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**The Least Square Values and the Shapley Value
for Cooperative TU Games**

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The Least Square Values and the Shapley Value for cooperative TU games.

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Abstract

The Least Square Values (briefly LS-values), represent a family of values for cooperative transferable utility games, introduced by L.Ruiz, F.Valenciano and J.Zarzuelo (1998). For a fixed set of players N , a set of weights $m \in \mathbb{R}^n_{++}$, $n = |N|$, and any TU-game v , M.Keane (1968) has defined and solved a quadratic programming problem: minimize the sum of weighted squares of deviations of the excesses from their average, on the preimputation set. He has shown that this problem has a unique solution which depends on the weights and has also given the new computational formula. Further, Ruiz/Valenciano/Zarzuelo have called this solution an LS-value, have characterized axiomatically this family of values and have also given many properties.

In the present paper, we start by showing that from this formula one can derive another new computational formula, similar to the formula given by the author for the Shapley value (see Dragan, 1992) and called an Average per capita formula. An algorithm for computing the LS-values in parallel and using only n operations more than in the case of the Shapley value is stated. A by product of the same formula is the result that an LS-value is the Shapley value of a game easily obtained by rescaling from the given game. This fact suggests that the LS-values have more properties similar to the Shapley value than the ones discovered by Ruiz/Valenciano/Zarzuelo. The LS-value is defined on any space of TU-games and its subspaces: we define a quasi-potential function of the Hart/Mas Colell type (1988, 1989). A potential basis for such a space, introduced by the author for the Shapley value and the weighted Shapley value (1991), and the Banzhaf value (1996) is allowing us to solve for the LS-values what we call the Inverse Problem: given a vector $\Phi \in \mathbb{R}^n$ and a weight vector $\gamma \in \mathbb{R}^n_{++}$, find out an explicit formula giving all games v with the set of players N for which the LS-value corresponding to γ is exactly Φ . In particular, for $\Phi = 0$ we find the null space of the LS-value: the nullity equals $2^n - n - 1$. The inverse problem in the cases shown above were solved earlier by the author.

In the first section, we prove the Average per capita formula for LS-values and corollaries, in the second section we introduce the quasi-potential and the potential basis, and in the last section we solve the inverse problem.

§ 1. The Average per capita formula for the Least Square Values.

A cooperative TU game is a pair $G = (N, v)$, where N is a finite set, the set of n players, $n = |N|$, and v is a function from the power set of N to the reals satisfying $v(\emptyset) = 0$. When N will be fixed we denote by G^N the set of all TU games with the set of players N and a game in G^N will be determined by its function v , so that the notation

$v \in G^N$ will be used with no confusion possible. It is well known that G^N can be identified with the euclidian space of dimension $2^n - 1$, by defining $(v_1 + v_2)(S) = v_1(S) + v_2(S)$, $\forall S \subseteq N$, and $(\alpha v)(S) = \alpha v(S)$, $\forall S \subseteq N$, for all v, v_1, v_2 in G^N and any $\alpha \in \mathbb{R}$. Beside the standard basis, given by $e_S \in G^N$, $\forall S \subseteq N$, with $e_S(S) = 1$ and $e_S(T) = 0$ otherwise, another popular basis is the unanimity basis, introduced by Shapley in [12], in order to derive the Shapley value formula: the unanimity basis is given by $u_S \in G^N$, $\forall S \subseteq N$, with $u_S(T) = 1$ for all $T \supseteq S$ and $u_S(T) = 0$ otherwise. In order to solve what will be called the inverse problem, other bases for G^N have been introduced in [2], [3], and [5], as tools in approaching the inverse problems for the Shapley value, the weighted Shapley value and the Banzhaf value, respectively. In this paper, we shall introduce another family of bases for G^N as a tool in solving the inverse problem for the Least Square Values.

M. Keane (1969), has introduced what has been later called the Least Square Values of cooperative TU games (briefly LS-values), as follows. For any game $v \in G^N$, and any $x \in \mathbb{R}^N$, denote the excess of S for x by $e(S, x) = v(S) - x(S)$, $\forall S \subseteq N$; assume that x is an efficient payoff, that is we have $e(N, x) = 0$. Then, denote the average of excesses by

$$(1.1) \quad e(v) = (2^n - 1)^{-1} \sum_{S \subseteq N} e(S, x) = (2^n - 1)^{-1} \left(\sum_{S \subseteq N} v(S) - 2^{n-1} v(N) \right)$$

which does not depend on x ; let $m: P(N) - \{\emptyset, N\} \rightarrow \mathbb{R}$ be a function which is positive and symmetric, that is for all coalitions of size s the value of m is the same, $m(s)$, a positive number. The weights may be viewed as components of a weight vector $m \in \mathbb{R}^{n-1}$. Consider the optimization problem (P):

$$(1.2) \quad \text{minimize} \quad \sum_{S \subseteq N} m(s) (e(S, x) - e(v))^2, \quad \text{s.t.} \quad x(N) = v(N).$$

Note that in general one can consider weights depending on the coalitions, as Lehrer (2002) does in trying to get a dynamic interpretation for the LS-values by means of a model of budget allocation. If the weight function depends only on the sizes of the coalitions, then we are saying that it is symmetric. Note also that in (1.2) it is possible to omit the term for $S=N$, as $e(N, v)=0$; therefore one can assume that $m(n)=1$. Obviously, the function m may also be assumed to be normalized, which would not change the optimal solution, but we do not need it here. In [8] it has been proved that: if m is positive and symmetric, then for any $v \in G^N$ the problem (P) has a unique optimal solution, namely

$$(1.3) \quad LS_i(m, v) = n^{-1} v(N) + (\alpha)^{-1} \left[a_i(m, v) - \frac{1}{n} \sum_{S \subseteq N} a_j(m, v) \right], \quad \forall i \in N,$$

where

$$(1.4) \quad \alpha = \sum_{s=1}^{n-1} m(s) C_{s-1}^{n-2}, \quad a_i(m, v) = \sum_{S: i \in S \subseteq N} m(s) v(S), \quad \forall i \in N.$$

Therefore, $LS(m, v)$ has been called a Least Square Value. Many interesting properties

of LS-values have been proved in [11]; among them it has been shown, as in Keane, that in the family of LS-values, for the weight function $m(s) = (n-1)^{-1} (C_{s-1}^{n-2})^{-1}$ and $s=1,2,\dots,n-1$, we get the Shapley value. This has been the starting point of our work, as we have tried to see whether an LS-value could be represented by an Average per capita formula similar to the Shapley value formula given by the author in [4]. Recall from [4] the notations

$$(1.5) \quad v_s = \frac{\sum_{|S|=s} v(S)}{C_s^n} \quad \text{and} \quad v_s^i = \frac{\sum_{|S|=s, i \notin S} v(S)}{C_s^{n-1}}, \quad \forall i \in N,$$

for $s=1,2,\dots,n-1$. In words, v_s is the average worth of coalitions of size s , while v_s^i is the average worth of coalitions of size s which do not contain player i . It has been shown in [4] that: for any $v \in G^N$ the Shapley value $SH(N,v)$ can be represented by the formula

$$(1.6) \quad SH_i(N,v) = n^{-1} v(N) + \sum_{s=1}^{n-1} \frac{v_s - v_s^i}{s}, \quad \forall i \in N,$$

which has been called the Average per capita formula.

In the following we shall consider another weight vector, denoted by γ , which is also positive and symmetric, so that it can be considered as a vector in R_{++}^n , and we shall take $\gamma(n)=1$; moreover the sum of the other components will be assumed equal to $n-1$.

Theorem 1: For the Least Square Value $LS(m,v)$, associated with a positive symmetric function $m: P(N)-\{\emptyset, N\} \rightarrow R$, there exists a positive symmetric function γ , which considered as a vector in R_{++}^n has the sum of components equal to n and $\gamma(n)=1$, such that

$$(1.7) \quad LS_i(\gamma,v) = n^{-1} v(N) + \sum_{s=1}^{n-1} \gamma(s) \frac{v_s - v_s^i}{s}, \quad \forall i \in N,$$

where $LS(\gamma,v)$ denotes the expression obtained when in $LS(m,v)$ the function m has been expressed in terms of γ .

Proof: A simple computation gives different expressions for the terms found in the bracket in (1.3). On the one hand, we have for all $i \in N$

$$(1.8) \quad a_i(m,v) = \sum_{S: i \in S} m(s) v(S) = \sum_{s=1}^{n-1} m(s) \left(\sum_{|S|=s, i \in S} v(S) \right);$$

on the other hand, taking into account that a coalition S enters in s sums $a_j(m,v)$, we can rewrite

$$\sum_{j \in N} a_j(m,v) = \sum_{j \in N} \left(\sum_{S: j \in S} m(s) v(S) \right) =$$

$$(1.9) \quad \sum_{s=1}^{n-1} \left(\sum_{|S|=s} m(s) v(S) \right) = \sum_{s=1}^{n-1} m(s) \left(\sum_{|S|=s} v(S) \right).$$

The difference of the two terms (1.8) and (1.9), multiplied each with the convenient factors, can be written as

$$(1.10) \quad \sum_{s=1}^{n-1} (n\alpha)^{-1} (n-s) m(s) \left(\sum_{|S|=s} v(S) \right) - \sum_{s=1}^{n-1} \alpha^{-1} m(s) \left(\sum_{|S|=s, i \notin S} v(S) \right),$$

where it has been used the fact that the sum of all worth for coalitions containing i equals the sum for all coalitions of that size minus the sum for coalitions which do not contain i .

Note that in (1.8), (1.9) and (1.10), in all sums over S we have only coalitions strictly included in N . Now, by introducing the averages (1.5) in (1.10), we obtain:

$$(1.11) \quad \sum_{s=1}^{n-1} (n\alpha)^{-1} (n-s) m(s) C_s^n v_s - \sum_{s=1}^{n-1} \alpha^{-1} m(s) C_s^{n-1} v_s^i = \sum_{s=1}^{n-1} \gamma(s) \frac{v_s - v_s^i}{s},$$

where we denoted for $s=1, 2, \dots, n-1$

$$(1.12) \quad \gamma(s) = \alpha^{-1} (n-1) m(s) C_{s-1}^{n-2}, \quad \alpha = \sum_{t=1}^{n-1} m(t) C_{t-1}^{n-2},$$

and we used relationships between combinations. Formula (1.12) shows that γ is a positive symmetric function with the sum of weights $\gamma(s)$ for $s=1, 2, \dots, n-1$ equal to $n-1$. Hence we proved Theorem 1.

Corollary 2: The Least Square value $LS(\gamma, v)$ associated with a weight vector $\gamma \in R_{++}^n$ having the sum of components equal to n and $\gamma(n)=1$, is the Shapley value, if and only if $\gamma(s)=1$, for $s=1, 2, \dots, n-1$.

Proof: The "if" part follows from (1.6) and (1.7). Conversely, from the equality $SH(N, v) - LS(\gamma, v) = 0$, $\forall v \in G^N$, we obtain

$$(1.13) \quad \sum_{s=1}^{n-1} (\gamma(s) - 1) \frac{v_s - v_s^i}{s} = 0, \quad \forall i \in N, \quad \forall v \in G^N.$$

Let $U \subset N$, $U \neq \emptyset$, contains player i , and let $v \in G^N$ be the game $v(U)=1$ and $v(S)=0$, $\forall S \neq U$. Then, $v_s=0$ and $v_s^i=0$, $\forall i \in N$, whenever $s \neq |U|$, and we have $v_u \neq 0$, and $v_u^i = 0$, $\forall i \in N$, and we get $\gamma(s)=1$, $s=1, 2, \dots, n-1$.

As shown in [4], the terms in formula (1.7) can be computed in parallel and very little computer memory is needed to compute one term of the sum in case that the numbers $v(S)$ are generated by the computer. The algorithm explained in [4] will work for LS-values exactly as for the Shapley value, except that each term of the sum should be multiplied by $\gamma(s)$ before making the summation.

Corollary 3: The LS-value associated with a weight vector $\gamma \in R_{++}^n$ having the sum of components equal to n , and $\gamma(n)=1$, is the Shapley value of the game $w \in G^N$ defined by

$$(1.14) \quad w(S) = \gamma(s) v(S), \quad \forall S \subseteq N.$$

Proof. This follows from Theorem 1, noticing that (1.14) implies $w_s = \gamma(s) v_s$, for all $s=1,2,\dots,n-1$, and $w'_i = \gamma(s) v'_i$ for all $s=1,2,\dots,n-1$ and all $i \in N$.

§ 2. The Potential

S.Hart and A.Mas Colell have introduced in [6] and [7] a potential function for the Shapley value; Corollary 3 in the previous section suggests that an LS-value has also a potential. We define this potential for a game $v \in G^N$ and a weight vector $\gamma \in R^{n++}$ with the sum of components equal to n and $\gamma(n)=1$, by a recursive formula: take

$$(2.1) \quad Q(\emptyset, v) = 0, \quad Q(\{i\}, v) = \gamma(1) v(\{i\}), \quad \forall i \in N,$$

and for all coalitions S with $|S| \geq 2$

$$(2.2) \quad Q(S, v) = s^{-1} \left[\gamma(s) v(S) + \sum_{j \in S} Q(S - \{j\}, v) \right],$$

where $s = |S|$. Note that the potential depends on the weight vector γ , but to keep the notation simple we did not mention it. Obviously, the LS-value will depend on the same weight function. Note also that by the choice of the potential function a lot of results will follow from the Hart/Mas Colell results on the Shapley value. The result which will justify the name of potential for the LS-value is given by:

Theorem 4: Let $\gamma \in R^{n++}$ with the sum of components equal to n and $\gamma(n)=1$, and let $v \in G^N$ be any game. Let $LS(\gamma, v)$ be the LS-value associated with γ , and Q the function defined by formulas (2.1) and (2.2). Then, we have

$$(2.3) \quad Q(N, v) - Q(N - \{i\}, v) = LS_i(\gamma, v), \quad \forall i \in N.$$

Proof: By definition and an easy induction, we have for the potential of the Shapley value for the game w connected to v by (1.14), denoted $P(S, w)$ for all $S \subseteq N$:

$$P(\emptyset, w) = 0 = Q(\emptyset, v),$$

$$(2.4) \quad P(\{i\}, w) = w(\{i\}) = \gamma(1) v(\{i\}) = Q(\{i\}, v), \quad \forall i \in N,$$

$$P(S, w) = s^{-1} \left[w(S) + \sum_{j \in S} P(S - \{j\}, w) \right] = s^{-1} \left[\gamma(s) v(S) + \sum_{j \in S} Q(S - \{j\}, v) \right] = Q(S, v),$$

for all $S \subseteq N$ with $|S| \geq 2$. Hence, for all coalitions $S \subseteq N$ we have $P(S, w) = Q(S, v)$. Then, our Corollary 3 and the Hart/Mas Colell Theorem on the potential for the Shapley value, give

$$(2.5) \quad P(N, w) - P(N - \{i\}, w) = Q(N, v) - Q(N - \{i\}, v) \leftrightarrow SH_i(N, w) = LS_i(\gamma, v), \quad \forall i \in N.$$

Obviously, other results could follow in a similar manner from the Hart/Mas Colell results; for example, it is easy to see that we have:

Theorem 5: If Q is the potential of the LS-value associated with the weight vector γ , then we have

$$(2.6) \quad Q(T, v) = \sum_{S \subseteq T} \frac{(s-1)!(t-s)!}{t!} \gamma(s) \cdot v(S), \quad \forall T \subseteq N.$$

Proof: We have from the proof of the previous theorem:

$$(2.7) \quad Q(T, v) = P(T, w) = \sum_{S \subseteq T} \frac{(s-1)!(t-s)!}{t!} w(S), \quad \forall T \subseteq N,$$

Where the last equality is given by the corresponding Hart/Mas Colell result; now, by using (1.14), the result (2.6) follows.

In the introduction we called the functional Q a quasi-potential because it depends on an weight vector, while in the case of potentials the values are numbers. Now, we continue this section with the determination of a potential basis for G^N relative to the LS-value.

Recall from [2], [3], [5], that for the Shapley value, the weighted Shapley value and the Banzhaf value which have a potential, we have been able to determine a potential basis for G^N , and by using this basis we have solved what we called the inverse problem for these values. Recall from [5] that a potential basis for G^N relative to a value χ , is a basis for G^N such that the coordinates of each $v \in G^N$ relative to that basis are exactly the potentials of χ for v and its subgames. Precisely, $C = \{c_S \in G^N : S \subseteq N, S \neq \emptyset\}$ is a potential basis for G^N relative to χ , if the coordinates of v in the expansion of v in basis C , call them α_S , are given by $\alpha_S = H(S, v)$, where H is the potential of χ for v and its subgames. Recall also from [5] the following result:

Theorem 6: If χ is a value which has a potential H , and there is a potential basis C for G^N relative to χ , then we must have for each coalition $S \subseteq N$:

$$(2.8) \quad H(S, c_S) = 1, \quad H(T, c_S) = 0, \quad \forall T \subseteq N, T \neq S,$$

where c_S is the vector in the basis C corresponding to the coalition S .

Note that the definition (2.1) and (2.2) gives the potentials if the game $c_S \in G^N$ is known, but they give also the game c_S for which the potentials of the value are known, when we write them as

$$(2.9) \quad \begin{aligned} c_S(\{i\}) &= [\gamma(1)]^{-1} Q(\{i\}, c_S), \quad \forall i \in N, \\ c_S(T) &= [\gamma(t)]^{-1} \sum_{j \in T} [Q(T, c_S) - Q(T - \{j\}, c_S)], \quad \forall T \subseteq N, |T| \geq 2. \end{aligned}$$

We know the potentials of the LS-value for the games c_S , given by Theorem 6, where H is the potential Q , hence we can use (2.9).

Theorem 7: For each $S \subseteq N$, $S \neq \emptyset$, there is a unique game $c_S \in G^N$, such that (2.8) with $H = Q$ hold, where Q is the potential of the LS-value associated with the weight function γ . Moreover, we have for each S :

$$(2.10) \quad \begin{aligned} c_S(T) &= [\gamma(s)]^{-1} s && \text{if } T = S, \\ c_S(T) &= -[\gamma(s+1)]^{-1} && \text{if } T = S \cup \{j\}, j \notin S, \\ &&& \text{and } c_S(T) = 0, \text{ otherwise.} \end{aligned}$$

Proof: Fix $i \in N$ and consider $S = \{i\}$; then, from (2.8) with $H=Q$ we get

$$(2.11) \quad \begin{aligned} c_{\{i\}}(\{i\}) &= [\gamma(1)]^{-1}, && c_{\{i\}}(\{j\}) = 0, j \neq i, j \in N, \end{aligned}$$

$$c_{\{i\}}(\{i,j\}) = -[\gamma(2)]^{-1}, \quad j \neq i, \quad j \in N,$$

and $c_{\{i\}}(T)=0$ otherwise. We proved (2.10) for $S = \{i\}$. Consider any coalition S with $|S| \geq 2$; obviously, from (2.8) and (2.9) we get $c_S(T)=0$ whenever $|T| \leq |S|$ but $T \neq S$, and for the other coalitions T with $|T| > |S|$ we get exactly (2.10).

Now, it takes only a repetition of the proof given in [5] for the similar result, to prove:

Theorem 8: The set of games $C = \{ c_S \in G^N : S \subseteq N, S \neq \emptyset \}$ given by (2.10) in the Theorem 7 forms a basis for G^N , and this is a potential basis relative to the LS-value.

Example 1: If $|N| = 3$, then the games in C given by (2.10) are

$$\begin{aligned} c_1 &= (1/\gamma(1), 0, 0, -1/\gamma(2), -1/\gamma(2), 0, 0), \\ c_2 &= (0, 1/\gamma(1), 0, -1/\gamma(2), 0, -1/\gamma(2), 0), \\ c_3 &= (0, 0, 1/\gamma(1), 0, -1/\gamma(2), -1/\gamma(2), 0), \\ c_{12} &= (0, 0, 0, 2/\gamma(2), 0, 0, -1/\gamma(3)), \\ c_{13} &= (0, 0, 0, 0, 2/\gamma(2), 0, -1/\gamma(3)), \\ c_{23} &= (0, 0, 0, 0, 0, 2/\gamma(2), -1/\gamma(3)), \\ c_{123} &= (0, 0, 0, 0, 0, 0, 3/\gamma(3)), \end{aligned}$$

where the components of each vector are corresponding to the coalitions taken in the order $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$, and in fact $\gamma(3)=1$, and $\gamma(1)+\gamma(2)=2$.

Note that these games form a basis for R^7 because they are linearly independent. Note also that for $\gamma(1) = \gamma(2) = 1$ this is the potential basis for the Shapley value, as shown in [2]. We can check that the above basis is a potential basis for G^N relative to the LS-value, by using the expansion in this basis and writing the game v in terms of the coordinates, as follows:

$$v(1) = [\gamma(1)]^{-1} \alpha_1, \quad v(2) = [\gamma(1)]^{-1} \alpha_2, \quad v(3) = [\gamma(1)]^{-1} \alpha_3,$$

$$v(1,2) = [\gamma(2)]^{-1} (2\alpha_{12} - \alpha_1 - \alpha_2), \quad v(1,3) = [\gamma(2)]^{-1} (2\alpha_{13} - \alpha_1 - \alpha_3),$$

$$v(2,3) = [\gamma(2)]^{-1} (2\alpha_{23} - \alpha_2 - \alpha_3), \quad v(1,2,3) = [\gamma(3)]^{-1} (3\alpha_{123} - \alpha_1 - \alpha_2 - \alpha_3);$$

then, we compute the potentials by means of (2.1) and (2.2). For example, we obtain

$$Q(\{1,2\}, v) = \frac{1}{2} [\gamma(2) v(1,2) + Q(\{1\}, v) + Q(\{2\}, v)] = \alpha_{12}.$$

§ 3. The inverse problem.

Consider the following problem: let $\gamma \in R_{++}^n$ with the sum of components equal to n and $\gamma(n)=1$ be given and let $L \in R^n$ be also given; determine the set of all n -person TU games $v \in G^N$ for which $LS(\gamma, v) = L$. This will be called the inverse problem, like in the previous works [2], [3] and [5]. To prove the result needed in solving the inverse problem we must first prove:

Theorem 9: Let $C = \{ c_S \in G^N : S \subseteq N, S \neq \emptyset \}$ be the potential basis for G^N relative to an LS-value; then, the LS-values of the basic games are:

$$(3.1) \quad \begin{aligned} &LS(\gamma, c_S) = 0, \quad \forall S \subset N, \quad |S| \leq n-2, \\ &LS(\gamma, c_{N-\{j\}}) = -e_j, \quad \forall j \in N, \quad LS(\gamma, c_N) = e, \end{aligned}$$

where $e_j, \forall j \in N$, are the games of the standard basis for R^n and $e \in R^n$ is the vector with all components equal to one.

Proof: To compute $LS(\gamma, c_S)$ we can use Theorem 4, because the potentials of the games c_S satisfy (2.8). Thus, if $|S| \leq n-2$, then both $Q(N, c_S) = 0$ and $Q(N-\{j\}, c_S) = 0, \forall j \in N$, hence $LS(\gamma, c_S) = 0$. If $S = N-\{j\}$, then $Q(N, c_{N-\{j\}}) = 0$ but $Q(N-\{i\}, c_{N-\{j\}}) = \delta_{ij}$ where $\delta_{ii} = 1$ and $\delta_{ij} = 0, i \neq j$, hence, we have $LS(\gamma, c_{N-\{j\}}) = -e_j$. Finally, $Q(N, c_N) = 1$ and $Q(N-\{i\}, c_N) = 0$, so that $LS(\gamma, c_N) = e$.

Now, the solution of the inverse problem is provided by:

Theorem 10: If $\gamma \in R^n_{++}$ with the sum of components equal to n and $\gamma(n) = 1$ is given, and $L \in R^n$ is also given, then any TU game $v \in G^N$ for which $LS(\gamma, v) = L$ is obtained for some numerical values of the parameters α_N and $\alpha_S, \forall S \subset N, |S| \leq n-2$, from the formula

$$(3.2) \quad v = \sum_{|S| \leq n-2} \alpha_S c_S + \alpha_N (c_N + \sum_{j \in N} c_{N-\{j\}}) - \sum_{j \in N} L_j c_{N-\{j\}},$$

where $C = \{c_S \in G^N : S \subseteq N, S \neq \emptyset\}$ is the potential basis for G^N relative to the LS-value $LS(\gamma, v)$.

Proof: In the expansion of v relative to the potential basis we take $Q(N, v) = \alpha_N$ and $Q(S, v) = \alpha_S, \text{ for all } S \subset N \text{ with } |S| \leq n-2$, and $Q(N-\{j\}, v) = Q(N, v) - L_j, \text{ for all } j \in N$, and we get (3.2). Conversely, if $v \in G^N$ is given by (3.2), then by the linearity of the LS-values we obtain for all $i \in N$ the equality

$$(3.3) \quad LS_i(\gamma, v) = \sum_{|S| \leq n-2} \alpha_S LS_i(\gamma, c_S) + \alpha_N LS_i(\gamma, c_N + \sum_{j \in N} c_{N-\{j\}}) - \sum_{j \in N} L_j LS_i(\gamma, c_{N-\{j\}}).$$

By Theorem 9, the first sum equals zero, and it is easy to see that the second term equals zero, and we get

$$(3.4) \quad LS_i(\gamma, v) = L_i, \quad \forall i \in N.$$

Note that for $L = 0$ we obtain the following:

Corollary 11: The null space of the linear operator $LS : G^N \rightarrow R^n$ is spanned by the set of games

$$(3.5) \quad \text{Nu}(LS) = \{c_S \in G^N : S \subset N, |S| \leq n-2\} \cup \{c_N + \sum_{j \in N} c_{N-\{j\}}\},$$

Hence the nullity is $2^n - n - 1$ for whatever γ .

Example 2 : If $|N| = 3$ and $L \in \mathbb{R}^3$ is given, then the solution of the inverse problem is obtained from:

$$(3.6) \quad v = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 + \alpha_{123} (c_{123} + c_{12} + c_{13} + c_{23}) - L_1 c_{23} - L_2 c_{13} - L_3 c_{12},$$

where the vectors c_s are those shown in the Example 1. Hence, the set of games having $LS(\gamma, v) = L$ is obtained for arbitrary numerical values of parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_{123}$, from

$$v(1) = [\gamma(1)]^{-1} \alpha_1, \quad v(2) = [\gamma(1)]^{-1} \alpha_2, \quad v(3) = [\gamma(1)]^{-1} \alpha_3,$$

$$v(1,2) = [\gamma(2)]^{-1} (2 \alpha_{123} - \alpha_1 - \alpha_2 - 2L_3),$$

$$v(1,3) = [\gamma(2)]^{-1} (2 \alpha_{123} - \alpha_2 - \alpha_3 - 2L_2),$$

$$v(2,3) = [\gamma(2)]^{-1} (2 \alpha_{123} - \alpha_3 - \alpha_1 - 2L_1),$$

$$v(1,2,3) = L_1 + L_2 + L_3,$$

derived from (3.2).

Note that in the paper by Ruiz/Valenciano/Zarzuelo several results about the consistency property of the LS-values relative to a Davis/Maschler reduced game (see [1]) are also given.

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