COMPARISON RESULTS FOR FIRST AND SECOND ORDER
BOUNDARY VALUE PROBLEMS AT RESONANCE[†]

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0. INTRODUCTION

It is well known that the comparison principle for the initial value problems has been very useful in the theory of differential equations [1, 2,5]. Recently, such types of comparison results were developed for boundary value problems [3,4] and were used in proving the existence of solutions.

It is natural to expect that comparison results for problems at resonance will be useful in proving, for example, existence results for periodic boundary value problems. Recently, existence of periodic solutions for first and second order differential equations have been considered by utilizing the method of upper and lower solutions and Lyapunov-Schmidt method, where certain simple comparison theorems have been proved [6,7]. In this paper, we develop systematically general comparison results of various types for boundary value problems at resonance for first and second order differential equations. We do hope that our general comparison theorems will play an important role in the existence theory of boundary value

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problems at resonance.

1. FIRST ORDER PERIODIC BOUNDARY VALUE PROBLEMS

Consider the periodic boundary value problem (PBVP for short)

$$(1.1) \quad u'' = g(t, u), \quad u(0) = u(2\pi),$$

where $g \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$. Let $\varphi \in C([0, 2\pi], \mathbb{R})$. Then a function $u(t, u)$

is said to be a modified function relative to φ if $G(t, u) = g(t, \varphi(t, u)) + \frac{u(t, u) - \varphi(t, u)}{1 + u^2}$, where $\varphi(t, u) = \max\{0, \varphi(t, u)\}$. In what follows we will assume that

and $D^\pm$ denote as usual the lower left and the upper right Dini-derivatives respectively. We shall first prove the following comparison theorem.

Theorem 1.1. Assume that

$$(1.2) \quad \varphi \in C([0, 2\pi], \mathbb{R}) \text{ and } \varphi(t) \leq g(t, \varphi(t)), \quad 0 \leq t \leq 2\pi,$$

$$(1.3) \quad \varphi(0) = \varphi(2\pi),$$

$$(1.4) \quad u'' = G(t, u), \quad u(0) = u(2\pi),$$

admits a solution u for every G satisfying (1.2).

(1.5) $r(t)$ is the maximal solution for the PBVP (1.1).

Then either (a) $m(0) \leq \pi(2\pi)$

$$\text{or } (b) \quad \varphi(t) \geq g(t, \varphi(t)) \text{ and } \varphi(t) \neq 0 \text{ implies } m(t) \leq r(t)$$

on $[0, 2\pi]$.

Proof. The assertion follows if we prove that $m(t) \leq r(t)$ for $t \in [0, 2\pi]$.

where u is a solution of the PBVP (1.2), where G is the modified function relative to φ . Suppose $m(t) \leq r(t)$, $t \in [0, 2\pi]$. If this is not true then there is a $\theta > 0$ and $\alpha \in [0, 2\pi]$ such that

$$(1.3) \quad m(t_0) = u(t_0) + \varepsilon \quad \text{and} \quad m(t) \leq u(t) + \varepsilon, \quad t \in [0, 2\pi].$$

Since $m(t_0) > u(t_0)$, we have $p(t_0, u(t_0)) = m(t_0)$. Hence if $t_0 \in (0, 2\pi]$, we obtain from (1.3) and the definition of G that

$$\begin{aligned} D_- m(t_0) &\geq u'(t_0) = g(t_0, p(t_0, u(t_0))) + \frac{p(t_0, u(t_0)) - u(t_0)}{1 + u^2(t_0)} \\ &= g(t_0, m(t_0)) + \frac{m(t_0) - u(t_0)}{1 + u^2(t_0)} > g(t_0, m(t_0)) \end{aligned}$$

which contradicts the hypothesis (i). If $t_0 = 0$, we have $m(0) = u(0) + \varepsilon$. Hence if (a) holds, we have $m(2\pi) \geq m(0) = u(0) + \varepsilon = u(2\pi) + \varepsilon$. This inequality, together with (1.3), yields $D_- m(2\pi) \geq u'(2\pi)$ which leads to a contradiction as before. Suppose next that (b) holds. Using the relations $m(0) = u(0) + \varepsilon$ and $m(h) \leq u(h) + \varepsilon$, $h > 0$, we obtain

$$\begin{aligned} D^+ m(0) &\leq u'(0) = g(0, m(0)) + \frac{m(0) - u(0)}{1 + u^2(0)} \\ &\leq g(0, m(0)) + |m(0)| + \frac{1}{2} \\ &< g(0, m(0)) + |m(0)| + 1 \end{aligned}$$

contradicting the hypothesis (b). Thus $m(t) \leq u(t)$ on $[0, 2\pi]$. But then we see from the definition of G that u is actually a solution of the PBVP (1.1). Since $u(t) \leq r(t)$ on $[0, 2\pi]$, the proof of the theorem is complete.

We shall next prove another type of comparison result which deals with upper and lower solutions.

Theorem 1.2. Assume that

(i) $m \in C[[0, 2\pi], R]$, $m(0) \leq m(2\pi)$ and

$$D_-m(t) \leq g(t, m(t)), \quad 0 < t \leq 2\pi;$$

(ii) $\beta \in C[[0, 2\pi], R]$, $\beta(0) \geq \beta(2\pi)$ and

$$D_- \beta(t) \geq g(t, \beta(t)), \quad 0 < t \leq 2\pi;$$

(iii) either (a) $g(t, u)$ is strictly decreasing in u for each t ;

or (b) there exists a function $z \in C^1[[0, 2\pi], R]$, such that,
 $z(t) > 0$, $t \in [0, 2\pi]$, $z(0) \geq z(2\pi)$ and for every $\lambda > 0$,
 $\lambda z' > g(t, \beta + \lambda z) - g(t, \beta)$, $0 < t \leq 2\pi$.

Then $m(t) \leq \beta(t)$, $t \in [0, 2\pi]$.

Proof. Suppose first that (iii) (a) holds. If the assertion is not true, then there exists an $\epsilon > 0$ and a $t_0 \in [0, 2\pi]$, such that

$$(1.4) \quad m(t_0) = \beta(t_0) + \epsilon \quad \text{and} \quad m(t) \leq \beta(t) + \epsilon, \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi]$, then since $m(t_0) > \beta(t_0)$, we have by (iii) (a) and (ii), the inequality

$$D_-m(t_0) \geq D_- \beta(t_0) \geq g(t_0, \beta(t_0)) > g(t_0, m(t_0))$$

which contradicts the hypothesis (i). If $t_0 = 0$, we have $m(0) = \beta(0) + \epsilon$ and hence by (i) and (ii) we get $m(2\pi) \geq m(0) = \beta(0) + \epsilon \geq \beta(2\pi) + \epsilon$.

Then (1.4) yields $D_-m(2\pi) \geq D_- \beta(2\pi)$ which again leads to a contradiction.

Suppose next that (iii) (b) holds. If $m(t) \leq \beta(t)$, $t \in [0, 2\pi]$ is not true, then we can find a minimal $\lambda > 0$ and a $t_0 \in [0, 2\pi]$, such that

$$(1.5) \quad m(t_0) = \beta(t_0) + \lambda z(t_0) \quad \text{and} \quad m(t) \leq \beta(t) + \lambda z(t), \quad t \in [0, 2\pi].$$

Hence if $t_0 \in (0, 2\pi]$, then from (1.5), (ii) and (iii) (b), we obtain

$$\begin{aligned}
D_- m(t_0) &\geq D_- \beta(t_0) + \lambda z'(t_0) \\
&> g(t_0, \beta(t_0)) + g(t_0, \beta(t_0) + \lambda z(t_0)) - g(t_0, \beta(t_0)) \\
&= g(t_0, m(t_0))
\end{aligned}$$

which is a contradiction to the hypothesis (1). If $t_0 = 0$, we have $m(0) = \beta(0) + \lambda z(0)$, and hence $m(2\pi) \geq m(0) = \beta(0) + \lambda z(0) \geq \beta(2\pi) + \lambda z(2\pi)$. Since by (1.5) we also have $m(2\pi) \leq \beta(2\pi) + \lambda z(2\pi)$, it follows that $m(2\pi) = \beta(2\pi) + \lambda z(2\pi)$ and this leads to a contradiction. The proof of the theorem is complete.

2. SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEMS

Consider the second order PBVP

$$(2.1) \quad -u'' = g(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

where $g \in C[[0, 2\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$. We list the following assumptions on functions $m, \beta, z \in C^2[[0, 2\pi], \mathbb{R}]$ for convenience.

$$\begin{aligned}
(B_0) \quad &m(0) = m(2\pi), \quad m'(0) \geq m'(2\pi) \quad \text{and} \\
&-m'' \leq g(t, m, m'), \quad 0 \leq t \leq 2\pi.
\end{aligned}$$

$$\begin{aligned}
(B_1) \quad &\beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi) \quad \text{and} \\
&-\beta'' \geq g(t, \beta, \beta'), \quad 0 \leq t \leq 2\pi.
\end{aligned}$$

$$\begin{aligned}
(B_2) \quad &\text{For any } u, v \in C^2[[0, 2\pi], \mathbb{R}], \quad g_u(t, u, v), \quad g_v(t, u, v) \text{ are continuous,} \\
&g_u(t, u, v) \leq 0 \quad \text{and} \quad g_u(t, u, v) \not\equiv 0.
\end{aligned}$$

(B₃) $z(t) > 0$ for $t \in [0, 2\pi]$, $z(0) = z(2\pi)$, $z'(0) \leq z'(2\pi)$ and for every $\lambda > 0$, $-\lambda z'' > g(t, \beta + \lambda z, \beta' + \lambda z') - g(t, \beta, \beta')$, $0 \leq t \leq 2\pi$.

Let $v \in C[[0, 2\pi], \mathbb{R}]$. Then a function $G(t, u, u')$ is said to be a modified function relative to v if

$$(2.2) \quad G(t, u, u') = g(t, p(t, u), u') + \frac{p(t, u) - u}{1 + u^2},$$

where $p(t, u) = \max(v(t), u)$. We are now equipped to prove the following comparison result.

Theorem 2.1. Consider the PBVP (2.1). Assume that

(i) the modified PBVP

$$-u'' = G(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

admits a solution u for every lower solution v of (2.1)

(ii) $r(t)$ is the maximal solution of the PBVP (2.1).

Then (B₀) implies $m(t) \leq r(t)$ on $[0, 2\pi]$.

Proof. Let u be a solution of the PBVP (2.3) where G is the modified function relative to m . If $m(t) \leq u(t)$ on $[0, 2\pi]$, then it follows from the definition of G that u is actually a solution of the PBVP (2.1).

It is therefore enough to establish $m(t) \leq u(t)$, $t \in [0, 2\pi]$. Suppose this is not true. Then there exists an $\varepsilon > 0$ and a $t_0 \in [0, 2\pi]$, such that

$$(2.4) \quad m(t_0) = u(t_0) + \varepsilon \quad \text{and} \quad m(t) \leq u(t) + \varepsilon, \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, we have $m'(t_0) = u'(t_0)$ and $m''(t_0) \leq u''(t_0)$. Since $m(t_0) > u(t_0)$, we also have $p(t_0, u(t_0)) = m(t_0)$. Consequently, we get

$$\begin{aligned}
-m''(t_0) &\geq -u''(t_0) = G(t_0, u(t_0), u'(t_0)) \\
&= g(t_0, m(t_0), m'(t_0)) + \frac{m(t_0) - u(t_0)}{1 + u^2(t_0)} \\
&> g(t_0, m(t_0), m'(t_0))
\end{aligned}$$

which contradicts the hypotheses (B_0) . If $t_0 = 0$ or 2π , then, by (2.4) we have $m(i) = u(i) + \varepsilon$, $i = 0, 2\pi$,

$$(2.5) \quad m'(0) \leq u'(0) \quad \text{and} \quad m'(2\pi) \geq u'(2\pi).$$

Hence if (B_0) holds, we get

$$(2.6) \quad m'(2\pi) = u'(2\pi)$$

As a consequence, we are led to a contradiction as before which proves $m(t) \leq u(t)$ on $[0, 2\pi]$. This implies that $u(t)$ is a solution of the PBVP (2.1) and since $u(t) \leq r(t)$ on $[0, 2\pi]$, the proof of the theorem is complete.

We shall next prove a comparison result which deals with upper and lower solutions of the PBVP (2.1).

Theorem 2.2. Let (B_1) and (B_2) hold. Then (B_0) implies $m(t) \leq \beta(t)$ on $[0, 2\pi]$.

Proof. We show that if one of the inequalities in (B_1) and (B_0)

is strict, then $m(t) < \beta(t)$ on $[0, 2\pi]$. Under this assumption, suppose that $m(t) < \beta(t)$, $t \in [0, 2\pi]$ is not true. Then there exists an $M \geq 0$ and a $t_0 \in [0, 2\pi]$ such that

$$(2.7) \quad m(t_0) - \beta(t_0) = M \quad \text{and} \quad m(t) - \beta(t) \leq M, \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, we have $m'(t_0) = \beta'(t_0)$ and $m''(t_0) \leq \beta''(t_0)$. Now, since one of the inequalities in (B_1) and (B_0) is assumed to be strict, we have by (B_2)

$$\begin{aligned} m''(t_0) - \beta''(t_0) &> -g(t_0, m(t_0), m'(t_0)) + g(t_0, \beta(t_0), \beta'(t_0)) \\ &= g_u(t_0, \xi(t_0), \xi'(t_0))(-M) \geq 0, \end{aligned}$$

where $\xi(t_0)$ is a point between $m(t_0)$ and $\beta(t_0)$, and $\xi'(t_0) = m'(t_0) = \beta'(t_0)$. Thus we get $m''(t_0) - \beta''(t_0) > 0$ which contradicts (2.7). If $t_0 = 0$ or 2π , we have $m(1) - \beta(1) = M$, $1 = 0, 2\pi$, $m'(0) \leq \beta'(0)$ and $m'(2\pi) \leq \beta'(2\pi)$. Hence if (B_0) holds, we get

$$(2.8) \quad m'(2\pi) = \beta'(2\pi),$$

which leads to a contradiction as before, proving that $m(t) < \beta(t)$ on $[0, 2\pi]$, provided one of the inequalities in (B_1) and (B_0) is strict.

Let us now suppose that none of the inequalities in (B_1) , and (B_0) is strict. Suppose that $m(t) \leq \beta(t)$, $t \in [0, 2\pi]$ is not true. Then there exists a $t_0 \in [0, 2\pi]$ and an $M > 0$ such that (2.7) holds. We show first that the existence of such a t_0 and M implies $m(t) - \beta(t)$ on $[0, 2\pi]$. If $m(t) - \beta(t) \not\equiv M$ on $[0, 2\pi]$, then we have a $t^* \in [0, 2\pi]$, such that, $m(t^*) - \beta(t^*) < M$. Let $t_0 \in (0, 2\pi)$ and $t^* \in (t_0, 2\pi]$. Define an auxiliary function $z(t) = e^{\alpha(t-t_0)} - 1$, $t \in [0, 2\pi]$, where $\alpha > 0$ will be soon chosen suitably. Note that $z(t) < 0$ for $t \in [0, t_0)$, $z(t_0) = 0$ and $z(t) > 0$ for $t \in (t_0, 2\pi]$. Consider now a function

$$(2.9) \quad W(t) = m(t) - \beta(t) + \epsilon z(t), \quad t \in [0, 2\pi],$$

where we choose $\epsilon > 0$, so that $0 < \epsilon < \frac{M - (m(t^*) - \beta(t^*))}{z(t^*)}$. Such a choice is possible because $z(t^*) > 0$ and $m(t^*) - \beta(t^*) < M$. From (2.9) we get

$$(2.10) \quad W''(t) \geq -g_u(t, \xi(t), \eta(t))W(t) - g_u(t, \xi(t), \eta(t))W'(t) - \epsilon g_u(t, \xi(t), \eta(t)) + \epsilon [g_u(t, \xi(t), \eta(t)) + g_u(t, \xi(t), \eta(t))\alpha + \alpha^2]e^{\alpha(t-t_0)}, \quad 0 < t < 2\pi$$

by (B_1) and (B_0) where $(t, \xi(t), \eta(t))$ is some point between

$(t, m(t), m'(t))$ and $(t, \beta(t), \beta'(t))$. Now choose $\alpha > 0$, so that $g_u(t, \xi(t), \eta(t)) + g_u(t, \xi(t), \eta(t)) \alpha + \alpha^2 > 0$. Such a choice is possible because $g_u(t, \xi(t), \eta(t))$ and $g_u(t, \xi(t), \eta(t))$ are bounded on $[0, 2\pi]$. It then follows from (2.10) that

$$(2.11) \quad W''(t) > -g_u(t, \xi(t), \eta(t))W(t) - g_u(t, \xi(t), \eta(t))W'(t), \quad 0 < t < 2\pi.$$

Now from the relations $W(t) < M$, $t \in [0, t_0)$, $W(t_0) = M$, $W(t^*) < M$ and $z(t) > 0$, $t \in (t_0, 2\pi]$, it follows that there exists a $\tilde{t} \in [t_0, t^*)$ at which W has a maximum $N \geq M > 0$. This implies a contradiction by (2.11) since $\tilde{t} \in (0, 2\pi)$. If $t_0 = 0$ or 2π , we have $m(1) - \beta(1) = M$, $1 = 0, 2\pi$, $m'(0) \leq \beta'(0)$ and $m'(2\pi) \geq \beta'(2\pi)$. If (B_0) holds, the foregoing inequalities imply

$$(2.12) \quad m'(0) = \beta'(0).$$

Consequently, from (2.9) we get $W'(0) = \epsilon \alpha > 0$. It then follows from the relations $W(0) = M$, $W'(0) > 0$ and $W(t^*) < M$ that there exists a $\tilde{t} \in (0, t^*) \subset (0, 2\pi)$ at which W has a maximum and this again implies a contradiction by (2.11). In the preceding discussion $t^* \in (t_0, 2\pi]$.

If on the other hand $t^* \in [0, t_0)$, we take $z(t) = e^{-\alpha(t-t_0)} - 1$, $t \in [0, 2\pi]$, where $\alpha > 0$ is chosen properly, and follow the same argument. Thus in all cases we have proved that the existence of a $t_0 \in [0, 2\pi]$ and an $M > 0$ satisfying (2.7) implies $m(t) - \beta(t) \equiv M$ on $[0, 2\pi]$. As a consequence of this we have by (B_2)

$$\begin{aligned} 0 \equiv m''(t) - \beta''(t) &\geq -g(t, m(t), m'(t)) + g(t, \beta(t), \beta'(t)) \\ &= g_u(t, \xi(t), m'(t))(-M) \geq 0, \quad t \in (0, 2\pi), \end{aligned}$$

which implies $g_u(t, \xi(t), m'(t)) \equiv 0$ on $[0, 2\pi]$ by the continuity of g_u , contradicting the hypothesis (B_2) . This implies that $m(t) \leq \beta(t)$ on $[0, 2\pi]$ and the proof is complete.

Theorem 2.3. The assertion of Theorem 2.2 remains valid if (B_2) is replaced by (B_3) .

Proof. Suppose that $m(t) \leq \beta(t)$, $t \in [0, 2\pi]$ is not true. Then we can find a minimal $\lambda > 0$ and a $t_0 \in [0, 2\pi]$, such that,

$$(2.13) \quad m(t_0) = \beta(t_0) + \lambda z(t_0) \quad \text{and} \quad m(t) \leq \beta(t) + \lambda z(t), \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, we have $m'(t_0) = \beta'(t_0) + \lambda z'(t_0)$ and $m''(t_0) \leq \beta''(t_0) + \lambda z''(t_0)$. As a consequence we get

$$\begin{aligned} m''(t_0) - \beta''(t_0) - \lambda z''(t_0) &> -g(t_0, m(t_0), m'(t_0)) + g(t_0, \beta(t_0), \beta'(t_0)) \\ &\quad + g(t_0, \beta(t_0) + \lambda z(t_0), \beta'(t_0) + \lambda z'(t_0)) \\ &\quad - g(t_0, \beta(t_0), \beta'(t_0)) \\ &= 0 \end{aligned}$$

by (B_1) , (B_3) and (B_0) . That is $m''(t_0) - \beta''(t_0) - \lambda z''(t_0) > 0$

which, in view of (2.13), implies a contradiction. If $t_0 = 0$ or 2π , we

have, in either case, $m(i) = \beta(i) + \lambda z(i)$, $i = 0, 2\pi$, and

$$(2.14) \quad m'(0) \leq \beta'(0) + \lambda z'(0), \quad m'(2\pi) \geq \beta'(2\pi) + \lambda z'(2\pi).$$

Hence if (B_0) holds, then from (2.14), we get

$$(2.15) \quad m'(2\pi) = \beta'(2\pi) + \lambda z'(2\pi)$$

which leads to a contradiction as before. This proves the assertion that $m(t) \leq \beta(t)$ on $[0, 2\pi]$.

Theorem 2.4. Assume that

$$(i) \quad m(0) = m(2\pi), \quad m'(0) \geq m'(2\pi) \quad \text{and} \quad -m'' \leq g(t, m, m'), \quad 0 \leq t \leq 2\pi;$$

(ii) either (a) (B_2) holds,

$$\text{or (b) } z(t) > 0 \text{ for } t \in [0, 2\pi], \quad z(0) = z(2\pi), \quad z'(0) \leq z'(2\pi),$$

and for every $\lambda > 0$, $-\lambda z'' > g(t, m, m') - g(t, m - \lambda z, m' - \lambda z')$, $0 \leq t \leq 2\pi$.

Then the assertion of Theorem 2.2 remains valid.

The proof follows on similar lines as those of Theorems 2.2 and 2.3 with suitable modifications and hence deleted.

We shall next consider the following second order boundary value problem with homogeneous Neumann conditions (NBVP for short)

$$(2.17) \quad -u'' = g(t, u, u'), \quad u'(0) = u'(1) = 0$$

where $g \in C[[0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$. Comparison results analogous to those for the PBVP (2.1) can be proved for the NBVP (2.17). But to avoid monotony, we shall only state the analogous theorems and mention the proof of one of them, where it slightly differs from the analogous theorem for the PBVP (2.1).

Let us now suppose that $m, \beta, z \in C^2[[0,1], \mathbb{R}]$. The following assumptions will be needed.

$$(C_0) \quad m'(0) \geq 0, \quad m'(1) \leq 0 \quad \text{and} \quad -m'' \leq g(t, m, m'), \quad 0 \leq t \leq 1.$$

$$(C_1) \quad \beta'(0) \leq 0, \quad \beta'(1) \geq 0 \quad \text{and} \quad -\beta'' \geq g(t, \beta, \beta'), \quad 0 \leq t \leq 1.$$

$$(C_2) \quad \text{For any } u, v \in C^2[[0,1], \mathbb{R}], \quad g_u(t, u, v), \quad g_v(t, u, v) \quad \text{are continuous,} \\ g_u(t, u, v) \leq 0 \quad \text{and} \quad g_u(t, u, v) \neq 0.$$

$$(C_3) \quad z(t) > 0 \quad \text{for } t \in [0,1], \quad z'(0) \leq 0, \quad z'(1) \geq 0 \quad \text{and for every } \lambda > 0$$

$$(i) \quad -\lambda z'' > g(t, \beta + \lambda z, \beta' + \lambda z') - g(t, \beta, \beta'), \quad 0 \leq t \leq 1,$$

$$(ii) \quad -\lambda z'' > g(t, m, m') - g(t, m - \lambda z, m' - \lambda z'), \quad 0 \leq t \leq 1.$$

For a given $v \in C[[0,1], \mathbb{R}]$, let $G(t, u, u')$ be defined as in (2.2).

Then we have the following results which we simply state without proofs.

Theorem 2.5. Consider the NBVP (2.17). Assume that

(i) the NBVP

$$(2.18) \quad -u'' = G(t, u, u'), \quad u'(0) = u'(1) = 0$$

has a solution u for every lower solution v of (2.17).

(ii) $r(t)$ is the maximal solution of the NBVP (2.17).

Then (C_0) implies $m(t) \leq r(t)$ on $[0, 2\pi]$.

Theorem 2.6. Let (C_1) and (C_2) hold. Then (C_0) implies $m(t) \leq \beta(t)$ on $[0, 1]$.

Theorem 2.7. The assertion of Theorem 3.2 remains valid if (C_2) is replaced by (C_3) .

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