R_n^k(x) \leq \max_{s^{n-1} \in S^{n-1}} \min_{x \in \hat{T}^{n-1}} R_{n-1}^k(x). \quad (3.12)$$

Now (3.12) will be true if it can be shown that for each $s^n \in S^n$ there exists $s^{n-1} \in S^{n-1}$ such that $\{R_n^k(z) \mid z \in \hat{T}^n\} = \{R_{n-1}^k(w) \mid w \in \hat{T}^{n-1}\}$ where $\hat{T}^n$ is determined by s^n and $\hat{T}^{n-1}$ is determined by s^{n-1} . Let $s^n \in S^n$ be fixed. Define the map $\phi: \hat{R}^n \rightarrow R^{n-1}$ by $\phi(x) = x^*$ where $x = (x_1, x_2, \dots, x_{n-1}, 0)$ and $x^* = (x_1, x_2, \dots, x_{n-1})$ and $\hat{R}^n = \{x \in R^n \mid \text{nth component of } x = 0\}$. Let $s^{n-1} = \phi(s^n) \in S^{n-1}$. Then

for all $z \in \hat{T}^n$, $\phi(z) = w \in T^{n-1}$ and furthermore $R_n^k(z) = R_{n-1}^k(w)$.
 Also for all $w \in T^{n-1}$, $\phi^{-1}(w) = z \in \hat{T}^n$ and again $R_n^k(z) = R_{n-1}^k(w)$.

Thus

$$\{R_n^k(z) | z \in \hat{T}^n\} = \{R_{n-1}^k(w) | w \in T^{n-1}\} \text{ and (3.12) is true.}$$

Thus it has been shown that $\lambda_{n,i}^k \leq \lambda_{n-1,i}^k$. If we use representation (3.5) for $\lambda_{n,i}^k$, an argument similar to the one above yields

$$\lambda_{n-1,i}^k \leq \lambda_{n,i+1}^k. \quad (3.13)$$

Combining (3.10) and (3.13), the proof is complete.

The remaining results have to do with moment displacements.

Theorem 3. As the moment displacements increase, so do the subeigenvalues.

That is; for fixed n ($n = 1, 2, \dots, N$) and for fixed i ($i = 1, 2, \dots, n$)

$$\lambda_{n,i}^k \leq \lambda_{n,i}^{k+1} \text{ for } k = 0, 1, 2, \dots.$$

Proof. Using (3.7) and letting $\beta_i(x) = \alpha_i \gamma_i(x)$, we have

$$(G_n^k x, x) = \sum_{i=1}^N \beta_i(x) \lambda_i^k. \quad (3.14)$$

That is $\{(G_n^k x, x)\}$ is a moment sequence for fixed n . Observe that the positive definiteness of G_n^k implies that $(G_n^k x, x) > 0$ for $x \neq 0$. Thus the moment sequence (3.14) is positive for $x \neq 0$. It follows that the 2×2 Hankel matrices formed by $\sigma_k = (G_n^k x, x)$ are positive definite [5] and hence

$$\begin{vmatrix} \sigma_k & \sigma_{k+1} \\ \sigma_{k+1} & \sigma_{k+2} \end{vmatrix} > 0. \quad (3.15)$$

This immediately yields

$$\frac{\left(G_n^{k+1} x, x \right)}{\left(G_n^k x, x \right)} \leq \frac{\left(G_n^{k+2} x, x \right)}{\left(G_n^{k+1} x, x \right)} \quad (3.16)$$

for $x \in R^n$, $x \neq 0$ and for $n = 1, 2, \dots, N$, $k = 0, 1, 2, \dots$.

Using (3.4), we now have the required result.

Theorems 1 and 3 imply that for fixed n and i , $\lambda_{n,i}^k$ converges as k increases. The next two theorems will be concerned with the convergence of the subeigenvalues.

From (3.1), $G_1^k = s_k$ and therefore solving equation (3.2), with $n = 1$, we have $\lambda_{1,1}^k = s_{k+1}/s_k$. Applying (2.1) we have

$$\lambda_{1,1}^k = \frac{\sum_{i=1}^N \alpha_i \lambda_i^{k+1}}{\sum_{i=1}^N \alpha_i \lambda_i^k} = \frac{\sum_{i=1}^N \alpha_i \left(\frac{\lambda_i}{\lambda_N} \right)^k \lambda_i}{\sum_{i=1}^N \alpha_i \left(\frac{\lambda_i}{\lambda_N} \right)^k}.$$

Letting k increase and recalling that $\lambda_1 < \lambda_2 < \dots < \lambda_N$, we have

$$\lambda_{1,1}^k \rightarrow \lambda_N \text{ as } k \text{ increases.} \quad (3.17)$$

Theorem 4. Assuming n ($n = 1, 2, \dots, N$) terms in model (2.1), then the largest of the n subeigenvalues derived from (3.3) converges to the

largest eigenvalue λ_N as the moment displacement k increases. That is,

$$\lambda_{n,n}^k \rightarrow \lambda_N \text{ as } k \rightarrow +\infty \text{ for } n = 1, 2, \dots, N.$$

Proof. Using theorem 2 we have $\lambda_{1,1}^k \leq \lambda_{n,n}^k \leq \lambda_N$. Combining this with (3.17), the proof is complete.

Theorem 5. Assuming n ($n = 1, 2, \dots, N$) terms in model (2.1), then the next to largest subeigenvalue derived from (3.3) converges to the next to largest eigenvalue λ_{N-1} as the moment displacement k increases. That is,

$$\lambda_{n,n-1}^k \rightarrow \lambda_{N-1} \text{ as } k \rightarrow +\infty \text{ for } n = 2, 3, \dots, N$$

Proof. We first show that $\lambda_{2,1}^k \rightarrow \lambda_{N-1}$. Consider the two roots of $\det \left[G_2^{k+1} - \lambda G_2^k \right] = 0$, denoted by $\lambda_{2,1}^k$ and $\lambda_{2,2}^k$. Using (3.3) we have $\lambda_{2,1}^k$ and $\lambda_{2,2}^k$ are the two roots of

$$\lambda^2 \left[s_k s_{k+2} - s_{k+1}^2 \right] - \lambda \left[s_k s_{k+3} - s_{k+1} s_{k+2} \right] + \left[s_{k+1} s_{k+3} - s_{k+2}^2 \right] = 0.$$

The sum of the two roots of this equation must equal $\frac{(s_k s_{k+3} - s_{k+1} s_{k+2})}{(s_k s_{k+2} - s_{k+1}^2)}$. Using (2.1), we have

$$\lambda_{2,1}^k + \lambda_{2,2}^k = \frac{\sum_{i,j=1}^N \alpha_i \alpha_j (\lambda_j - \lambda_i) \lambda_i^k \lambda_j^{k+2}}{\sum_{i,j=1}^N \alpha_i \alpha_j (\lambda_i - \lambda_j) \lambda_i^k \lambda_j^{k+1}}$$

Dividing each term by λ_{N-1} and letting k increase we have

$$\lim_{k \rightarrow +\infty} \left(\lambda_{2,1}^k + \lambda_{2,2}^k \right) = \lambda_N + \lambda_{N-1}$$

Using Theorem 4, we then have

$$\lim_{k \rightarrow +\infty} \lambda_{2,1}^k = \lambda_{N-1}. \quad (3.18)$$

Now applying the separation result in Theorem 2 yields

$$\lambda_{2,1}^k \leq \lambda_{n,n-1}^k \leq \lambda_{N-1}. \quad (3.19)$$

(3.18) and (3.19) yield the required result.

In conclusion, we have shown that assuming n terms in model (2.1) and generating n subeigenvalues $\lambda_{n,1}^k, \lambda_{n,2}^k, \dots, \lambda_{n,n}^k$, the two largest subeigenvalues converge to the two largest eigenvalues λ_N and λ_{N-1} as the moment displacement increases. We hope that future work will generalize these results to show that these n subeigenvalues converge to the n largest eigenvalues of (2.1). The results obtained here have been applied in the analysis of fluorescence decay data, as can be seen in [6]. Numerical examples that illustrate these results will now be presented.

IV. NUMERICAL EXAMPLE

Consider the model (2.1) with three components.

Let the eigenvalues, λ_i , be .5, 2.0, and 3.0 with respective amplitudes, α_i , of 2.0, 1.0, and 2.0. In this example we will assume the above model to have only 1 and 2 terms and generate subeigenvalues under these assumptions.

Assume $n = 1$

<i>moment displacement</i>	$\lambda_{1,1}^k$
$k = 0$	1.8000
$k = 1$	2.5000
$k = 2$	2.7667
$k = 3$	2.8614
$k = 4$	2.9084
$k = 5$	2.9379
$k = 6$	2.9579
$k = 7$	2.9716
$k = 8$	2.9809
$k = 9$	2.9872
$k = 10$	2.9914
$k = 11$	2.9943
$k = 12$	2.9962

Assume $n = 2$

<i>moment displacement</i>	$\lambda_{2,1}^k$	$\lambda_{2,2}^k$
$k = 0$	.6013	2.8511
$k = 1$	.8468	2.9032
$k = 2$	1.3304	2.9492
$k = 3$	1.7527	2.9827
$k = 4$	1.9299	2.9962
$k = 5$	1.9817	2.9996
$k = 6$	1.9954	3.0000

For additional examples, see [6].

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