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**MOLECULAR STUDY OF TURBULENCE IN
THREE DIMENSIONAL CAVITY FLOW**

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MOLECULAR STUDY OF TURBULENCE IN THREE DIMENSIONAL CAVITY FLOW

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Abstract

Three dimensional molecular cavity problems are formulated and solved numerically. The fluid considered is water at 15°C . The turbulent flows generated are characterized by strong crosscurrents over the usual primary vortex direction.

I - Mathematical, Physical, and Computational Considerations

1.1. The N-Body Problem

The fundamental mathematical problem considered is a noncontinuum problem called the *N-body problem*, which is described in complete generality as follows. In cgs units and for $i = 1, 2, \dots, N$, let P_i of mass m_i be at $\vec{r}_i = (x_i, y_i, z_i)$, have velocity $\vec{v}_i = (v_{i,x}, v_{i,y}, v_{i,z})$, and have acceleration $\vec{a}_i = (a_{i,x}, a_{i,y}, a_{i,z})$ at time $t \geq 0$. Let the positive distance between P_i and P_j , $i \neq j$, be $r_{ij} = r_{ji} \neq 0$. Let the force on P_i due to P_j be $\vec{F}_{ij} = \vec{F}_{ij}(r_{ij})$, so that the force depends only on the *distance* between P_i and P_j . Also, assume that the force $\vec{F}_{ji}$ on P_j due to P_i satisfies $\vec{F}_{ji} = -\vec{F}_{ij}$. Then, given the initial positions and velocities of all the P_i , $i = 1, 2, 3, \dots, N$, the general *N*-body problem is to determine the motion of the system if each P_i interacts with all the other P_j 's in the system.

1.2 A Classical Molecular Potential for Water.

Since our attention will be directed to water, let us consider an approximate Lennard-Jones potential for water molecules, that is,

$$(1.2) \quad \phi(r_{ij}) = (1.9646)10^{-13} \left[\frac{2.725^{12}}{r_{ij}^{12}} - \frac{2.725^6}{r_{ij}^6} \right] \text{ erg} \quad \left(\frac{\text{gcm}^2}{\text{sec}^2} \right)$$

in which r_{ij} is measured in angstroms ($\text{\AA}$). The force $\vec{F}_{ij}$ exerted on P_i by P_j is then

$$(1.3) \quad \vec{F}_{ij} = (1.9646)10^{-5} \left[\frac{12(2.725^{12})}{r_{ij}^{13}} - \frac{6(2.725^6)}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}} \text{ dynes} \quad \left(\frac{\text{gcm}}{\text{sec}^2} \right).$$

Note that in deriving (1.3) from (1.2), one must use the chain rule

$$F_{ij} = - \frac{d\phi(r_{ij})}{dR} = - \frac{d\phi(r_{ij})}{dr_{ij}} \frac{dr_{ij}}{dR}$$

and the fact that $R \text{ cm} = 10^8 R \text{\AA}$, so that, $r_{ij} = 10^8 R$.

Note also that $F_{ij} = \|\vec{F}_{ij}\| = 0$ implies that $r_{ij} = 3.06 \text{\AA}$, which is called the *equilibrium distance*.

1.3 Molecular Mechanics

Molecular mechanics is the simulation of molecular interaction as an N-body problem using classical molecular potentials and Newtonian mechanics. Most fluid studies using molecular mechanics are concerned with the *physics* of fluids (see, e.g., Koplik and Banavar (1998)) and are concerned with flows at low Reynolds numbers. The related thermodynamic considerations are not applicable to the study of turbulence and so our only concern will be with temperature. Recall then that the temperature T on the Kelvin scale of a molecule of mass m gr and speed v cm/sec is given in three dimensions by

$$(1.4) \quad \frac{3}{2} kT = \frac{1}{2} m v^2,$$

in which k is the Boltzman constant $(1.381)10^{-16}$ erg deg $^{-1}$. Also recall that T degrees Kelvin and C degrees centigrade are related by

$$(1.5) \quad T = C + 273.$$

1.4 Equations of Motion for Water Molecules.

From the discussion in Section 1.2, it follows that the equation of motion for a single water molecule P_i acted on by a single water molecule P_j , $i \neq j$, is

$$(1.6) \quad m_i \vec{a}_i = (1.9646)10^{-5} \left[\frac{12(2.725^{12})}{r_{ij}^{13}} - \frac{6(2.725^6)}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}}.$$

Since the mass of a water molecule is $(30.103)10^{-24}$ gr, the latter equation is equivalent to

$$\vec{a}_i = (160.33)10^{19} \left[\frac{818.90}{r_{ij}^{13}} - \frac{1}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}} \quad \left(\frac{\text{cm}}{\text{sec}^2} \right).$$

Recasting the latter equation in $\text{\AA}/(\text{ps}^2)$ yields, since $1 \text{ sec} = 10^{12} \text{ ps}$,

$$(1.7) \quad \vec{a}_i = (160330) \left[\frac{818.90}{r_{ij}^{13}} - \frac{1}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}} \quad \left(\frac{\text{\AA}}{\text{ps}^2} \right).$$

On the molecular level, however, the effective force on P_i is local, that is, it is determined only by close molecules, by which we mean only those molecules within a distance D determined by

$$\frac{dF_{ij}}{dr_{ij}} = 0.$$

The solution to this equation yields $r_{ij} = 3.39 \text{ \AA}$. Thus, for $r_{ij} \geq D = 3.39 \text{ \AA}$, we choose $\vec{F}_{ij} = \vec{0}$. From (1.7), then, the dynamical equation for water molecule P_i will be

$$(1.8) \quad \frac{d^2 \vec{r}_i}{dt^2} = (160330) \sum_{\substack{j \\ j \neq i}} \left[\frac{818.90}{r_{ij}^{13}} - \frac{1}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}} ; r_{ij} < D.$$

The equations of motion for a system of water molecules are then

$$(1.9) \quad \frac{d^2 \vec{r}_i}{dt^2} = (160330) \sum_{\substack{j \\ j \neq i}} \left[\frac{818.90}{r_{ij}^{13}} - \frac{1}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}} ; i = 1, 2, 3, \dots, N; r_{ij} < D.$$

Note that on the molecular level gravity can be neglected since its effect is negligible because $980 \text{ cm/sec}^2 = (980)10^{-16} \text{ \AA/ps}^2$. Note also that in simulating the motion of a system of molecules, nonlinear equations (1.9) forewarn us that we will probably have to solve *large* systems of nonlinear, second order, ordinary differential equations and that these will have to be solved numerically.

1.5 The Leap Frog Formulas

Classical molecular force formulas require small time steps in any numerical simulation in order to yield physically correct results for the effect of repulsion, which is unbounded when the distance between the molecules is close to zero. Because we are restricted physically to small time steps and because the number of equations is usually exceptionally large, Runge-Kutta and Taylor expansion methods, for example, prove to be most unwieldy for related problems. Hence, in this section we will describe a simplistic, efficient, but low order method, called the Leap Frog Method, for molecular mechanics simulations and it is given as follows.

Choose a positive time step h and let $t_k = kh$, $k = 0, 1, 2, \dots$. For $i = 1, 2, 3, \dots, N$, let P_i have mass m_i and at t_k let it be at $\vec{r}_{i,k} = (x_{i,k}, y_{i,k}, z_{i,k})$, have velocity $\vec{v}_{i,k} = (v_{i,k,x}, v_{i,k,y}, v_{i,k,z})$, and have acceleration $\vec{a}_{i,k} = (a_{i,k,x}, a_{i,k,y}, a_{i,k,z})$. The leap frog formulas, which relate position, velocity and acceleration are

$$(1.10) \quad \frac{\vec{v}_{i,\frac{1}{2}} - \vec{v}_{i,0}}{\frac{1}{2}h} = \vec{a}_{i,0} \quad (\text{Starter})$$

$$(1.11) \quad \frac{\vec{v}_{i,k+\frac{1}{2}} - \vec{v}_{i,k-\frac{1}{2}}}{h} = \vec{a}_{i,k}, \quad k = 1, 2, 3, \dots$$

$$(1.12) \quad \frac{\vec{r}_{i,k+1} - \vec{r}_{i,k}}{h} = \vec{v}_{i,k+\frac{1}{2}}, \quad k = 0, 1, 2, 3, \dots,$$

or, explicitly,

$$(1.13) \quad \vec{v}_{i,\frac{1}{2}} = \vec{v}_{i,0} + \frac{1}{2}h\vec{a}_{i,0}, \quad (\text{Starter})$$

$$(1.14) \quad \vec{v}_{i,k+\frac{1}{2}} = \vec{v}_{i,k-\frac{1}{2}} + h\vec{a}_{i,k}, \quad k = 1, 2, 3, \dots$$

$$(1.15) \quad \vec{r}_{i,k+1} = \vec{r}_{i,k} + h\vec{v}_{i,k+\frac{1}{2}}, \quad k = 0, 1, 2, 3, \dots$$

Note that (1.11) and (1.12) are two point central difference formulas. The name *leap frog* derives from the way position and velocity are defined at alternate, sequential time values. The r values are defined at the times $t_0, t_1, t_2, t_3, \dots$, while the v values are defined at the times $t_{\frac{1}{2}}, t_{\frac{3}{2}}, t_{\frac{5}{2}}, t_{\frac{7}{2}}, \dots$.

1.6 Turbulence.

Since turbulence is the most commonly observed form of fluid behavior, interest has been exceptional in the modelling of related phenomena. Beginning with Taylor's seminal paper (Taylor (1921)), one school of study (Favre (1964)) has emphasized the statistical approach. A second major school of thought, following fundamental papers of Landau (1944), and Ruelle and Takens (1971) uses Galerkin approximations to simplify the Navier-Stokes equations and bifurcation theory to analyze the resulting ordinary differential system (Barenblatt, Iooss, and Joseph (1983)). A more recent computer approach is to solve the full Navier-Stokes equations numerically, assuming that these equations represent turbulent flow in some average sense (Markatos (1986)). Unfortunately, important and realistic aspects of turbulent motions have defied inclusion in all the models just described (Barenblatt, Iooss, and Joseph (1983), Markatos (1986), Zabusky (1968)). Thus, for example, whereas homogeneous turbulence has received intensive theoretical study, it is not known to exist anywhere in Nature.

Our attention will be directed in three dimensions to a prototype fluid flow problem called the *cavity problem*. For such problems there are two significant markers for turbulent flows. First, turbulent flow follows when a strong crosscurrent develops and imposes itself on a laminar flow (Schlichting (1960)). Second, turbulent flow exhibits the rapid appearance and disappearance of many small vortices (Kolmogorov (1964)). Both these markers are observed in the large. However, since we will work on the molecular level, we *may* find somewhat different results. Indeed, researchers in both nanophysics and nanotechnology are finding that this is exactly the case.

II - MOLECULAR CAVITY FLOW OF WATER IN THREE DIMENSIONS

2.1 Introduction

To simulate the flow of molecular water in three dimensions, vector equations (1.9) remain valid and for convenience are recalled now as

$$(2.1) \quad \frac{d^2 \vec{r}_i}{dt^2} = (160330) \sum_{\substack{j \\ j \neq i}} \left[\frac{818.90}{r_{ij}^{13}} - \frac{1}{r_{ij}^7} \right] \frac{\vec{r}_{ji}}{r_{ij}}; \quad i = 1, 2, 3, \dots, N; \quad r_{ij} < D.$$

Recall also that associated with (2.1) are the equilibrium distance $3.06 \text{ \AA}$ and the local interaction distance $D = 3.39 \text{ \AA}$.

We will formulate a three dimensional cavity problem and will then find that the numerical solution of the dynamical equations can be exceptionally arduous. The computations to be described were executed on a Digital Alpha 533. However, supercomputer simulation is really in order.

It is most important to note that forces exist in three dimensional cavity flow which are not present in two dimensions (Roache (1972)).

2.2 Molecular Arrangement and the Cavity Problem

Using the equilibrium value $3.06 \text{ \AA}$, we first construct a cubic grid of points as follows. Consider the three dimensional cube with

$$-52.02 \leq x \leq 52.02, \quad 0 \leq y \leq 104.04, \quad 0 \leq z \leq 104.04,$$

as shown in Figure 2.1. The edge of this cube has length $104.04 \text{ \AA}$ and it is symmetrical about the YZ plane, that is, about the plane $x = 0$. For later discussion, note that the length OA is called the span, the length CG is called the height, and the length BC is called the width. In Figure 2.1, all three have the same length, namely, $104.04 \text{ \AA}$.

In this cube a three dimensional cubic grid is constructed with $\Delta x = \Delta y = \Delta z = 3.06 \text{ \AA}$ as follows:

$$\begin{aligned} x(1) &= -52.02, \quad y(1) = 0.0, \quad z(1) = 0.0 \\ x(i+1) &= x(i) + 3.06, \quad y(i) = 0.0, \quad z(i) = 0.0, \quad i = 1, 34 \\ x(36) &= 0.0, \quad y(36) = 3.06, \quad z(36) = 0.0 \end{aligned}$$

Fig. 2.1 The cubic basin.

$$\begin{aligned}
x(i+1) &= x(i) + 3.06, y(i+1) = 3.06, z(i+1) = 0.0, i = 36, 69 \\
x(i) &= x(i-70), y(i) = y(i-70) + 6.12, z(i) = 0.0, i = 71, 1225 \\
x(i) &= x(i-1225), y(i) = y(i-1225), z(i) = 3.06, i = 1226, 2450 \\
x(i) &= x(i-2450), y(i) = y(i-2450), z(i) = z(i-2450) + 6.12, i = 2451, 42872.
\end{aligned}$$

There result 42875 interior and boundary grid points of which 1225 lie in the YZ plane. Renumber the points in the YZ plane to be

$$x(i), y(i), z(i); \quad i = 20826, 22050$$

Renumber the remaining points so that

$$x(i) = -x(42876 - i), y(i) = y(42876 - i), z(i) = z(42876 - i), i = 22051, 42875.$$

At grid point numbered i , set a molecule P_i ; $i = 1, 42875$. To complete the initial data for the molecules, note that at 15°C the speed in $\text{\AA}/\text{ps}$ for a water molecule in three dimensions, from (1.4), is $6.3 \text{ \AA}/\text{ps}$. This speed and its sign are assigned to each molecule in the X , Y , or Z direction, at random, but with the reasonable restrictions (Freitas, Street, Findikakis, and Koseff (1985)) that if P_i and P_j are not in the YZ plane, but are symmetrical relative to the YZ plane, then

$$v_x(P_i) = -v_x(P_j), v_y(P_i) = v_y(P_j), v_z(P_i) = v_z(P_j),$$

while if P_i is in the YZ plane then $v_x(P_i) = 0$. These assumptions are allowable because the cavity problem to be described yields, experimentally, a flow which is symmetrical about the YZ plane. In this fashion, through the use of symmetry, the dynamics reduces to a 22050-body problem.

For the 42875 water molecules with dynamical equations (2.1) and the initial data described above, we now describe a first three dimensional cavity, or, basin, problem to be studied. The top face of the cube is the only face allowed to move. It moves in the Y direction and is allowed an extended length so that the fluid is always enclosed by six faces. The constant speed V of the top face is called the wallspeed. Our problems are to describe the gross fluid motion within the cube for various values of V , and, of course, to produce turbulence.

Observe also that the connection between wallspeed V and Reynolds number Re is given (Pan and Acrivos (1967)) by $Re = |V|B/\nu$, in which B is the span and ν is the average kinematic viscosity for water. Thus, for $V = 25 \text{ \AA}/\text{ps}$, $B = 104.04 \text{ \AA}$, and $\nu = 113.12 \text{ \AA}^2/\text{ps}$ at 15°C (Streeter (1962)), then $Re = 23$.

2.3 Computational Considerations

In applying the leap frog formulas, we must first describe a protocol to apply when a molecule has crossed a face into the exterior of the cavity. For each of the lower five faces, the position will be reflected back symmetrically relative to that face into the interior of the cavity, the velocity components tangent to the face will be set to zero, and the velocity component perpendicular to the face will be multiplied by -1 . If the molecule crosses then moving face, then its position will be reflected back symmetrically, its x component of velocity will be set to zero, its z component of velocity will be multiplied by -1 , and its y component of velocity will be increased by the wall speed V .

Next, since molecular motion is Brownian, we introduce average velocities $\vec{v}_{i,k,J}$ as follows. For J a positive integer, let molecule P_i be at $(x(i,k), y(i,k), z(i,k))$ at $t_k = k\Delta t$ and at $(x(i,k-J), y(i,k-J), z(i,k-J))$ at $t_{k-J} = (k-J)\Delta t$. Then, the average velocity $\vec{v}_{i,k,J}$ of P_i at t_k is defined by

$$(2.2) \quad \vec{v}_{i,k,J} = \left(\frac{x(i,k) - x(i,k-J)}{J\Delta t}, \frac{y(i,k) - y(i,k-J)}{J\Delta t}, \frac{z(i,k) - z(i,k-J)}{J\Delta t} \right).$$

2.4 Examples

It will be important first to understand the nature of nonturbulent flow. Consider then $V = 25$, $\Delta t = 0.00008$, and $J = 25200$. Figures 2.2-2.6 show vortex development for $x = 0$, that is, in the YZ plane at the respective times $t = 2.90, 3.35, 4.02, 7.04, 10.07$. Figure 2.2 reveals the development of a compression wave moving down and to the left. This results in compression and molecular repulsion upwards, as is apparent in the upper left sections of Figures 2.3 and 2.4. Continuation of this motion results in the primary vortex flows shown in Figure 2.5 and 2.6. For graphical clarity, the velocities are reduced by the factor $vm=0.51$. Calculations using $J = 19600, 22400, 28000, 30800$ yielded results entirely similar to those displayed in Figures 2.2-2.6.

Figure 2.7 shows a direct, two dimensional simulation of the flow in the YZ plane for wall speed 25, $J = 25200$, and $\Delta t = 0.00008$ at $T = 7.04$. The initial data are exactly those for the molecules in the YZ plane in the three dimensional example described above. The flow is analyzed using the very same parameters as those used to display Figure 2.5. Comparison of the results in Figure 2.5 with those in Figure 2.7 show that there is a distinct difference in the two dimensional and three dimensional simulations. The average speed of the molecules in Figure 2.7 is $4.3574 \text{ } \overset{\circ}{A}/ps$, while those in Figure 2.5 is $8.2368 \text{ } \overset{\circ}{A}/ps$. In addition, the molecular distributions, though similar, are different. The differences are the result of the three dimensional forces which play no role in the two dimensional simulation.

2.5 Turbulent Flow.

Fig. 2.2 $T = 2.90$

Fig. 2.3 $T = 3.35$

Fig. 2.4 $T = 4.02$

Fig. 2.5 $T = 7.04$

Fig. 2.6 $T = 10.07$

Fig. 2.7 Two dimensional simulation at $T = 7.04$

In turning to turbulent flow we will concentrate primarily on $V = 6524$ for which related computations by others are available (Zang, Street and Koseff (1993)). Assuming the configuration described in Section 2.2, wallspeed 6524 corresponds to $Re=6000$. However, it became apparent readily that for $V=6524$, the problem of roundoff error accumulation required the exceptionally small time step $\Delta t = 2(10)^{-7}$. The projected time in order to reach meaningful results on our Digital Alpha 533 was excessively large. So, we were forced to decide to decrease the number of molecules and consider the following second cavity problem. In a cube of edge length $91.8 \text{ \AA}$, a cubic grid of 29791 points was generated as follows:

$$\begin{aligned} x(1) &= -45.9, y(1) = 0.0, z(1) = 0.0 \\ x(i+1) &= x(i) + 3.06, y(i) = 0.0, z(i) = 0.0, i = 1, 30 \\ x(32) &= 0.0, y(32) = 3.06, z(32) = 0.0 \\ x(i+1) &= x(i) + 3.06, y(i+1) = 3.06, z(i+1) = 0.0, i = 32, 61 \\ x(i) &= x(i-62), y(i) = y(i-62) + 6.12, z(i) = 0.0, i = 63, 961 \\ x(i) &= x(i-961), y(i) = y(i-961), z(i) = 3.06, i = 962, 1922 \\ x(i) &= x(i-1922), y(i) = y(i-1922), z(i) = z(i-1922) + 6.12, i = 1923, 29791. \end{aligned}$$

There result 29791 interior and boundary grid points of which 961 lie in the YZ plane. Renumber the points in the YZ plane to be

$$x(i), y(i), z(i); \quad i = 14416, 15376$$

Renumber the remaining points so that

$$x(i) = -x(29792-i), y(i) = y(29792-i), z(i) = z(29792-i), i = 15377, 29791.$$

At grid point numbered i , set a molecule P_i ; $i = 1, 29791$. To complete the initial data for the molecules, recall that at $15^\circ C$ the speed in $\text{\AA}/ps$ for a water molecule in three dimensions is $6.3 \text{ \AA}/ps$. This speed and its sign are assigned to each molecule in the X , Y , or Z direction, at random, but with the restrictions that if P_i and P_j are not in the YZ plane, but are symmetrical relative to the YZ plane, then

$$v_x(P_i) = -v_x(P_j), v_y(P_i) = v_y(P_j), v_z(P_i) = v_z(P_j),$$

while if P_i is in the YZ plane then $v_x(P_i) = 0$. The resulting symmetry reduces the problem to a 15376-body problem.

Figures 2.8-2.10 show the very rapid development of turbulence in the YZ plane at the times $t = 0.0561, 0.1100, 0.1650$, with $J = 77000$. Figure 2.10 shows very clearly the large crosscurrent over the center bottom, center right, center left, and top left sections of the usual primary vortex direction. Entirely similar results followed for $J = 44000, 55000, 66000, 88000$.

Fig. 2.8 $T = 0.0561$

Fig. 2.9 $T = 0.1100$

Fig. 2.10 $T = 0.1650$

Figures 2.11-2.14 show typical molecular motion through the trajectories of P_{4000} , P_{13500} , P_{19000} , P_{31500} during the time interval $0.0627 \leq t \leq 0.1914$. The initial point of P_{4000} in Figure 2.11 is $(-26.37, 64.84, 1.60)$ and its terminal point is $(-3.08, 90.23, 24.51)$. The motion is extensive for such a short period of time and is characterized by multiple positions of unstable equilibrium. The motion of P_{13500} in Figure 2.12 begins at $(-3.06, 0.00, 88.92)$ and terminates at $(-2.65, 61.43, 40.17)$. The motion of P_{19000} in Figure 2.13 shows the same complexities as those in Figures 2.11, 2.12. Figure 2.14 shows that even the molecules at the bottom of the cavity have extensive motion. The motion of P_{31500} is entirely in the plane $Z = 0$. It begins at $(71.44, 13.33, 0.00)$ and terminates at $(26.65, 35.04, 0.00)$.

Note that in this example, $V = 6524$ corresponds to $Re = 5300$, since the edge length of the cube is $91.8 \text{ \AA}$.

2.6 Remarks

Using the same procedure as that described in Section 2.5, Figures 2.15-2.17 show the results at the times $t = 0.06$, 0.12 , and 0.18 with $J = 60000$ for the wallspeed 3479 , which corresponds to $Re=3200$. Figure 2.17 clearly shows the development of turbulence in the center bottom portion of the YZ plane at $t = 0.18$. A strong crosscurrent has developed at this time. Entirely similar results followed for $J = 40000$, 50000 , 70000 . Using the Navier-Stokes equations with $Re=3200$, Perng and Street (1989) showed that, in the large, the resulting flow in a cube is not only stable but also yields the interesting Taylor-Görtler vortices, which may or may not be contradictory to our results in the small. It is also interesting to observe, relative to the use of the Navier-Stokes equations, that there is no uniform agreement as to which values of Re are admissible for these laminar equations. Mills (1965) allows $Re = 100000$, while Chiang, Sheu and Hwang (1998) doubt results outside the range $1 < Re < 2000$. Perhaps a major problem in reaching agreement is that related experiments are usually performed with very different span-to-width aspect ratios, thus resulting in greater or lesser side wall effects.

It should be observed also that we have not used the Kolmogorov criterion for turbulence. The reason is that on the molecular level, a small vortex is defined as follows. For $3 \leq M \leq 6$, a *small vortex* is defined as a flow in which M molecules nearest to an $(M + 1)$ st molecule rotate either clockwise or counterclockwise about the $(M + 1)$ st molecule and, in addition, the $(M + 1)$ st molecule lies *interior* to a simple polygon determined by the given M molecules. On the molecular level every flow has a large number of small, rapidly appearing and disappearing small vortices. Figure 2.18 shows, for example, the locations of 113 small vortices which are present in the nonturbulent flow shown in Figure 2.6.

Fig. 2.11 P_{4000}

Fig. 2.12 P_{13500}

Fig. 2.13 P_{19000}

Fig. 2.14 P_{31500}

Fig. 2.15 $T = 0.06$

Fig. 2.16 $T = 0.12$

Fig. 2.17 $T = 0.18$

Fig. 2.18 The locations of 113 small vortices in Figure 2.6.

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