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**BANZHAF PERMISSION VALUES FOR
GAMES WITH PERMISSION STRUCTURE**

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Abstract

A *game with a permission structure* describes a situation in which cooperation possibilities in a cooperative game with transferable utility are limited because there are players that need permission from other players before they are allowed to cooperate. In the *conjunctive approach* it is assumed that every player needs permission from *all* its direct superiors. In the *disjunctive approach* it is assumed that each player only needs permission from *at least one* of its direct superiors. Both approaches yield different modified games which take account of the limited cooperation possibilities. Applying the *Banzhaf value* to these modified games yields allocation rules that distribute the payoffs in games with a permission structure over the individual players. In this paper we provide axiomatic characterizations of these *Banzhaf permission values*.

Keywords: TU-game, Banzhaf value, permission structure, permission value.

1 Introduction

A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v: 2^N \rightarrow \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. The collection of all characteristic functions on a particular player set N is denoted by $\mathcal{G}^N$.

In a TU-game there are no restrictions on the cooperation possibilities of the players, i.e., every coalition $E \subset N$ is formable and can generate a payoff. An example of models in which there are restrictions on the possibilities of cooperation are the *games in coalition structure* as considered in, e.g., Aumann and Drèze (1974), Owen (1977), Hart and Kurz (1983) and Winter (1989). In these models it is assumed that the players are part of a coalition structure being a partition of the players into disjoint sets. These sets can be seen as social groups such that for a particular player it is more easy to cooperate with players in his own group than to cooperate with players in other groups. Another example of models with restricted cooperation possibilities are the games with *limited communication structure* in which the edges of an undirected graph on the set of players represent binary communication links between the players. Whether players can cooperate or not then depends on their position in the communication graph. For

such models we refer to, e.g., Myerson (1977), Kalai, Postlewaite, and Roberts (1978), Owen (1986) and Borm, Owen, and Tijs (1992).

This paper considers *games with a permission structure* in which it is assumed that players in a TU-game are part of a hierarchical organization in which there are players that need permission from other players before they are allowed to cooperate within a coalition¹. Thus the possibilities of coalition formation are determined by the positions of the players in this so-called *permission structure*. Various assumptions can be made about how a permission structure affects the cooperation possibilities. In the *conjunctive approach*, as developed in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996), it is assumed that every player needs permission from *all* its direct superiors before it is allowed to cooperate with other players. Alternatively, in the *disjunctive approach*, as considered in Gilles and Owen (1994) and van den Brink (1997) it is assumed that every player needs permission from *at least one* of its direct superiors before it is allowed to cooperate with other players.

Given a game and a permission structure a modified game is defined which takes account of the limited cooperation possibilities. The conjunctive and disjunctive approach yield different modified games. An *allocation rule* for games with a permission structure is a function that assigns to every game with a permission structure a distribution of the payoffs that can be obtained by cooperation over the individual players. Applying solutions for TU-games (being function that assign a payoff distribution to every TU-game) to these modified games yields allocation rules for games with a permission structure. Applying, for example, the *Shapley value* (Shapley (1953)) to these modified games yields the *conjunctive* and *disjunctive permission value* which have been characterized in van den Brink and Gilles (1996) and van den Brink (1997), respectively.

Another well known solution for TU-games is the *Banzhaf value* which is based on the Banzhaf index for voting games (Banzhaf (1965)), and is generalized to arbitrary games by, e.g., Owen (1975) and Dubey and Shapley (1979). Axiomatizations of the Banzhaf value for TU-games can be found in, e.g., Lehrer (1988) and Haller (1994).

¹Another model in which cooperation possibilities in a TU-game are limited by some hierarchical structure on the set of players can be found in Faigle and Kern (1993).

Applying the Banzhaf value to the modified games in the conjunctive and disjunctive approach yields the *conjunctive*, respectively, *disjunctive Banzhaf permission value*. In this paper we give axiomatic characterizations of these Banzhaf permission values.

The paper is organized as follows. In Section 2 we briefly discuss the conjunctive and disjunctive approach to games with a permission structure and restate known characterizations of the conjunctive and disjunctive (Shapley) permission values on games with a *hierarchical* (i.e, acyclic and quasi-strongly connected) permission structure. In these characterizations the disjunctive and conjunctive permission values differ with respect to the ‘fairness’ axiom that is used. *Disjunctive fairness* (just referred to as *fairness*) implies that, under certain conditions, deleting the relation between a player and one of its direct superiors changes the payoffs of these two players by the same amount. It is satisfied by the disjunctive permission value, but not by the conjunctive permission value. Instead, the conjunctive permission value satisfies *conjunctive fairness* implying that deleting the relation between a player and one of its direct superiors changes the payoffs of this player and each of its *other* direct superiors by the same amount. This is not satisfied by the disjunctive permission value.

In Section 3 we show that for monotone games with a permission structure the changes in the disjunctive and conjunctive (Shapley) permission values of two direct superiors of a player change in opposite direction if we delete its relation with one of the two superiors. This is not true for arbitrary games with a permission structure. However, the conjunctive and disjunctive Banzhaf permission values satisfy this property for all games with a hierarchical permission structure. Moreover, the absolute values of the changes in the Banzhaf permission values of two direct superiors of a player are the same after deleting its relation with one of the two. We refer to this axiom as *predecessor fairness*.

In Section 4 we characterize the disjunctive Banzhaf permission value using, a.o, fairness, predecessor fairness, and two axioms that are related to the *neutrality* axioms used in characterizing the Banzhaf value for TU-games in Lehrer (1988) and Haller (1994).

Section 5 provides a characterization of the conjunctive Banzhaf permission

value.

Finally, in Section 6 we show how predecessor fairness and the neutrality axioms can be adapted to give new characterizations of the (Shapley) permission values.

2 Preliminaries on games with a permission structure

We assume that players who participate in a TU-game are part of a hierarchical organization in which there are players that need permission from certain other players before they are allowed to cooperate. For a finite set of players $N \subset \mathbb{N}$ such a hierarchical organization is represented by a pair (N, S) , where the mapping $S: N \rightarrow 2^N$ is called a *permission structure* on N . The players in $S(i)$ are called the *successors* of player $i \in N$ in permission structure S . The players in $S^{-1}(i) := \{j \in N \mid i \in S(j)\}$ are called the *predecessors* of i in S . By $\hat{S}$ we denote the *transitive closure* of the permission structure S , i.e., $j \in \hat{S}(i)$ if and only if there exists a sequence of players $(h_1, \dots, h_t)$ such that $h_1 = i$, $h_{k+1} \in S(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = j$. The players in $\hat{S}(i)$ are called the *subordinates* of i in S , and the players in $\hat{S}^{-1}(i) := \{j \in N \mid i \in \hat{S}(j)\}$ are called the *superiors* of i in S . In this paper we restrict our attention to a special class of permission structures.

Definition 2.1 *A permission structure S on N is **hierarchical** if it is*

- (i) **acyclic**, i.e., $i \notin \hat{S}(i)$ for all $i \in N$, and
- (ii) **quasi-strongly connected**, i.e., there exists an $i \in N$ such that $\hat{S}(i) = N \setminus \{i\}$.

We denote the collection of all hierarchical permission structures on a particular player set N by $\mathcal{S}_H^N$. In van den Brink and Gilles (1994) it is shown that in a hierarchical permission structure there exists a *unique* player i_0 such that $\hat{S}(i_0) = N \setminus \{i_0\}$. Moreover, $S^{-1}(i_0) = \emptyset$ for this player. We call this player the *top-player* in the permission structure. Since in this paper we only consider hierarchical permission structures we will

often refer to these simply as permission structures². A triple (N, v, S) with $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$ is called a *game with a (hierarchical) permission structure*.

2.1 Disjunctive and conjunctive restrictions

In the *disjunctive approach*, as developed in Gilles and Owen (1994) and van den Brink (1997), it is assumed that each player needs permission from *at least one* of its predecessors before it is allowed to cooperate with other players. Consequently, a coalition is formable if and only if every player in the coalition, except the top-player i_0 , has a predecessor who also belongs to the coalition. (Note that this implies that the unique top-player i_0 belongs to the coalition.) Thus, the formable coalitions are the ones in the set

$$\Phi_{N,S}^d := \left\{ E \subset N \mid S^{-1}(i) \cap E \neq \emptyset \text{ or } S^{-1}(i) = \emptyset \text{ for all } i \in E \right\}. \quad (1)$$

The coalitions in $\Phi_{N,S}^d$ are called the *disjunctive autonomous* coalitions in S . The largest disjunctive autonomous subset of $E \subset N$ in $S \in \mathcal{S}_H^N$ is denoted by $\sigma_{N,S}^d(E) = \cup \{ F \in \Phi_{N,S}^d \mid F \subset E \}$, and is called the *disjunctive sovereign part* of E in S . It consists of those players in E that can be reached by a directed ‘permission path’ starting from the top-player such that all players on this path belong to coalition E . Using this concept we can transform the characteristic function v into a modified characteristic function which takes account of the limited cooperation possibilities as determined by the permission structure S . Given a game with a permission structure (N, v, S) , the *disjunctive restriction* of v on S is the characteristic function $r_{N,v,S}^d: 2^N \rightarrow \mathbb{R}$ given by $r_{N,v,S}^d(E) = v(\sigma_{N,S}^d(E))$ for all $E \subset N$, i.e., in the restricted game each coalition obtains the worth of its disjunctive sovereign part.

Alternatively, in the *conjunctive approach*, as developed in Gilles, Owen, and van den Brink (1992) and van den Brink and Gilles (1996), it is assumed that each player needs permission from *all* its predecessors before it is allowed to cooperate. This implies that a coalition E is formable if and only if for every player in the coalition it

²The results in this paper could be stated for acyclic permission structures that not necessarily are quasi-strongly connected. For notational convenience we restrict attention to permission structures satisfying both conditions of Definition 2.1.

holds that all its predecessors belong to the coalition. The set of formable coalitions in this approach thus is given by

$$\Phi_{N,S}^c := \{E \subset N \mid S^{-1}(i) \subset E \text{ for all } i \in E\}. \quad (2)$$

The coalitions in the set $\Phi_{N,S}^c$ are called the *conjunctive autonomous* coalitions in S . The largest conjunctive autonomous subset of E is denoted by $\sigma_{N,S}^c(E) = \cup\{F \in \Phi_{N,S}^c \mid F \subset E\}$, and is referred to as the *conjunctive sovereign part* of E in S . It consists of all players in E whose superiors all belong to E . Given game with permission structure (N, v, S) , the *conjunctive restriction* of v on S is the characteristic function $r_{N,v,S}^c: 2^N \rightarrow \mathbb{R}$ given by $r_{N,v,S}^c(E) = v(\sigma_{N,S}^c(E))$ for all $E \subset N$.

Example 2.2 Consider the game with permission structure (N, v, S) with $N = \{1, 2, 3, 4\}$, $v \in \mathcal{G}^N$ given by $v = u_{\{4\}}$, where u_T denotes the *unanimity game*³ of coalition $T \subset N$, and $S \in \mathcal{S}_H^N$ given by $S(1) = \{2, 3\}$, $S(2) = S(3) = \{4\}$, $S(4) = \emptyset$.

The disjunctive and conjunctive restrictions of v on S , respectively, are given by

$$r_{N,v,S}^d(E) = \begin{cases} 1 & \text{if } E \in \{\{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$r_{N,v,S}^c(E) = \begin{cases} 1 & \text{if } E = \{1, 2, 3, 4\} \\ 0 & \text{otherwise.} \end{cases}$$

2.2 The (Shapley) permission values

An *allocation rule* for games with a permission structure is a function that assigns to every game with a permission structure (N, v, S) a distribution of the payoffs that can be obtained by cooperation according to v taking into account the limited cooperation possibilities determined by S . The *disjunctive permission value* φ^d is obtained by applying the *Shapley value* (Shapley (1953)) to the disjunctive restricted games, while

³The unanimity game $u_T \in \mathcal{G}^N$ of $T \subset N$ is given by $u_T(E) = 1$ if $E \supset T$, and $u_T(E) = 0$ otherwise.

the *conjunctive permission value* φ^c is obtained by applying the Shapley value to the conjunctive restricted games, i.e.,

$$\varphi^d(N, v, S) = Sh(N, r_{N,v,S}^d) \text{ and } \varphi^c(N, v, S) = Sh(N, r_{N,v,S}^c),$$

where

$$Sh_i(N, v) = \sum_{\substack{E \subset N \\ E \ni i}} \frac{(|N| - |E|)! (|E| - 1)!}{|N|!} (v(E) - v(E \setminus \{i\})), \text{ for all } i \in N.$$

Example 2.3 The disjunctive and conjunctive permission values of the game with permission structure given in Example 2.2 are

$$\varphi^d(N, v, S) = \frac{1}{12}(5, 1, 1, 5) \text{ and } \varphi^c(N, v, S) = \frac{1}{4}(1, 1, 1, 1).$$

The disjunctive permission value is axiomatized in van den Brink (1997), while axiomatizations of the conjunctive permission value can be found in van den Brink and Gilles (1996) and van den Brink (1999)⁴. The first two axioms that are satisfied by both allocation rules are straightforward generalizations of the efficiency and additivity axioms of solutions for TU-games.

Axiom 2.4 (Efficiency) For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$,

$$\sum_{i \in N} f_i(N, v, S) = v(N).$$

Axiom 2.5 (Additivity) For every $N \subset \mathbb{N}$, $v, w \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$,

$$f(N, v + w, S) = f(N, v, S) + f(N, w, S),$$

where $(v + w) \in \mathcal{G}^N$ is defined by $(v + w)(E) = v(E) + w(E)$ for all $E \subset N$.

⁴The results cited in this section originally are stated for games with a permission structure on a fixed player set N . For notational convenience and uniformity with other results of this paper we restate all results for a variable player set.

Player $i \in N$ is a *null player* in game (N, v) if $v(E) = v(E \setminus \{i\})$ for all $E \subset N$. Player $i \in N$ is *inessential* in game with permission structure (N, v, S) if i and all its subordinates are null players in (N, v) , i.e., if $v(E) = v(E \setminus \{j\})$ for all $E \subset N$ and $j \in \{i\} \cup \widehat{S}(i)$. The next axiom requires that inessential players earn a zero payoff.

Axiom 2.6 (Inessential player property) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $i \in N$ is an inessential player in (N, v, S) , then $f_i(N, v, S) = 0$.*

The next two axioms are stated for monotone characteristic functions. A characteristic function $v \in \mathcal{G}^N$ is *monotone* if $v(E) \leq v(F)$ for all $E \subset F \subset N$. The class of all monotone characteristic functions on N is denoted by $\mathcal{G}_M^N$. Player $i \in N$ is called *necessary* in game (N, v) if any coalition not containing player i generates a zero worth, i.e., if $v(E) = 0$ for all $E \subset N \setminus \{i\}$. The next axiom states that a necessary player in a monotone game with permission structure always earns at least as much as any other player.

Axiom 2.7 (Necessary player property) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}_M^N$ and $S \in \mathcal{S}_H^N$, if $i \in N$ is a necessary player in v then $f_i(N, v, S) \geq f_j(N, v, S)$ for all $j \in N$.*

We say that player $i \in N$ dominates player $j \in N$ ‘completely’ if all directed ‘permission paths’ from the top-player i_0 to player j contain player i . We denote the set of players that player i dominates ‘completely’ by $\overline{S}(i)$, i.e.,

$$\overline{S}(i) = \left\{ j \in \widehat{S}(i) \mid \begin{array}{l} i \in \{i_0, i_1, \dots, i_{t-1}\} \text{ for every sequence of nodes } i_0, i_1, \dots, i_t \\ \text{such that } i_t = j \text{ and } i_k \in S(i_{k-1}), k \in \{1, \dots, t\}, \end{array} \right\}.$$

We also define $\overline{S}^{-1}(i) = \{j \in \widehat{S}^{-1}(i) \mid i \in \overline{S}(j)\}$. The next axiom states that in a monotone game with permission structure a player always earns at least as much as any of the players it dominates completely.

Axiom 2.8 (Weak structural monotonicity) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}_M^N$ and $S \in \mathcal{S}_H^N$, if $i, j \in N$ are such that $j \in \overline{S}(i)$, then $f_i(N, v, S) \geq f_j(N, v, S)$.*

The five axioms given above are satisfied by both the disjunctive and conjunctive permission values. Next we mention an axiom that is satisfied by the disjunctive but not by the conjunctive permission value. It states that deleting the relation between two players h and $j \in S(h)$ (with $|S^{-1}(j)| \geq 2$) changes the payoffs of players h and j by the same amount. Moreover, also the payoffs of all players i that ‘completely’ dominate player h , in the sense that $h \in \bar{S}(i)$, change by this same amount. We refer to this axiom as *fairness* because it is related to the fairness axiom introduced in Myerson (1977) for games with a limited communication structure⁵. For $S \in \mathcal{S}_H^N$, $h \in N$ and $j \in S(h)$ (with $|S^{-1}(j)| \geq 2$) we denote $S_{-(h,j)}(h) = S(h) \setminus \{j\}$ and $S_{-(h,j)}(i) = S(i)$ for $i \in N \setminus \{h\}$.

Axiom 2.9 (Fairness) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h, j \in N$ are such that $j \in S(h)$ and $|S^{-1}(j)| \geq 2$, then*

$$f_i(N, v, S) - f_i(N, v, S_{-(h,j)}) = f_j(N, v, S) - f_j(N, v, S_{-(h,j)}) \text{ for all } i \in \{h\} \cup \bar{S}^{-1}(h).$$

Theorem 2.10 (van den Brink (1997)) *An allocation rule f is equal to the disjunctive permission value φ^d if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and fairness.*

The conjunctive permission value does not satisfy fairness. However, it satisfies the alternative axiom which states that deleting the relation between two players h and $j \in S(h)$ (with $|S^{-1}(j)| \geq 2$) changes the payoffs of player j and any *other* predecessor $g \in S^{-1}(j) \setminus \{h\}$ by the same amount. Moreover, also the payoffs of all players that ‘completely’ dominate the other predecessor g change by this same amount.

Axiom 2.11 (Conjunctive fairness) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h, j, g \in N$ are such that $h \neq g$ and $j \in S(h) \cap S(g)$, then*

$$f_i(N, v, S) - f_i(N, v, S_{-(h,j)}) = f_j(N, v, S) - f_j(N, v, S_{-(h,j)}) \text{ for all } i \in \{g\} \cup \bar{S}^{-1}(g).$$

⁵In that model fairness means that deleting a communication relation between two players in a symmetric communication graph has the same effect on both their payoffs. Note that in our fairness property we require that the successor on the relation to be deleted has at least two predecessors.

Theorem 2.12 (van den Brink (1999)) *An allocation rule f is equal to the conjunctive permission value φ^c if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.*

In van den Brink and Gilles (1996) it is shown that on the class of all games with an arbitrary (not necessarily hierarchical) permission structure, the conjunctive permission value is characterized by efficiency, additivity, the inessential player property, the necessary player property and *structural monotonicity*. This last property is a strengthening of weak structural monotonicity and states that a player in a monotone game with a permission structure always earns at least as much as any of its successors, i.e., $f_i(N, v, S) \geq f_j(N, v, S)$ for all $i \in N$ and $j \in S(i)$. The disjunctive permission value does not satisfy structural monotonicity.

3 Predecessor fairness and Banzhaf permission values

Fairness and conjunctive fairness compare the effects of deleting the relation between players h and $j \in S(h)$ on the payoffs of players h and j , and on the payoffs of players $g \in S^{-1}(j) \setminus \{h\}$ and j , respectively. These properties do not compare the change in payoffs of players h and $g \in S^{-1}(j) \setminus \{h\}$ after deleting the relation between h and j . For monotone games the changes in the disjunctive and conjunctive permission values of these players are opposite.

Proposition 3.1 *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h, g, j \in N$ with $h \neq g$ and $j \in S(h) \cap S(g)$,*

1. $(\varphi_j^d(v, S) - \varphi_j^d(v, S_{-(h,j)})) \geq 0$, $(\varphi_h^d(v, S) - \varphi_h^d(v, S_{-(h,j)})) \geq 0$, and
 $(\varphi_g^d(N, v, S) - \varphi_g^d(N, v, S_{-(h,j)})) \leq 0$;
2. $(\varphi_j^c(N, v, S) - \varphi_j^c(N, v, S_{-(h,j)})) \leq 0$, $(\varphi_h^c(N, v, S) - \varphi_h^c(N, v, S_{-(h,j)})) \geq 0$, and
 $(\varphi_g^c(N, v, S) - \varphi_g^c(N, v, S_{-(h,j)})) \leq 0$.

PROOF

Let $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h, g, j \in N$ with $h \neq g$ and $j \in S(h) \cap S(g)$. For every $E \subset N$ we denote $p_E^N = \frac{(|N|-|E|)!(|E|-1)!}{|N|!}$.

Since $\sigma_{N,S}^d(E) \supset \sigma_{N,S_{-(h,j)}}^d(E)$ for all $E \subset N$, with $\sigma_{N,S}^d(E) = \sigma_{N,S_{-(h,j)}}^d(E)$ if $E \not\ni \{h, j\}$, it follows for monotone v that

$$\begin{aligned} & (\varphi_j^d(N, v, S) - \varphi_j^d(N, v, S_{-(h,j)})) = Sh_j(N, r_{N,v,S}^d) - Sh_j(N, r_{N,v,S_{-(h,j)}}^d) \\ &= \sum_{E \ni j} p_E^N (v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S}^d(E \setminus \{j\})) - v(\sigma_{N,S_{-(h,j)}}^d(E)) + v(\sigma_{N,S_{-(h,j)}}^d(E \setminus \{j\}))) \\ &= \sum_{E \ni j} p_E^N (v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S_{-(h,j)}}^d(E))) \geq 0, \end{aligned}$$

the same for player h , and since also $\sigma_{N,S}^d(E) = \sigma_{N,S_{-(h,j)}}^d(E)$ if $E \ni g$,

$$\begin{aligned} & (\varphi_g^d(N, v, S) - \varphi_g^d(N, v, S_{-(h,j)})) = Sh_g(N, r_{N,v,S}^d) - Sh_g(N, r_{N,v,S_{-(h,j)}}^d) \\ &= \sum_{E \ni g} p_E^N (v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S}^d(E \setminus \{g\})) - v(\sigma_{N,S_{-(h,j)}}^d(E)) + v(\sigma_{N,S_{-(h,j)}}^d(E \setminus \{g\}))) \\ &= \sum_{E \ni g} p_E^N (v(\sigma_{N,S_{-(h,j)}}^d(E \setminus \{g\})) - v(\sigma_{N,S}^d(E \setminus \{g\}))) \leq 0. \end{aligned}$$

For the conjunctive approach, $\sigma_{N,S}^c(E) \subset \sigma_{N,S_{-(h,j)}}^c(E)$ for all $E \subset N$, with $\sigma_{N,S}^c(E) = \sigma_{N,S_{-(h,j)}}^c(E)$ if $E \not\ni \{g, j\}$. Thus, for monotone v

$$(\varphi_j^c(N, v, S) - \varphi_j^c(N, v, S_{-(h,j)})) = \sum_{E \ni j} p_E^N (v(\sigma_{N,S}^c(E)) - v(\sigma_{N,S_{-(h,j)}}^c(E))) \leq 0,$$

the same for player g , and since also $\sigma_{N,S}^c(E) = \sigma_{N,S_{-(h,j)}}^c(E)$ if $E \ni h$,

$$(\varphi_h^c(N, v, S) - \varphi_h^c(N, v, S_{-(h,j)})) = \sum_{E \ni h} p_E^N (v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{h\})) - v(\sigma_{N,S}^c(E \setminus \{h\}))) \geq 0.$$

□

As a consequence we have that deleting the relation between player h and $j \in S(h)$ (with $|S^{-1}(j)| \geq 2$) in a monotone game with a hierarchical permission structure

changes the disjunctive and conjunctive permission values of players h and $g \in S^{-1}(j) \setminus \{h\}$ in opposite direction⁶. A similar result holds for games which characteristic function v satisfies $v(E) \geq v(F)$ if $E \subset F$. The fairness properties discussed in the previous section hold for all games with a hierarchical permission structure. This is not the case for Proposition 3.1 and the ‘opposite change’ consequence described here.

Example 3.2 Consider the game with permission structure (N, v, S) given by $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{4,5\}} - \frac{7}{10}u_{\{4\}}$, where u_T again denotes the unanimity game of $T \subset N$, and $S(1) = \{2, 3, 5\}$, $S(2) = S(3) = \{4\}$ and $S(4) = S(5) = \emptyset$. In this example, $\varphi_2^d(N, v, S) - \varphi_2^d(N, v, S_{-(2,4)}) = -\frac{1}{120} - 0 < 0$ and $\varphi_3^d(N, v, S) - \varphi_3^d(N, v, S_{-(2,4)}) = -\frac{1}{120} - \frac{1}{60} = -\frac{1}{40} < 0$. Thus, deleting the relation between players 2 and 4 changes the disjunctive permission values of player 4’s predecessors 2 and 3 in the same direction. (By fairness the change in the disjunctive permission value of player 4 is the same as for player 2: $\varphi_4^d(N, v, S) - \varphi_4^d(N, v, S_{-(2,4)}) = \frac{1}{120} - \frac{1}{60} = -\frac{1}{120}$.) For the conjunctive permission values $\varphi_2^c(N, v, S) - \varphi_2^c(N, v, S_{-(2,4)}) = \frac{1}{40} - 0 > 0$ and $\varphi_3^c(N, v, S) - \varphi_3^c(N, v, S_{-(2,4)}) = \frac{1}{40} - \frac{1}{60} = \frac{1}{120} > 0$. Thus, deleting the relation between players 2 and 4 also changes the conjunctive permission values of player 4’s predecessors 2 and 3 in the same direction. (By conjunctive fairness the change in the conjunctive permission value of player 4 is the same as for player 3: $\varphi_4^c(N, v, S) - \varphi_4^c(N, v, S_{-(2,4)}) = \frac{1}{40} - \frac{1}{60} = \frac{1}{120}$.)

It turns out that applying the *Banzhaf value* to the conjunctive and disjunctive restricted games yields allocation rules for games with a permission structure that satisfy the ‘opposite change’ property for every game with a hierarchical permission structure. The Banzhaf value for TU-games is given by

$$B_i(N, v) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni i}} (v(E) - v(E \setminus \{i\})), \text{ for all } i \in N. \quad (3)$$

⁶Together with fairness and conjunctive fairness, this implies that deleting the relation between players h and $j \in S(h) \cap S(g)$, $h \neq g$, changes the disjunctive permission values of players g and j and the conjunctive permission values of players h and j in opposite directions.

The *disjunctive Banzhaf permission value* β^d and the *conjunctive Banzhaf permission value* β^c are obtained by applying the Banzhaf value to the disjunctive and conjunctive restricted games, respectively, yielding

$$\beta^d(N, v, S) = B(N, r_{N,v,S}^d) \quad \text{and} \quad \beta^c(N, v, S) = B(N, r_{N,v,S}^c).$$

Example 3.3 The disjunctive and conjunctive Banzhaf permission values of the game with permission structure given in Example 2.2 are

$$\beta^d(N, v, S) = \frac{1}{8}(3, 1, 1, 3) \quad \text{and} \quad \beta^c(N, v, S) = \frac{1}{8}(1, 1, 1, 1).$$

The disjunctive and conjunctive Banzhaf permission values of players h and g , $h \neq g$, always change in opposite direction after deleting the relation between players h and $j \in S(h) \cap S(g)$. Moreover, the absolute values of the changes in the Banzhaf permission values of these two players are the same.

Axiom 3.4 (Predecessor fairness) For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h, g, j \in N$ are such that $h \neq g$ and $j \in S(h) \cap S(g)$, then

$$f_h(N, v, S) - f_h(N, v, S_{-(h,j)}) = f_g(N, v, S_{-(h,j)}) - f_g(N, v, S).$$

Theorem 3.5 The disjunctive and conjunctive Banzhaf permission values satisfy predecessor fairness.

PROOF

Let $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h, g, j \in N$ be such that $j \in S(h) \cap S(g)$ and $h \neq g$. Since $\sigma_{N,S}^d(E) = \sigma_{N,S_{-(h,j)}}^d(E)$ if $E \not\ni h$ or $E \ni g$, it follows that

$$\begin{aligned} \beta_h^d(N, v, S) - \beta_h^d(N, v, S_{-(h,j)}) &= B_h(N, r_{N,v,S}^d) - B_h(N, r_{N,v,S_{-(h,j)}}^d) = \\ &= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni h}} \left(v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S}^d(E \setminus \{h\})) - v(\sigma_{N,S_{-(h,j)}}^d(E)) + v(\sigma_{N,S_{-(h,j)}}^d(E \setminus \{h\})) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni h}} \left(v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S_{-(h,j)}}^d(E)) \right) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni h, E \not\ni g}} \left(v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S_{-(h,j)}}^d(E)) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \not\ni g}} \left(v(\sigma_{N,S}^d(E)) - v(\sigma_{N,S_{-(h,j)}}^d(E)) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni g}} \left(v(\sigma_{N,S}^d(E \setminus \{g\})) - v(\sigma_{N,S_{-(h,j)}}^d(E \setminus \{g\})) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni g}} \left(v(\sigma_{N,S_{-(h,j)}}^d(E)) - v(\sigma_{N,S_{-(h,j)}}^d(E \setminus \{g\})) - v(\sigma_{N,S}^d(E)) + v(\sigma_{N,S}^d(E \setminus \{g\})) \right) \\
&= B_g(N, r_{N,v,S_{-(h,j)}}^d) - B_g(N, r_{N,v,S}^d) = \beta_g^d(N, v, S_{-(h,j)}) - \beta_g^d(N, v, S).
\end{aligned}$$

This shows that β^d satisfies predecessor fairness.

To show that β^c satisfies predecessor fairness note that $\sigma_{N,S}^c(E) = \sigma_{N,S_{-(h,j)}}^c(E)$ if $E \ni h$ or $E \not\ni g$. But then, in a similar way as above for β^d , it follows that

$$\begin{aligned}
&\beta_h^c(N, v, S) - \beta_h^c(N, v, S_{-(h,j)}) = B_h(N, r_{N,v,S}^c) - B_h(N, r_{N,v,S_{-(h,j)}}^c) = \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni h}} \left(v(\sigma_{N,S}^c(E)) - v(\sigma_{N,S}^c(E \setminus \{h\})) - v(\sigma_{N,S_{-(h,j)}}^c(E)) + v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{h\})) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni h}} \left(v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{h\})) - v(\sigma_{N,S}^c(E \setminus \{h\})) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \supset \{h,g\}}} \left(v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{h\})) - v(\sigma_{N,S}^c(E \setminus \{h\})) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni g, E \not\ni h}} \left(v(\sigma_{N,S_{-(h,j)}}^c(E)) - v(\sigma_{N,S}^c(E)) \right) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni g}} \left(v(\sigma_{N,S_{-(h,j)}}^c(E)) - v(\sigma_{N,S}^c(E)) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni g}} \left(v(\sigma_{N,S_{-(h,j)}}^c(E)) - v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{g\})) - v(\sigma_{N,S}^c(E)) + v(\sigma_{N,S}^c(E \setminus \{g\})) \right)
\end{aligned}$$

$$= B_g(N, r_{N,v,S_{-(h,j)}}^c) - B_g(N, r_{N,v,S}^c) = \beta_g^c(N, v, S_{-(h,j)}) - \beta_g^c(N, v, S).$$

This shows that β^c satisfies predecessor fairness. □

Since β^d satisfies fairness and predecessor fairness, $\beta_j^d(N, v, S) - \beta_j^d(N, v, S_{-(h,j)}) = \beta_g^d(N, v, S_{-(h,j)}) - \beta_g^d(N, v, S)$ for every game with permission structure (N, v, S) and $h, g, j \in N$ such that $j \in S(h) \cap S(g)$, $h \neq g$. Note that β^c does not satisfy this property. However, by β^c satisfying conjunctive fairness and predecessor fairness, $\beta_j^c(N, v, S) - \beta_j^c(N, v, S_{-(h,j)}) = \beta_h^c(N, v, S_{-(h,j)}) - \beta_h^c(N, v, S)$ for every game with permission structure (N, v, S) and $h, g, j \in N$ such that $j \in S(h) \cap S(g)$, $h \neq g$, which is not satisfied by β^d .

4 An axiomatization of the disjunctive Banzhaf permission value

Main purpose of this paper is to axiomatize the Banzhaf permission values. In this section we focus on the disjunctive Banzhaf permission value. Compared to the disjunctive (Shapley) permission value it satisfies the axioms stated in Theorem 2.10 except efficiency⁷. However, it satisfies efficiency for one player games with a permission structure. Note that $S \in \mathcal{S}_H^N$ with $|N| = 1$ implies that $S(i) = \emptyset$ for $i \in N$, and thus $r_{N,v,S}^d = v$ in that case.

Axiom 4.1 (One player efficiency) *For every $N \subset \mathbb{N}$ with $|N| = 1$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, $f_i(N, v, S) = v(\{i\})$ for $i \in N$.*

Replacing efficiency in Theorem 2.10 by one player efficiency and predecessor fairness is not sufficient to characterize the disjunctive Banzhaf permission value as can be seen from Example 4.9.2 and 4.9.3 at the end of this section. To characterize the disjunctive

⁷Remember that the Banzhaf value is not efficient for TU-games. The efficient *normalized Banzhaf value* for TU-games which distributes the worth of the ‘grand coalition’ proportional to the Banzhaf values of the players, is characterized in van den Brink and van der Laan (1998).

Banzhaf permission value we introduce two more axioms that, in particular, will be applied to permission *trees*, i.e., hierarchical permission structures in which each player has at most one predecessor. These two axioms are related to *amalgamation neutrality* and *collusion neutrality* as used, respectively, in Lehrer (1988) and Haller (1994) to characterize the Banzhaf value for TU-games. First, we introduce a similar neutrality property that is satisfied by the Banzhaf value for TU-games. Suppose that player $h \in N \setminus \mathbb{N}$ is entering the game as a veto player for player $j \in N$. Then the sum of the Banzhaf values of players h and j in the new game is equal to the Banzhaf value of player j in the original game. To be more precise, for game (N, v) and players $j \in N$ and $h \in \mathbb{N} \setminus N$, let the characteristic function $v_{hj} \in \mathcal{G}^{N \cup \{h\}}$ be given by

$$v_{hj}(E) = \begin{cases} v(E \setminus \{j\}) & \text{if } E \subset N \\ v(E \cap N) & \text{if } E \subset N \cup \{h\}, E \ni h. \end{cases} \quad (4)$$

Proposition 4.2 *If $N \subset \mathbb{N}$, $j \in N$, $h \in \mathbb{N} \setminus N$ and $v \in \mathcal{G}^N$, then*

$$B_h(N \cup \{h\}, v_{hj}) + B_j(N \cup \{h\}, v_{hj}) = B_j(N, v),$$

where $v_{hj} \in \mathcal{G}^{N \cup \{h\}}$ is given by (4).

PROOF

If $N \subset \mathbb{N}$, $j \in N$, $h \in \mathbb{N} \setminus N$ and $v \in \mathcal{G}^N$, then $v_{hj}(E) - v_{hj}(E \setminus \{h\}) = 0$ if $E \not\ni j$, and $v_{hj}(E) - v_{hj}(E \setminus \{j\}) = 0$ if $E \not\ni h$. Thus

$$\begin{aligned} & B_h(N \cup \{h\}, v_{hj}) + B_j(N \cup \{h\}, v_{hj}) = \\ &= \frac{1}{2^{|N|}} \left(\sum_{\substack{E \subset N \cup \{h\} \\ E \ni h}} (v_{hj}(E) - v_{hj}(E \setminus \{h\})) + \sum_{\substack{E \subset N \cup \{h\} \\ E \ni j}} (v_{hj}(E) - v_{hj}(E \setminus \{j\})) \right) \\ &= \frac{1}{2^{|N|}} \left(\sum_{\substack{E \subset N \cup \{h\} \\ E \supset \{h, j\}}} (v_{hj}(E) - v_{hj}(E \setminus \{h\})) + \sum_{\substack{E \subset N \cup \{h\} \\ E \supset \{h, j\}}} (v_{hj}(E) - v_{hj}(E \setminus \{j\})) \right) \\ &= \frac{1}{2^{|N|}} \sum_{\substack{E \subset N \cup \{h\} \\ E \supset \{h, j\}}} (v(E \cap N) - v((E \cap N) \setminus \{j\}) + v(E \cap N) - v((E \cap N) \setminus \{h\})) \end{aligned}$$

$$= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni j}} (v(E) - v(E \setminus \{j\})) = B_j(N, v).$$

□

Next we apply this idea to permission structures in two ways. First, suppose that all subordinates of player $j \in N$ are null players (note that this does not imply that j is an inessential player because j need not be a null player itself). Now, let $h \in \mathbb{N} \setminus N$ be a new player whose only task is to supervise player j , i.e., h will be a null player in the new game and in the permission structure he becomes the only predecessor of j and gets all previous predecessors of j as its predecessors. Then the sum of the payoffs of players h and j in the new game with permission structure is equal to the payoff of player j in the original game with permission structure.

Axiom 4.3 (Vertical neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h \in \mathbb{N} \setminus N$, if $j \in N$ and all $i \in \hat{S}(j)$ are null players in (N, v) , then*

$$f_j(N \cup \{h\}, v', S') + f_h(N \cup \{h\}, v', S') = f_j(N, v, S),$$

where $v' \in \mathcal{G}^{N \cup \{h\}}$ is given by $v'(E) = v(E \cap N)$ for all $E \subset N \cup \{h\}$, and $S' \in \mathcal{S}_H^{N \cup \{h\}}$ is given by

$$S'(i) = \begin{cases} \{j\} & \text{if } i = h \\ (S(i) \setminus \{j\}) \cup \{h\} & \text{if } i \in S^{-1}(j) \\ S(i) & \text{if } i \in N \setminus S^{-1}(j). \end{cases}$$

For the second neutrality property, suppose that player $j \in N$, with $S(j) = \emptyset$ and $S^{-1}(j) \neq \emptyset$, is split in two in the sense that a new player h enters who gets the same successors (none) and predecessors as j , and in the new game h and j veto each other. Then, again, the sum of the payoffs of players h and j in the new game with permission structure is equal to the payoff of player j in the original game with permission structure.

Axiom 4.4 (Horizontal neutrality) For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h \in \mathbb{N} \setminus N$, if $S(j) = \emptyset$ and $S^{-1}(j) \neq \emptyset$ for $j \in N$, then

$$f_j(N \cup \{h\}, v'', S'') + f_h(N \cup \{h\}, v'', S'') = f_j(N, v, S),$$

where $v'' \in \mathcal{G}^{N \cup \{h\}}$ is given by $v''(E) = \begin{cases} v(E \setminus \{j\}) & \text{if } E \subset N \\ v(E \cap N) & \text{if } E \subset N \cup \{h\}, E \ni h, \end{cases}$

and $S'' \in \mathcal{S}_H^{N \cup \{h\}}$ is given by $S''(i) = \begin{cases} \emptyset & \text{if } i = h \\ S(i) \cup \{h\} & \text{if } i \in S^{-1}(j) \\ S(i) & \text{if } i \in N \setminus S^{-1}(j). \end{cases}$

Replacing efficiency in Theorem 2.10 by one player efficiency, predecessor fairness, vertical neutrality and horizontal neutrality characterizes the disjunctive Banzhaf permission value β^d . First, we show that β^d satisfies the axioms.

Theorem 4.5 *The disjunctive Banzhaf permission value β^d satisfies one player efficiency, vertical neutrality, horizontal neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity, fairness and predecessor fairness.*

PROOF

Proving that the disjunctive Banzhaf permission value satisfies additivity, the inessential player property, the necessary player property and weak structural monotonicity is along the same lines as this is shown for the (Shapley) disjunctive permission value in van den Brink (1997). One player efficiency is evident. Predecessor fairness follows from Theorem 3.5.

Vertical neutrality follows from Proposition 4.2 and the fact that $r_{N, v', S'}^d = (r_{N, v, S}^d)_{hj}$ where v' and S' are as given in Axiom 4.3, and v_{hj} is as given in (4). Similarly, horizontal neutrality follows from Proposition 4.2 and the fact that $r_{N, v'', S''}^d = (r_{N, v, S}^d)_{hj}$ where v'' and S'' are as given in Axiom 4.4.

So, we are left to show that β^d satisfies fairness. Let $S \in \mathcal{S}_H^N$ and $h, j \in N$ be such that $j \in S(h)$ and $|S^{-1}(j)| \geq 2$. Further, let $i \in \{h\} \cup \overline{S}^{-1}(h)$.

Since $E \not\supset \{i, j\}$ implies that $\sigma_{N,S}^d(E) = \sigma_{N,S_{-(h,j)}}^d(E)$, it follows that

$$\begin{aligned}
\beta_i^d(N, v, S) - \beta_i^d(N, v, S_{-(h,j)}) &= B_i(N, r_{N,v,S}^d) - B_i(N, r_{N,v,S_{-(h,j)}}^d) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni i}} \left(v(\sigma_{N,v,S}^d(E)) - v(\sigma_{N,v,S}^d(E \setminus \{i\})) - v(\sigma_{N,v,S_{-(h,j)}}^d(E)) + v(\sigma_{N,v,S_{-(h,j)}}^d(E \setminus \{i\})) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni i}} \left(v(\sigma_{N,v,S}^d(E)) - v(\sigma_{N,v,S_{-(h,j)}}^d(E)) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \supset \{i,j\}}} \left(v(\sigma_{N,v,S}^d(E)) - v(\sigma_{N,v,S_{-(h,j)}}^d(E)) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni j}} \left(v(\sigma_{N,v,S}^d(E)) - v(\sigma_{N,v,S}^d(E \setminus \{j\})) - v(\sigma_{N,v,S_{-(h,j)}}^d(E)) + v(\sigma_{N,v,S_{-(h,j)}}^d(E \setminus \{j\})) \right) \\
&= \beta_j^d(N, v, S) - \beta_j^d(N, v, S_{-(h,j)}).
\end{aligned}$$

So, β^d satisfies fairness. □

Next we show that the axioms mentioned in Theorem 4.5 characterize the disjunctive Banzhaf permission value. We do this in some steps. First, we prove the following lemma for positively scaled unanimity games with a permission tree that has no inessential players. We denote by $\mathcal{S}_T^N = \{S \in \mathcal{S}_H^N \mid \text{for all } i \in N, |S^{-1}(i)| \leq 1\}$ the class of all permission trees on N . For $S \in \mathcal{S}_H^N$ and $h \in N$ the permission structure $S_{-h} \in \mathcal{S}^{N \setminus \{h\}}$ is given by

$$S_{-h}(i) = \begin{cases} (S(i) \setminus \{h\}) \cup S(h) & \text{if } i \in S^{-1}(h) \\ S(i) & \text{if } i \in N \setminus (\{h\} \cup S^{-1}(h)). \end{cases} \quad (5)$$

Lemma 4.6 *If the allocation rule f satisfies one player efficiency, vertical neutrality, horizontal neutrality, the necessary player property, weak structural monotonicity and predecessor fairness, then $f(N, v, S)$ is uniquely determined for all $S \in \mathcal{S}_T^N$ and $v = c_T u_T$ for some $T \subset N$ satisfying $T \cup \hat{S}^{-1}(T) = N$ and $c_T \geq 0$.*

PROOF

Suppose that the allocation rule f satisfies the six axioms. Consider a hierarchical permission (tree) structure $S \in \mathcal{S}_T^N$ and the monotone characteristic function $w_T = c_T u_T$, $T \subset N$, $c_T \geq 0$, i.e.,

$$w_T(E) = \begin{cases} c_T & \text{if } E \supset T \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Suppose that $T \cup \widehat{S}^{-1}(T) = N$. (So, there are no inessential players.) Clearly, f satisfying the necessary player property implies that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(N, w_T, S) = c^*$ for all $i \in T$, and $f_i(N, w_T, S) \leq c^*$ for all $i \in N \setminus T$. But then weak structural monotonicity and the fact that $\widehat{S}(i) = \overline{S}(i)$ for all $i \in N$ and $S \in \mathcal{S}_T^N$ imply that $f_i(N, w_T, S) = c^*$ for all $i \in T \cup \overline{S}^{-1}(T) = T \cup \widehat{S}^{-1}(T) = N$. So, we have determined $f(N, w_T, S)$ if we determine c^* . We do this by induction on $|N|$.

If $|N| = 1$ then one player efficiency implies that $f_i(N, w_T, S) = c_T$ for $i \in N$.

Proceeding by induction assume that $f_i(N', w_{T'}, S') = \frac{c_{T'}}{2^{|N'|-1}}$ for all $(N', w_{T'}, S')$ with $S' \in \mathcal{S}_{T'}^{N'}$, $c_{T'} \geq 0$ and $|N'| < |N|$.

We distinguish the following three cases (of which at least one must occur).

- (i) Suppose there exist $h, g, j \in N$ such that $S(h) = S(j) = \emptyset$ and $\{h, j\} \subset S(g)$. (Note that this case can only occur if $|N| \geq 3$.) By the assumption that $T \cup \widehat{S}^{-1}(T) = N$, we have $\{h, j\} \subset T$. Horizontal neutrality implies that

$$f_j(N, w_T, S) + f_h(N, w_T, S) = f_j(N \setminus \{h\}, c_T u_{T \setminus \{h\}}, S_{-h}).$$

With the induction hypothesis and the fact that $f_j(N, w_T, S) = f_h(N, w_T, S) = c^*$ this yields that $2c^* = \frac{c_T}{2^{|N|-2}}$, and thus $f_i(N, w_T, S) = c^* = \frac{c_T}{2^{|N|-1}}$ for all $i \in N$.

- (ii) Suppose that there exist $j \in T$ and $h \in N \setminus T$ with $S(j) = \emptyset$ and $S(h) = \{j\}$. Vertical neutrality implies that

$$f_j(N, w_T, S) + f_h(N, w_T, S) = f_j(N \setminus \{h\}, w_T, S_{-h}).$$

With the induction hypothesis and the fact that $f_j(N, w_T, S) = f_h(N, w_T, S) = c^*$ this yields that $2c^* = \frac{c_T}{2^{|N|-2}}$, and thus $f_i(N, w_T, S) = c^* = \frac{c_T}{2^{|N|-1}}$ for all $i \in N$.

- (iii) Suppose that there exist $h, j \in T$ with $S(j) = \emptyset$ and $S(h) = \{j\}$. Then take a $g \in N \setminus N$, and define $S', S'' \in \mathcal{S}^{N \cup \{g\}}$ by

$$S'(i) = \begin{cases} \{h\} & \text{if } i = g \\ (S(i) \setminus \{h\}) \cup \{g\} & \text{if } i \in S^{-1}(h) \\ S(i) & \text{if } i \in N \setminus S^{-1}(h) \end{cases}$$

and

$$S''(i) = \begin{cases} \{h, j\} & \text{if } i = g \\ (S(i) \setminus \{h\}) \cup \{g\} & \text{if } i \in S^{-1}(h) \\ S(i) & \text{if } i \in N \setminus S^{-1}(h) \end{cases}$$

(Note that $S'' \notin \mathcal{S}_T^{N \cup \{g\}}$.) The necessary player property and weak structural monotonicity imply that

$$f_h(N \cup \{g\}, w_T, S'') = f_j(N \cup \{g\}, w_T, S'') = f_g(N \cup \{g\}, w_T, S''). \quad (7)$$

Since $((N \cup \{g\}) \setminus \{h\}, c_T u_{T \setminus \{h\}}, S''_{-h}) \in \mathcal{S}_T^{(N \cup \{g\}) \setminus \{h\}}$ (with $|(N \cup \{g\}) \setminus \{h\}| = |N|$) is as considered in case (ii) it follows from that case that

$$f_j((N \cup \{g\}) \setminus \{h\}, c_T u_{T \setminus \{h\}}, S''_{-h}) = f_g((N \cup \{g\}) \setminus \{h\}, c_T u_{T \setminus \{h\}}, S''_{-h}) = \frac{c_T}{2^{|N|-1}}.$$

But then horizontal neutrality implies that $f_h(N \cup \{g\}, w_T, S''_{-(h,j)}) + f_j(N \cup \{g\}, w_T, S''_{-(h,j)}) = f_j((N \cup \{g\}) \setminus \{h\}, c_T u_{T \setminus \{h\}}, S''_{-h}) = \frac{c_T}{2^{|N|-1}}$. With the necessary player property and weak structural monotonicity this yields

$$f_h(N \cup \{g\}, w_T, S''_{-(h,j)}) = f_j(N \cup \{g\}, w_T, S''_{-(h,j)}) = f_g(N \cup \{g\}, w_T, S''_{-(h,j)}) = \frac{c_T}{2^{|N|}}. \quad (8)$$

Now, predecessor fairness implies that

$$f_h(N \cup \{g\}, w_T, S'') - f_h(N \cup \{g\}, w_T, S''_{-(h,j)}) = f_g(N \cup \{g\}, w_T, S''_{-(h,j)}) - f_g(N \cup \{g\}, w_T, S''),$$

which with (7) and (8) gives

$$f_h(N \cup \{g\}, w_T, S'') = f_g(N \cup \{g\}, w_T, S'') = f_j(N \cup \{g\}, w_T, S'') = \frac{c_T}{2^{|N|}}. \quad (9)$$

Again applying predecessor fairness yields

$$f_g(N \cup \{g\}, w_T, S'') - f_g(N \cup \{g\}, w_T, S') = f_h(N \cup \{g\}, w_T, S') - f_h(N \cup \{g\}, w_T, S''),$$

which with the necessary player property, weak structural monotonicity and (9) yields

$$f_g(N \cup \{g\}, w_T, S') = f_h(N \cup \{g\}, w_T, S') = f_j(N \cup \{g\}, w_T, S') = \frac{c_T}{2^{|N|}}.$$

Finally, vertical neutrality yields

$$c^* = f_h(N, w_T, S) = f_h(N \cup \{g\}, w_T, S') + f_g(N \cup \{g\}, w_T, S') = \frac{c_T}{2^{|N|-1}},$$

and thus $f_i(N, w_T, S) = c^* = \frac{c_T}{2^{|N|-1}}$ for all $i \in N$.

So, for $S \in \mathcal{S}_T^N$ and $w_T = c_T u_T$ with $c_T \geq 0$ and $T \cup \hat{S}^{-1}(T) = N$, we conclude that $f_i(N, w_T, S) = c^* = \frac{c_T}{2^{|N|-1}} = \beta_i^d(N, w_T, S)$ for all $i \in T \cup \hat{S}^{-1}(T) = N$. □

The next step is to show that adding the inessential player property and fairness to the axioms implies that f is uniquely determined for all positively scaled unanimity games with a permission tree.

Lemma 4.7 *If the allocation rule f satisfies one player efficiency, vertical neutrality, horizontal neutrality, the inessential player property, the necessary player property, weak structural monotonicity, fairness and predecessor fairness, then $f(N, v, S)$ is uniquely determined whenever $S \in \mathcal{S}_T^N$ and $v = c_T u_T$ for some $T \subset N$ and $c_T \geq 0$.*

PROOF

Suppose that the allocation rule f satisfies the eight axioms. Consider a permission (tree) structure $S \in \mathcal{S}_T^N$ and monotone characteristic function $w_T = c_T u_T$ as given in (6) for some $c_T \geq 0$. Denote $\alpha_S(T) = T \cup \hat{S}^{-1}(T)$.

Since all players in $N \setminus \alpha_S(T)$ are inessential players in (N, w_T, S) , the inessential player property implies that $f_i(N, w_T, S) = 0$ for all $i \in N \setminus \alpha_S(T)$. Further, f satisfying the necessary player property and weak structural monotonicity and the fact that $\hat{S}(i) = \bar{S}(i)$ for all $i \in N$ and $S \in \mathcal{S}_T^N$ imply that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(N, w_T, S) = c^*$ for all $i \in \alpha_S(T)$. So,

$$f_i(N, w_T, S) = \begin{cases} c^* & \text{if } i \in \alpha_S(T) = T \cup \hat{S}^{-1}(T) \\ 0 & \text{if } i \in N \setminus \alpha_S(T) \end{cases} \quad (10)$$

If $T = \emptyset$ then $\alpha_S(T) = \emptyset$, and thus $f_i(N, w_\emptyset, S) = 0$ for all $i \in N$.

Suppose that $T \neq \emptyset$. We prove that c^* (and thus $f(N, w_T, S)$) is uniquely determined by induction on $|N \setminus \alpha_S(T)|$.

If $|N \setminus \alpha_S(T)| = 0$ then c^* (and thus $f(N, w_T, S)$) is uniquely determined by Lemma 4.6.

Proceeding by induction assume that $c^* = f_{i_0}(N', w_{T'}, S')$ is uniquely determined for all $(N', w_{T'}, S')$ with $S' \in \mathcal{S}_{T'}^{N'}$, $c_{T'} \geq 0$ and $|N' \setminus \alpha_{S'}(T')| < |N \setminus \alpha_S(T)|$.

Since $N \setminus \alpha_S(T) \neq \emptyset$ there exists a $j \in N \setminus \alpha_S(T)$ with $S(j) = \emptyset$. We distinguish the following three cases (of which at least one must occur).

- (i) Suppose there exists an $h \in \alpha_S(T) \cap S(i_0)$ such that $j \notin \hat{S}(h)$. (Remember that i_0 denotes the unique top-player in the permission structure.) Define $S' \in \mathcal{S}_H^N$ by

$$S'(i) = \begin{cases} \{h\} & \text{if } i = j \\ S(i) & \text{otherwise.} \end{cases}$$

(Note that $S' \notin \mathcal{S}_T^N$.) Fairness implies that

$$f_{i_0}(N, w_T, S') - f_{i_0}(N, w_T, S) = f_j(N, w_T, S') - f_j(N, w_T, S). \quad (11)$$

Predecessor fairness implies that

$$f_{i_0}(N, w_T, S) - f_{i_0}(N, w_T, S') = f_j(N, w_T, S') - f_j(N, w_T, S). \quad (12)$$

So, from (11) and (12) it follows that

$$f_{i_0}(N, w_T, S) = f_{i_0}(N, w_T, S') \text{ and } f_j(N, w_T, S) = f_j(N, w_T, S'). \quad (13)$$

With the inessential player property it then follows that

$$f_j(N, w_T, S') = f_j(N, w_T, S) = 0. \quad (14)$$

Predecessor fairness also yields that

$$f_{i_0}(N, w_T, S') - f_{i_0}(N, w_T, S'_{-(i_0, h)}) = f_j(N, w_T, S'_{-(i_0, h)}) - f_j(N, w_T, S'). \quad (15)$$

Since $|N \setminus \alpha_{S'_{-(i_0, h)}}(T)| < |N \setminus \alpha_S(T)|$ and $S'_{-(i_0, h)} \in \mathcal{S}_T^N$, the induction hypothesis, (14) and (15) yield that $f_{i_0}(N, w_T, S') = f_{i_0}(N, w_T, S'_{-(i_0, h)}) + f_j(N, w_T, S'_{-(i_0, h)})$ is uniquely determined. So, with (13) also $c^* = f_{i_0}(N, w_T, S)$ is uniquely determined.

- (ii) Suppose that there exists an $h \in \alpha_S(T) \cap S(i_0)$ such that $j \in \widehat{S}(h)$. Take $g \in \widehat{S}(h) \cap (S^{-1}(j) \cup \{j\})$, and define $S' \in \mathcal{S}_H^N$ by

$$S'(i) = \begin{cases} S(i) \cup \{g\} & \text{if } i = i_0 \\ S(i) & \text{otherwise.} \end{cases}$$

(Note that again $S' \notin \mathcal{S}_T^N$.) Fairness implies that

$$f_{i_0}(N, w_T, S') - f_{i_0}(N, w_T, S'_{-(h, g)}) = f_h(N, w_T, S') - f_h(N, w_T, S'_{-(h, g)}),$$

while predecessor fairness implies that

$$f_{i_0}(N, w_T, S'_{-(h, g)}) - f_{i_0}(N, w_T, S') = f_h(N, w_T, S') - f_h(N, w_T, S'_{-(h, g)}).$$

So, $f_{i_0}(N, w_T, S') = f_{i_0}(N, w_T, S'_{-(h,g)})$ and $f_h(N, w_T, S') = f_h(N, w_T, S'_{-(h,g)})$. Since $S'_{-(h,g)}$ is as in case (i), we determined $f_{i_0}(N, w_T, S')$ and $f_h(N, w_T, S')$.

With predecessor fairness we then also determined

$$c^* = f_{i_0}(N, w_T, S) = f_h(N, w_T, S') - f_h(N, w_T, S) + f_{i_0}(N, w_T, S').$$

- (iii) Finally, we must consider the case $\alpha_S(T) = \{i_0\}$. Define the game with permission structure $(N \cup \{h\}, v', S')$ where, for some $h \in \mathbb{N} \setminus N$, $v' \in \mathcal{G}^{N \cup \{h\}}$ is given by $v'(E) = v(E \setminus \{i_0\}) = 0$ for all $E \subset N$, and $v'(E) = v(E \cap N)$ for all $E \subset N \cup \{h\}$ with $E \ni h$, and $S' \in \mathcal{S}_T^{N \cup \{h\}}$ is given by $S'(h) = \{i_0\}$ and $S'(i) = S(i)$ for all $i \in N$. Vertical neutrality, the necessary player property and weak structural monotonicity imply that $f_{i_0}(N, v, S) = f_{i_0}(N \cup \{h\}, v', S') + f_h(N \cup \{h\}, v', S') = 2f_{i_0}(N \cup \{h\}, v', S')$.

Since $(N \cup \{h\}, v', S')$ is as considered in case (ii), we determined $c^* = f_{i_0}(N, v, S)$.

So, in all three cases we uniquely determined c^* and thus, with (10) we uniquely determined $f(N, w_T, S)$.

□

Finally, adding additivity to the axioms of Lemma 4.7, we state the main result.

Theorem 4.8 *An allocation rule f is equal to the disjunctive Banzhaf permission value β^d if and only if it satisfies one player efficiency, vertical neutrality, horizontal neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity, fairness and predecessor fairness.*

PROOF

With Theorem 4.5, in order to prove the theorem we now have to show that there can be at most one allocation rule that satisfies the nine axioms stated in the theorem.

Therefore, suppose that allocation rule f satisfies the nine axioms.

Consider the hierarchical permission structure $S \in \mathcal{S}_H^N$ and the monotone characteristic function $w_T = c_T u_T$, $c_T \geq 0$, as given in (6).

If $T = \emptyset$ then the inessential player property implies that $f_i(N, w_T, S) = 0$ for all $i \in N$. Suppose that $T \neq \emptyset$.

Again, we denote by $\alpha_S(T) = T \cup \widehat{S}^{-1}(T)$ the set consisting of all players in T and all their superiors. For $S \in \mathcal{S}_H^N \setminus \mathcal{S}_T^N$ it need not hold that $\widehat{S}_i^{-1}(T)$ equals $\overline{S}_i^{-1}(T)$ for all $i \in N$. Therefore, by $\gamma_S(T) := \{i \in \alpha_S(T) \mid T \cap (\{i\} \cup \overline{S}(i)) \neq \emptyset\}$ we denote the set of those players in $\alpha_S(T)$ who belong to T or have subordinates in T that they dominate ‘completely’.

Again, the inessential player property implies that $f_i(N, w_T, S) = 0$ for all $i \in N \setminus \alpha_S(T)$. Further, f satisfying the necessary player property and weak structural monotonicity implies that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(N, w_T, S) = c^*$ for all $i \in \gamma_S(T)$. We prove that c^* and all $f_i(N, w_T, S)$, $i \in \alpha_S(T)$, are uniquely determined by induction on the number $\sum_{i \in N} |S(i)|$.

If $\sum_{i \in N} |S(i)| = |N| - 1$ then $S \in \mathcal{S}_T^N$ and c^* and all $f_i(N, w_T, S)$, $i \in \alpha_S(T)$, are uniquely determined by Lemma 4.7. (Note that $\sum_{i \in N} |S(i)| \geq |N| - 1$ for all $S \in \mathcal{S}_H^N$.)

Proceeding by induction assume that $f(N, w_T, S')$ is uniquely determined for all $S' \in \mathcal{S}_H^N$ with $\sum_{i \in N} |S'(i)| < \sum_{i \in N} |S(i)|$.

Next we recursively define the sets L_k , $k \in \{0\} \cup \mathbb{N}$, by

$$L_0 := \emptyset, \text{ and}$$

$$L_k := \left\{ i \in N \setminus \bigcup_{t=1}^{k-1} L_t \mid S(i) \subset \bigcup_{t=1}^{k-1} L_t \right\}, \text{ for all } k \in \mathbb{N}.$$

In van den Brink and Gilles (1994) it is shown that for hierarchical permission structures there exists an $M < \infty$ such that the sets $L_1, \dots, L_M$ form a partition of N consisting of non-empty sets only.

Next we describe a procedure which determines the values $f_i(N, w_T, S)$ as linear functions of the constant c^* for all $i \in N$.

STEP 1 For every $i \in L_1$ one of the following two conditions is satisfied:

- (i) If $i \in N \setminus \alpha_S(T)$ then $f_i(N, w_T, S) = 0$ as mentioned before.
- (ii) If $i \in \alpha_S(T)$ then $i \in T$ since $S(i) = \emptyset$. Thus $f_i(N, w_T, S) = c^*$.

Let $k = 2$.

STEP 2 If $L_k = \emptyset$ then STOP.

Else, for every $i \in L_k$ one of the following three conditions is satisfied:

- (i) If $i \in N \setminus \alpha_S(T)$ then $f_i(N, w_T, S) = 0$.
- (ii) If $i \in \gamma_S(T)$ then $f_i(N, w_T, S) = c^*$.
- (iii) If $i \in \alpha_S(T) \setminus \gamma_S(T)$ then by definition of $\alpha_S(T)$ and $\gamma_S(T)$ there exists an $h \in \{i\} \cup \bar{S}(i)$ and a $j \in S(h)$ such that $|S^{-1}(j)| \geq 2$. Fairness then implies that

$$f_i(N, w_T, S) - f_i(N, w_T, S_{-(h,j)}) = f_j(N, w_T, S) - f_j(N, w_T, S_{-(h,j)}).$$

Using the induction hypothesis and the fact that $j \in \hat{S}(i)$ implies that we already determined $f_j(N, w_T, S)$ as a linear function of c^* (since $j \in L_l$ with $l < k$), it follows that we have determined

$$f_i(N, w_T, S) = f_j(N, w_T, S) - f_j(N, w_T, S_{-(h,j)}) + f_i(N, w_T, S_{-(h,j)}) = c^* + c_i,$$

where $c_i = c_j + f_i(N, w_T, S_{-(h,j)}) - f_j(N, w_T, S_{-(h,j)})$ is known (for the particular choice of h and j).

STEP 3 Let $k = k + 1$. GOTO STEP 2.

Since there exists an $M < \infty$ such that the sets $L_1, \dots, L_M$ form a partition of N consisting of non-empty sets only the procedure described above determines the values $f_i(N, w_T, S)$ as linear functions of c^* for all $i \in \alpha_S(T)$, and $f_i(N, w_T, S) = 0$ for all $i \in N \setminus \alpha_S(T)$.

Next, we distinguish the following two cases.

- (i) Suppose there exists $j \in \alpha_S(T)$ with $|S^{-1}(j)| \geq 2$. Then take $h, g \in N$, $h \neq g$, such that $j \in S(h) \cap S(g)$. In the procedure above, we used fairness to show that

$$f_h(N, w_T, S) = c^* + c_j - f_j(N, w_T, S_{-(h,j)}) + f_h(N, w_T, S_{-(h,j)}), \quad (16)$$

where the induction hypothesis determines c_j , $f_j(N, w_T, S_{-(h,j)})$ and $f_h(N, w_T, S_{-(h,j)})$.

Predecessor fairness implies that

$$f_h(N, w_T, S) - f_h(N, w_T, S_{-(g,j)}) = f_g(N, w_T, S_{-(g,j)}) - f_g(N, w_T, S),$$

which with fairness yields

$$f_h(N, w_T, S) - f_h(N, w_T, S_{-(g,j)}) = f_j(N, w_T, S_{-(g,j)}) - f_j(N, w_T, S),$$

and thus

$$f_h(N, w_T, S) = f_j(N, w_T, S_{-(g,j)}) - c^* - c_j + f_h(N, w_T, S_{-(g,j)}).$$

With (16) this yields

$$c^* = \frac{1}{2}(-2c_j + f_h(N, w_T, S_{-(g,j)}) - f_h(N, w_T, S_{-(h,j)}) + f_j(N, w_T, S_{-(g,j)}) + f_j(N, w_T, S_{-(h,j)}),$$

and thus with the induction hypothesis c^* is uniquely determined.

- (ii) Suppose that $|S^{-1}(j)| = 1$ for all $j \in \alpha_S(T) \setminus \{i_0\}$. Then, by assumption, there exists $j \in N \setminus \alpha_S(T)$ with $|S^{-1}(j)| \geq 2$. Take $h \in S^{-1}(j)$. Then fairness and the inessential player property imply

$$f_{i_0}(N, w_T, S) - f_{i_0}(N, w_T, S_{-(h,j)}) = f_j(N, w_T, S) - f_j(N, w_T, S_{-(h,j)}) = 0.$$

So, $f_{i_0}(N, w_T, S) = f_{i_0}(N, w_T, S_{-(h,j)})$ is uniquely determined by the induction hypothesis. Since we already determined $f_{i_0}(N, w_T, S)$ as a linear function of c^* , we have determined c^* .

In both cases we determined c^* . Since we already determined all values $f_i(N, w_T, S)$, $i \in N$, as linear functions of c^* , we uniquely determined $f(N, w_T, S)$.

Now, let $S \in \mathcal{S}_H^N$ and consider the characteristic function $w_T = c_T u_T$ with $c_T < 0$. (Note that we cannot apply the null player property and weak structural monotonicity

to this game with permission structure, because $c_T u_T$ is not monotone if $c_T < 0$.) Let $v_0 \in \mathcal{G}^N$ denote the *null game*, i.e., $v_0(E) = 0$ for all $E \subset N$. From the inessential player property it follows that $f_i(N, v_0, S) = 0$ for all $i \in N$. Since $-w_T = -c_T u_T$ with $-c_T \geq 0$ is monotone (and thus $f(N, -w_T, S)$ is uniquely determined as shown above) and $r_{N, w_T, S}^d + r_{N, -w_T, S}^d = r_{N, v_0, S}^d$ for all $T \subset N$, additivity of f implies that $f(N, w_T, S) = f(N, v_0, S) - f(N, -w_T, S) = -f^d(N, -w_T, S)$. Since $f^d(N, -w_T, S)$ is uniquely determined, $f^d(N, w_T, S)$ is uniquely determined.

Finally, since every characteristic function $v \in \mathcal{G}^N$ can be expressed as a linear combination of unanimity games it follows with additivity of f that $f(N, v, S)$ is uniquely determined for every $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$. □

Next we show the independence of the axioms of Theorem 4.8.

Example 4.9 We show independence of the axioms of Theorem 4.8 by the following nine allocation rules for games with a hierarchical permission structure.

1. The allocation rule f given by $f_i(N, v, S) = 0$ for all $i \in N$ satisfies all axioms except one player efficiency.
2. The allocation rule f given by $f_{i_0}(N, v, S) = v(\{i_0\})$, and $f_i(N, v, S) = 0$ for all $i \in N \setminus \{i_0\}$ satisfies all axioms except vertical neutrality. (Remember that i_0 denotes the top-player in (N, S) .)
3. The allocation rule f given by $f(N, v, S) = \beta^d(N, \sum_{i \in N} v(\{i\})u_{\{i\}}, S)$ satisfies all axioms except horizontal neutrality.
4. For game (N, v) define $d(N, v) = \max\{\Delta_{(N, v)}(T) \mid T \subset N\}$, where $\Delta_{(N, v)}(T) = \sum_{F \subset T} (-1)^{|T|-|F|} v(F)$ are the Harsanyi dividends (see Harsanyi (1959)), and $D(N, v) = \{T \subset N \mid \Delta_{(N, v)}(T) = d(N, v)\}$. The allocation rule f given by $f(N, v, S) = \beta^d(N, \sum_{T \in D(N, v)} \Delta_{(N, v)}(T) u_T, S)$ satisfies all axioms except additivity.
5. The allocation rule f given by $f_i(N, v, S) = \frac{v(N)}{2^{|N|-1}}$ for all $i \in N$, satisfies all axioms except the inessential player property.

6. The allocation rule f given by $f_{i_0}(N, v, S) = v(N)$ and $f_i(N, v, S) = 0$ for all $i \in N \setminus \{i_0\}$ satisfies all axioms except the necessary player property. (Remember that i_0 denotes the top-player in (N, S) .)
7. The allocation rule f given by $f(N, v, S) = B(N, v)$ satisfies all axioms except weak structural monotonicity.
8. The conjunctive Banzhaf permission value β^c satisfies all axioms except fairness.
9. For $S \in \mathcal{S}_H^N$ and $T \subset N$ let $\mathcal{Z}_T^S = \{Z \in \mathcal{S}_H^N \mid Z^{-1}(i) = S^{-1}(i) \text{ for all } i \in N \setminus \alpha_S(T), Z^{-1}(i) \subset S^{-1}(i) \text{ for all } i \in \alpha_S(T), \text{ and } |Z^{-1}(i)| = 1 \text{ for all } i \in \alpha_S(T) \setminus \{i_0\}\}$. The allocation rule f given by $f(N, v, S) = \sum_{T \subset N} \sum_{Z \in \mathcal{Z}_T^S} \beta_i^d(N, \Delta_{(N,v)}(T) u_T, Z)$ satisfies all axioms except predecessor fairness.

We end this section by remarking that we can adapt the axioms so that the disjunctive Banzhaf permission value is axiomatized for games with a fixed player set. In that case also the notion of hierarchical permission structure should be adapted. Also we can restate the axiomatization for games with an acyclic permission structure (which is not necessarily quasi-strongly connected). For notational convenience we discussed games with a variable player set and a hierarchical permission structure.

5 The conjunctive Banzhaf permission value

As mentioned in Example 4.9.8, the conjunctive Banzhaf permission value satisfies all axioms stated in Theorem 4.8 except fairness. Similarly as with the disjunctive and conjunctive (Shapley) permission values, we can characterize the conjunctive Banzhaf permission value by replacing fairness by conjunctive fairness.

Theorem 5.1 *An allocation rule f is equal to the conjunctive Banzhaf permission value β^c if and only if it satisfies one player efficiency, vertical neutrality, horizontal neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity, conjunctive fairness and predecessor fairness.*

PROOF

Proving that the conjunctive Banzhaf permission permission value satisfies additivity, the inessential player property, the necessary player property and weak structural monotonicity is along the same lines as this is shown for the conjunctive (Shapley) permission value in van den Brink and Gilles (1996). One player efficiency is evident. Vertical and horizontal neutrality of the conjunctive Banzhaf permission value follow from similar arguments given for the disjunctive Banzhaf permission value in the proof of Theorem 4.5. Predecessor fairness follows from Theorem 3.5.

So, we are left to show that β^c satisfies conjunctive fairness. Let $S \in \mathcal{S}_H^N$ and $h, g, j \in N$ be such that $j \in S(h) \cap S(g)$, $h \neq g$. Further, let $i \in \{g\} \cup \overline{S}^{-1}(g)$. The proof is similar to the corresponding proof for fairness of the disjunctive Banzhaf permission value, but with the roles of h and g exchanged. Since $E \not\supset \{i, j\}$ implies that $\sigma_{N,S}^c(E) = \sigma_{N,S_{-(h,j)}}^c(E)$, it follows that

$$\begin{aligned}
\beta_i^c(N, v, S) - \beta_i^c(N, v, S_{-(h,j)}) &= B_i(N, r_{N,v,S}^c) - B_i(N, r_{N,v,S_{-(h,j)}}^c) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni i}} \left(v(\sigma_{N,S}^c(E)) - v(\sigma_{N,S}^c(E \setminus \{i\})) - v(\sigma_{N,S_{-(h,j)}}^c(E)) + v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{i\})) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni i}} \left(v(\sigma_{N,S}^c(E)) - v(\sigma_{N,S_{-(h,j)}}^c(E)) \right) = \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \supset \{i,j\}}} \left(v(\sigma_{N,S}^c(E)) - v(\sigma_{N,S_{-(h,j)}}^c(E)) \right) \\
&= \frac{1}{2^{|N|-1}} \sum_{\substack{E \subset N \\ E \ni j}} \left(v(\sigma_{N,S}^c(E)) - v(\sigma_{N,S}^c(E \setminus \{j\})) - v(\sigma_{N,S_{-(h,j)}}^c(E)) + v(\sigma_{N,S_{-(h,j)}}^c(E \setminus \{j\})) \right) \\
&= \beta_j^c(N, v, S) - \beta_j^c(N, v, S_{-(h,j)})
\end{aligned}$$

So, β^c satisfies conjunctive fairness.

In order to prove the theorem it must be shown that there can be at most one allocation rule that satisfies the nine axioms stated in the theorem. Since this is along the same lines as the uniqueness proof in Theorem 4.8 it is omitted.

□

6 Predecessor fairness and neutrality properties of the (Shapley) permission values

In Section 2 we mentioned some known characterizations of the (Shapley) permission values. In Sections 4 and 5 we gave characterizations of the Banzhaf permission values. From the new axioms introduced in Sections 3 and 4 the (disjunctive and conjunctive) Shapley permission values satisfy one player efficiency. They do not satisfy predecessor fairness, vertical neutrality nor horizontal neutrality. However, these three axioms can be easily adapted so that they are satisfied by the (Shapley) permission values. Instead of stating that the change in payoffs of players h and g , $h \neq g$, are the same in absolute value but have opposite sign after deleting the relation between players h and $j \in S(h) \cap S(g)$, the (Shapley) permission values satisfy the property that after deleting the relation between players h and $j \in S(h) \cap S(g)$, the payoffs of player h and the sum of the payoffs of *all* other players are the same in absolute value but have opposite sign. (In other words, the sum of the payoffs over all players does not change.)

Axiom 6.1 (Shapley predecessor fairness) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$, if $h, j \in N$ are such that $j \in S(h)$ and $|S^{-1}(j)| \geq 2$, then*

$$f_h(N, v, S) - f_h(N, v, S_{-(h,j)}) = \sum_{i \in N \setminus \{h\}} (f_i(N, v, S_{-(h,j)}) - f_i(N, v, S)).$$

In a similar way we adapt vertical and horizontal neutrality. Let player $h \in \mathbb{N} \setminus N$ be a new player in the game who becomes a supervisor for player j as described by the game with permission structure $(N \cup \{h\}, v', S')$ given in Axiom 4.3. Then the sum of the payoffs of all players (including player h) in the new game with permission structure is equal to the sum of the payoffs of all players (excluding player h) in the original game with permission structure.

Axiom 6.2 (Shapley vertical neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h \in \mathbb{N} \setminus N$, if $j \in N$ and all $i \in \hat{S}(j)$ are null players in (N, v) , then*

$$\sum_{i \in N \cup \{h\}} f_i(N \cup \{h\}, v', S') = \sum_{i \in N} f_i(N, v, S),$$

where $v' \in \mathcal{G}^{N \cup \{h\}}$ and $S' \in \mathcal{S}_H^{N \cup \{h\}}$ are as given in Axiom 4.3.

In a similar way we adapt horizontal neutrality.

Axiom 6.3 (Shapley horizontal neutrality) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $h \in \mathbb{N} \setminus N$, if $S(j) = \emptyset$ and $S^{-1}(j) \neq \emptyset$ for $j \in N$, then*

$$\sum_{i \in N \cup \{h\}} f_i(N \cup \{h\}, v'', S'') = \sum_{i \in N} f_i(N, v, S),$$

where $v'' \in \mathcal{G}^{N \cup \{h\}}$ and $S'' \in \mathcal{S}_H^{N \cup \{h\}}$ are as given in Axiom 4.4.

We conclude by giving these new characterizations of the disjunctive and conjunctive (Shapley) permission values.

Theorem 6.4 *An allocation rule f is equal to*

- (i) *the disjunctive permission value φ^d if and only if it satisfies one player efficiency, Shapley vertical neutrality, Shapley horizontal neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity, fairness and Shapley predecessor fairness;*
- (ii) *the conjunctive permission value φ^c if and only if it satisfies one player efficiency, Shapley vertical neutrality, Shapley horizontal neutrality, additivity, the inessential player property, the necessary player property, weak structural monotonicity, conjunctive fairness and Shapley predecessor fairness.*

PROOF

Since $v'(N) = v''(N) = v(N)$ for every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$, $j \in N$ and $h \in \mathbb{N} \setminus N$ such that $j \in S(h)$ and $|S^{-1}(j)| \geq 2$, φ^d and φ^c satisfying efficiency implies that they satisfy one player efficiency, Shapley vertical neutrality and Shapley horizontal neutrality. Efficiency of φ^d and φ^c also implies that they satisfy Shapley predecessor fairness. The Shapley permission values satisfying the other axioms mentioned in the Theorem follows directly from the results of Section 2.

Showing that there can be at most one allocation rule satisfying the axioms can be done in a similar way as in the proof of Lemma's 4.6 and 4.7 and Theorem 4.8.

□

Appendix: The Banzhaf value for TU-games

In Section 3 we stated a neutrality property for the Banzhaf value for TU-games with a variable player set (see Proposition 4.2) which is closely related to the neutrality properties used in Lehrer (1988) and Haller (1994) in characterizing the Banzhaf value. For games with fixed player set this property can be restated as follows. Suppose that player $h \in N$ is a null player in $v \in \mathcal{G}^N$. Then the sum of the Banzhaf values of players h and j does not change if we change v to $v_{hj} \in \mathcal{G}^N$ given by

$$v_{hj}(E) = \begin{cases} v(E \setminus \{j\}) & \text{if } E \subset N \setminus \{h\} \\ v(E) & \text{if } E \subset N \cup \{h\}, E \ni h. \end{cases}$$

Axiom 6.5 (Neutrality) *For every $N \subset \mathbb{N}$ and $v \in \mathcal{G}^N$, if $h, j \in N$ are such that h is a null player in (N, v) , then $f_h(N, v) + f_j(N, v) = f_h(N, v_{hj}) + f_j(N, v_{hj})$.*

The following axioms for TU-games are well-known. A solution $f: \mathcal{G}^N \rightarrow \mathbb{R}^N$ satisfies

- (i) **one player efficiency** if $\sum_{i \in N} f_i(N, v) = v(N)$ whenever there is at most one non-null player in (N, v) ;
- (ii) **symmetry** if $v(E \cup \{i\}) = v(E \cup \{j\})$ for all $E \subset N \setminus \{i, j\}$ implies that $f_i(N, v) = f_j(N, v)$;
- (iii) **the null player property** if $f_i(N, v) = 0$ whenever $i \in N$ is a null player in (N, v) ;
- (iv) **additivity** if $f(N, v + w) = f(N, v) + f(N, w)$ for all $v, w \in \mathcal{G}^N$.

The proof of the following theorem is straightforward and is therefore omitted.

Theorem 6.6 *The solution $f: \mathcal{G}^N \rightarrow \mathbb{R}^N$ is equal to the Banzhaf value if and only if it satisfies one player efficiency, symmetry, the null player property, neutrality and additivity.*

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