ON THE EXISTENCE OF PERIODIC QUASI SOLUTIONS
FOR FIRST ORDER SYSTEMS*

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Introduction.

Recently the method of upper and lower solutions and Lyapunov-Schmitt method have been fruitfully employed to prove the existence of periodic solutions for scalar first and second order equations in [2,4]. In this paper we shall use this technique to prove the existence of periodic solutions for first order systems which is the generalisation of Müller's result [3] for periodic case. We shall also develop monotone iterative technique to obtain coupled minimal and maximal periodic quasisolutions for system of first order equations. Further, under a uniqueness assumption, our results yield a unique periodic solution for the first order system.

2. An abstract existence result.

In this section we recall an earlier known result the form of which is present in [4].

Let E be a real Hilbert space. Consider the nonlinear operator equation

$$(2.1) \quad Lu = Nu$$

where $L: D(L) \subset E \rightarrow E$ is a linear operator and $N: D(N) \subset E \rightarrow E$ a nonlinear operator with $D(L) \cap D(N) \neq \emptyset$. Suppose that $P: E \rightarrow E_0$ is the projection operator and $H: E_1 \rightarrow E$ the partial inverse of L on E_1 . Then it is well known the problem (2.1) is equivalent to the coupled system of operator equations

$$(2.2) \quad u_1 = H(I-P)N(u_0 + u_1)$$

$$(2.3) \quad 0 = PN(u_0 + u_1).$$

Concerning the problem (2.1), we have the following existence result [1].

Theorem 2.1. Suppose that

- (i) $\|Nu\| \leq J_0$, $u \in D(N)$
- (ii) there exist $r_0, R_0 > 0$ such that $\langle N(u_0 + u_1), u_0 \rangle \geq 0$ (or ≤ 0) whenever $\|u_0\| = R_0$ and $\|u_1\| \leq r_0$ where $u_0 \in E_0$ and $u_1 \in E_1$.

Then the problem (2.1) admits at least one solution.

3. Existence of Periodic Solutions.

Consider the periodic boundary value problem (PBVP for short),

$$(3.1) \quad u' = f(t, u), \quad u(0) = u(2\pi)$$

where $f \in C([0, 2\pi] \times \mathbb{R}^n, \mathbb{R}^n)$. We assume that $i \in \{1, 2, \dots, n\}$ and all inequalities between vectors are componentwise. We list the following assumptions for convenience.

(A₁) $\alpha, \beta \in C[[0, 2\pi], \mathbb{R}^n]$, $\alpha(t) \leq \beta(t)$ on $[0, 2\pi]$ and $D_{-}\alpha_1(t) \leq f_1(t, \sigma)$,
for all σ such that $\alpha(t) \leq \sigma \leq \beta(t)$ and $\sigma_1 = \alpha_1(t)$, $t \in (0, 2\pi]$.

$D_{-}\beta_1(t) \geq f_1(t, \sigma)$, for all σ such that $\alpha(t) \leq \sigma \leq \beta(t)$ and
 $\sigma_1 = \beta_1(t)$, $t \in (0, 2\pi]$.

(A₂) (i) $\alpha(0) \leq \alpha(2\pi)$ and $\alpha \in C^1[(0, 2\pi], \mathbb{R}^n]$

(ii) $\beta(0) \geq \beta(2\pi)$ and $\beta \in C^1[(0, 2\pi], \mathbb{R}^n]$.

(A₃) (i) (a) $D^+\alpha_1(0) \geq f_1(0, \sigma) + |\alpha_1(0)| + 1$ for all σ such that

$\sigma_1 = \alpha_1(0)$ and $\alpha(0) \leq \sigma \leq \beta(0)$.

(i) (b) $\int_0^{2\pi} f_1(s, \sigma) ds \geq 0$ for all σ such that $\alpha(s) \leq \sigma \leq \beta(s)$ and
 $\sigma_1 = \alpha_1(s)$.

(ii) (a) $D^+\beta_1(0) \leq f_1(0, \sigma) - |\beta_1(0)| - 1$ for all σ such that

$\sigma_1 = \beta_1(0)$ and $\alpha(0) \leq \sigma \leq \beta(0)$.

(ii) (b) $\int_0^{2\pi} f_1(s, \sigma) ds \leq 0$ for all σ such that $\alpha(s) \leq \sigma \leq \beta(s)$ and

$\sigma_1 = \beta_1(s)$.

(A₄) (A₂)(i) (A₃)(ii)

(A₅) (A₂)(ii) and (A₃)(i)

As an application of Theorem 2.1, we have the following existence result.

Theorem 3.1. Consider the PBVP (3.1) and let (A₁) hold. Then any one of the conditions (A₂), (A₃), (A₄) or (A₅) implies that there exists a solution u of (3.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 2\pi]$.

Proof. Relative to the couple (α, β) we define a modified function $F(t, u)$ as follows. $F(t, u) = f(t, P(t, u)) + r(t, u)$, where $P_1(t, u) = \max[\alpha_1(t), \min\{u_1, \beta_1(t)\}]$ and

$$r_1(t, u) = \begin{cases} \frac{\beta_1(t) - u_1}{1 + u_1^2} & \text{if } u_1 > \beta_1(t) \\ 0 & \text{if } \alpha_1(t) \leq u_1 \leq \beta_1(t) \\ \frac{\alpha_1(t) - u_1}{1 + u_1^2} & \text{if } u_1 < \alpha_1(t) \end{cases}$$

Obviously, we have $\|F(t, u)\| \leq J$ on $[0, 2\pi] \times \mathbb{R}^n$, where J depends only on (α, β) . Let us now consider the modified PBVP

$$(3.2) \quad u' = F(t, u), \quad u(0) = u(2\pi).$$

We let $E = [L_2[0, 2\pi]]^n$, $Lu = u'$, $D(L) = [u \in E: u \text{ is absolutely continuous, } u' \in E \text{ and } u(0) = u(2\pi)]$ and N be a nonlinear operator generated by F . Then the PBVP (3.1) is equivalent to the operator equation (2.1). We note that $E_0 = N(L)$ consists of constant functions and hence E_1 , where $E = E_0 \oplus E_1$ is the class of all functions whose average is zero. The projection operator P can be defined by $Pu = \int_0^{2\pi} u(s) ds$. It is easily seen that the partial inverse operator H also exists and the assumptions required in Theorem 2.1 are satisfied. Since F is bounded we can find a positive constant A which depends only on (α, β) such that any solution u_1 of (2.2)-(2.3) satisfies the estimate $\|u_1(t)\| \leq A$ on $[0, 2\pi]$. By Theorem 2.1, it is sufficient to choose an $R_0 > 0$ such that

$$(3.3) \quad \langle N(u_0 + u_1), u_0 \rangle \geq 0 \quad (\text{or } \leq 0)$$

for all u_0 and u_1 where $|u_0| = R_0$ and $\|u_1(t)\| \leq A$ on $[0, 2\pi]$. Since $u_0 \in E_0$ and $E_0 = \langle 1 \rangle$, the relation (3.3) with ≤ 0 is equivalent to

$$(3.4) \quad \int_0^{2\pi} F_1(s, R_0 + u_1(s)) ds \leq 0 \quad \text{and} \quad \int_0^{2\pi} F_1(s, -R_0 + u_1(s)) ds \geq 0.$$

Let us now choose $R_0 > 0$, $R_0 \in \mathbb{R}_+^n$ large enough so that we have $R_0 + u_1(t) > \alpha$ on $[0, 2\pi]$ and $-R_0 + u_1(t) < \beta$ on $[0, 2\pi]$.

and $-R_0 + u_1(t) \leq \min_{[0, 2\pi]} \alpha(t)$. This choice of R_0 , in view of definition of F , reduces the inequalities in (3.4) to

$$\int_0^{2\pi} f_1(s, \sigma) ds \leq 0 \text{ for all } \sigma \text{ such that } \alpha(s) \leq \sigma \leq \beta(s) \text{ and } \sigma_1 = \beta_1(s).$$

$$\text{Also } \int_0^{2\pi} f_1(s, \sigma) ds \geq 0 \text{ for all } \sigma \text{ such that } \alpha(s) \leq \sigma \leq \beta(s) \text{ and } \sigma_1 = \alpha_1(s).$$

Thus, if (A_2) or (A_3) holds, then the PBVP (3.2) admits at least one solution $u(t)$.

We shall now show that $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 2\pi]$ where $u(t)$ is a solution of the PBVP (3.2) in which case, $u(t)$ will actually be a solution of the PBVP (3.1). We shall only prove $\alpha(t) \leq u(t)$ since proving $u(t) \leq \beta(t)$ uses similar arguments. Suppose $\alpha(t) \leq u(t)$ on $[0, 2\pi]$ is not true. Then there exists an index k , $k \in \{1, 2, \dots, n\}$, a $t_0 \in [0, 2\pi]$ and an $\epsilon > 0$ such that

$$(3.5) \quad \alpha_k(t_0) = u_k(t_0) + \epsilon \text{ and } \alpha_k(t) \leq u_k(t) + \epsilon, t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi]$, we have $D_- \alpha_k(t_0) \geq u'_k(t_0)$. Also from (3.5) we see that $\alpha_k(t_0) > u_k(t_0)$ which implies that $P_k(t_0, u(t_0)) = \alpha_k(t_0)$. Hence

$$f_k(t_0, \sigma) \geq D_- \alpha_k(t_0) \geq u'_k(t_0) = f_k(t_0, P(t_0, u(t_0))) + \frac{\alpha_k(t_0) - u_k(t_0)}{1 + u_k^2(t_0)} > f_k(t_0, \sigma),$$

for all σ such that $\alpha(t) \leq \sigma \leq \beta(t)$ and $\sigma_k = \alpha_k(t_0)$ which leads to a contradiction. If $t_0 = 0$, we obtain on the otherhand $D^+ \alpha_k(0) \leq u'_k(0)$. However, if (A_2) holds we have $\epsilon = \alpha_k(0) - u_k(0) \leq \alpha_k(2\pi) - u_k(2\pi)$. This together with $\alpha_k(t) \leq u_k(t) + \epsilon$ implies $D_- \alpha'_k(2\pi) \geq u'_k(2\pi)$ which gives a contradiction to (A_1) as before. If $(A_3)(i)$ holds, one gets,

$$\begin{aligned} f_k(0, \sigma) + |\alpha_k(0)| + 1 &\leq D^+ \alpha_k(0) \leq u'_k(0) = f_k(0, \sigma) + \frac{\alpha_k(0) - u_k(0)}{1 + u_k^2(0)} \\ &\leq f_k(0, \sigma) + |\alpha_k(0)| + \frac{1}{2} \\ &< f_k(0, \sigma) + |\alpha_k(0)| + 1. \end{aligned}$$

for all σ such that $\alpha(t) \leq \sigma \leq \beta(t)$ and $\sigma_k = \alpha_k(0)$. This leads to a contradiction. Thus, we have proved $\alpha(t) \leq u(t) \leq \beta(t)$. It is now easy to see that the theorem is true when (A_4) or (A_5) holds. Hence the proof of the theorem is complete.

4. Monotone Iterative Technique.

We shall develop in this section, monotone iterative technique which offers a constructive method to obtain coupled maximal and minimal quasi solutions of (3.1). In the special case, when f satisfies uniqueness condition, we can show that coupled maximal and minimal quasisolutions coincide and give the unique solution of the system (3.1).

To define quasisolutions of (3.1), we fix for each $i \in \{1, 2, \dots, n\}$, two nonnegative integers p_i, q_i such that $p_i + q_i = n - 1$ and split $u \in \mathbb{R}^n$ into $u = (u_i, [u]_{p_i}, [u]_{q_i})$. Then the system (3.1) becomes

$$(4.1) \quad u'_i = f_i(t, u_i, [u]_{p_i}, [u]_{q_i}), \quad u_i(0) = u_i(2\pi).$$

As before, we assume that $i = 1, 2, \dots, n$ and all inequalities between vectors are componentwise.

The functions $u, v \in C^1[[0, 2\pi], \mathbb{R}^n]$ are said to be coupled periodic quasisolutions of (4.1) if

$$(4.2) \quad u'_i = f_i(t, u_i, [u]_{p_i}, [v]_{q_i}), \quad u_i(0) = u_i(2\pi)$$

$$v'_i = f_i(t, v_i, [v]_{p_i}, [u]_{q_i}), \quad v_i(0) = v_i(2\pi).$$

The function f is said to possess a mixed quasimonotone property (mqmp for short) if for each i , $f_i(t, u_i, [u]_{p_i}, [u]_{q_i})$ is monotone nondecreasing in $[u]_{p_i}$ and monotone nonincreasing in $[u]_{q_i}$.

In order to develop monotone method for system (4.1) we need the following

assumptions which we list below.

(B₁) $\alpha, \beta \in C[[0, 2\pi], R^n]$, $\alpha(t) \leq \beta(t)$ on $[0, 2\pi]$ and

$$\begin{aligned} D_{-}\alpha_1(t) &\leq f_1(t, \alpha_1, [\alpha]_{p_1}, [\beta]_{q_1}) \\ D_{-}\beta_1(t) &\geq f_1(t, \beta_1, [\beta]_{p_1}, [\alpha]_{q_1}) \end{aligned} \quad t \in (0, 2\pi]$$

(B₂) (i) $\alpha_1(0) \leq \alpha_1(2\pi)$, $\alpha \in C^1[(0, 2\pi], R^n]$

(ii) $\beta_1(0) \geq \beta_1(2\pi)$, $\beta \in C^1[(0, 2\pi], R^n]$.

(B₃) (i) $D^{+}\alpha_1(0) > f_1(0, \alpha_1(0), [\alpha(0)]_{p_1}, [\beta(0)]_{q_1}) + M_1(\alpha_1(0) - \underline{\alpha}_{1,0})$

where $M_1 > 0$ and $\underline{\alpha}_{1,0} = \min_{t \in [0, 2\pi]} \alpha_1(t)$

$$(i)(b) \int_0^{2\pi} f_1(s, \alpha_1(s), [\alpha(s)]_{p_1}, [\beta(s)]_{q_1}) ds \geq 0$$

(ii)(a) $D^{+}\beta_1(0) < f_1(0, \beta_1(0), [\beta(0)]_{p_1}, [\alpha(0)]_{q_1}) - M_1(\bar{\beta}_{1,0} - \beta_1(0))$

where $M_1 > 0$ and $\bar{\beta}_{1,0} = \max_{t \in [0, 2\pi]} \beta_1(t)$

$$(ii)(b) \int_0^{2\pi} f_1(s, \beta_1(s), [\beta(s)]_{p_1}, [\alpha(s)]_{q_1}) ds \leq 0$$

(B₄) (B₂)(i), (B₃)(ii)(a), (b)

(B₅) (B₂)(ii), (B₃)(i)(a)(b),

(B₆) $f_1(t, u_1, [u]_{p_1}, [u]_{q_1}) - f_1(t, v_1, [u]_{p_1}, [u]_{q_1}) \geq -M_1(u_1 - v_1)$

whenever $\alpha(t) \leq u$, $v \leq \beta(t)$, $t \in [0, 2\pi]$.

Remark 4.1. We remark that when f possesses a mixed quasimonotone property the

conditions (B₁), (B₃)(ii)(a), (b) imply the corresponding condition (A₁), (A₃)

(ii)(a), (b).

To proceed further we require the following lemma on which our main discussion of this section depends on.

Lemma 4.1. Let $m \in C^1[[0, 2\pi], R]$ and $m'(t) \leq -Lm(t)$ for $0 < t \leq 2\pi$ where $L > 0$. Then (i) $m(0) \leq m(2\pi)$ implies $m(t) \leq 0$ on $[0, 2\pi]$, (ii) $m(0) > m(2\pi)$,

and $m'(0) > 0$ implies $m(t) \leq 0$ on $[0, 2\pi]$.

Proof. If the conclusion is false, there exists a $t_0 \in [0, 2\pi]$ and an $\epsilon > 0$ such that

$$m(t_0) = \epsilon, m(t) \leq \epsilon \text{ on } [0, 2\pi].$$

Let $t_0 \in (0, 2\pi]$, then $m'(t_0) \geq 0$. It then follows that

$$0 \leq m'(t_0) \leq -L m(t_0) = -L \epsilon < 0$$

which is a contradiction. If $t_0 = 0$, if (i) holds, then $m(2\pi) \geq m(0) = \epsilon$ and hence $0 \leq m'(2\pi) \leq -M m(2\pi) \leq -M \epsilon < 0$, which is a contradiction. If (ii) holds then $0 < m'(0) \leq 0$ which is a contradiction. This proves the lemma.

We are now in a position to prove the main result.

Theorem 4.1. Let (B_1) , (B_6) hold and let f possess mqmp property. Further, let any one of the condition B_2 , B_3 , B_4 or B_5 hold. Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n \rightarrow \infty} \alpha_n(t) = \rho(t)$, $\lim_{n \rightarrow \infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$ and that (ρ, r) are coupled minimal and maximal periodic quasi solutions of the PBVP (4.1). That is, if (u, v) are any coupled periodic quasi solutions of (4.1) such that $\alpha \leq u$, $v \leq w$, then

$$(4.3) \quad \alpha = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \rho \leq u, v \leq r \leq \beta_n \leq \dots \leq \beta_1 \leq \beta_0 = \beta \text{ on } [0, 2\pi]$$

Furthermore, if u is any solution of (4.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 2\pi]$ also satisfies the inequality (4.3).

Proof. For any $\eta, \mu \in [\alpha, \beta] = \{u \in C[0, 2\pi], \mathbb{R}; \alpha \leq u \leq \beta\}$, consider the PBVP

$$(4.4) \quad u_1' = G_1(t, u), u_1(0) = u_1(2\pi),$$

where $G_1(t, u) = f_1(t, \eta_1, [\eta]_{p_1}, [\mu]_{q_1}) - M_1(u_1 - \eta_1)$.

By a direct computation, it is easy to calculate the unique solution of (4.4) which is given by

$$(4.5) \quad u_1(t) = [u_1(0) + \int_0^t [f_1(s, \eta_1(s), [\eta(s)]_{p_1}, [\mu(s)]_{q_1}) + M_1 \eta_1(s)] e^{M_1 s} ds] e^{-M_1 t}$$

where

$$(4.6) \quad u_1(0) = \frac{1}{e^{M_1 \pi} - 1} \int_0^{2\pi} [f_1(s, \eta_1(s), [\eta(s)]_{p_1}, [\mu(s)]_{q_1}) + M_1 \eta_1(s)] e^{M_1 s} ds = u_1(2\pi)$$

We will show that $\underline{\alpha}_{1,0} \leq u_1(0) = u_1(2\pi) \leq \bar{\beta}_{1,0}$. It is enough to prove $\underline{\alpha}_{1,0} \leq u_{1,0}$ since a similar computation is true for $u_1(0) \leq \bar{\beta}_{1,0}$. Now using either (B_6) , and $(B_2)(i)$ or (B_6) , and $(B_3)(i)(b)$ we get

$$\begin{aligned} u_1(0) &= \frac{1}{e^{M_1 \pi} - 1} \int_0^{2\pi} [f_1(s, \eta_1(s), [\eta(s)]_{p_1}, [\mu(s)]_{q_1}) + M_1 \eta_1(s)] e^{M_1 s} ds \\ &\geq \frac{1}{e^{M_1 \pi} - 1} \int_0^{2\pi} [f_1(s, \alpha_1(s), [\alpha(s)]_{p_1}, [\beta(s)]_{q_1}) + M_1(\eta_1 - \alpha_1) + M_1 \eta_1(s)] e^{M_1 s} ds \\ &\geq \frac{M}{e^{M_1 \pi} - 1} \int_0^{2\pi} \alpha_1 e^{M_1 s} ds \geq \underline{\alpha}_{1,0}. \end{aligned}$$

This estimate on $u_1(0)$ is useful for the next part of our result.

We now define a mapping $A(\eta, \mu) = u$, where u is the unique solution of (4.4). It can be shown that

$$(a) \quad \alpha \leq A(\alpha, \beta), \quad \beta \geq A(\beta, \alpha).$$

$$(b) \quad A \text{ possesses mixed monotone property on the segment } [\alpha, \beta].$$

To prove (a) set $A(\alpha, \beta) = \alpha_1$ where α_1 is the unique solution of the PBVP (4.4) with $\eta = \alpha, \mu = \beta$. Set $p_1(t) = \alpha_1 - \alpha_{1,1}$, then

$$p_1' = \alpha_1' - \alpha_{1,1}' \leq f_1(t, \alpha_1, [\alpha]_{p_1}, [\beta]_{q_1}) - f_1(t, \alpha_1, [\alpha]_{p_1}, [\beta]_{q_1}) + M_1(\alpha_{1,1} - \alpha_1) = -M_1 p_1.$$

Now if (B_2) holds then $p_1(0) \leq p_1(2\pi)$, whereas if $(B_3)(1)(a)$ holds, we have

$$\begin{aligned} p_1'(0) &= D^+ \alpha_1(0) - D^+ \alpha_{1,1}(0) > f_1(0, \alpha_1(0), [\alpha(0)]_{p_1}, [\beta(0)]_{q_1}) + M_1(\alpha_1(0) - \alpha_{1,1}(0)) \\ &\quad - f_1(0, \alpha_1(0), [\alpha(0)]_{p_1}, [\beta(0)]_{q_1}) + M_1(\alpha_{1,1}(0) - \alpha_1(0)) \\ &= M_1(\alpha_{1,1}(0) - \alpha_1(0)) \geq 0. \end{aligned}$$

Using the estimate on $\alpha_{1,1}(0)$. Further using Lemma 4.1 we can conclude $p_1(t) \leq 0$. This proves $\alpha \leq A(\alpha, \beta)$. A similar proof holds for $\beta \geq A(\beta, \alpha)$.

To prove (b), let $\eta, \mu \in [\alpha, \beta]$ such that $\eta \leq \mu$. Suppose that $u_1 = A(\eta, \mu)$, $u_2 = A(\mu, \eta)$. Setting $p_1(t) = u_{1,1}(t) - u_{2,1}(t)$, we see that

$$\begin{aligned} p_1'(t) &= u_{1,1}'(t) - u_{2,1}'(t) = f_1(t, \eta_1, [\eta]_{p_1}, [\mu]_{q_1}) - M_1(u_{1,1} - \eta_1) \\ &\quad - f_1(t, \mu_1, [\mu]_{p_1}, [\eta]_{q_1}) + M_1(u_{2,1} - \mu_1) \\ &\leq M_1(\mu_1 - \eta_1) - M_1(u_{1,1} - u_{2,1}) + M_1(\eta_1 - \mu_1) \\ &= -M_1 p_1(t). \end{aligned}$$

using max of f and (B_6) . Also $p_1(0) = u_{1,1}(0) - u_{2,1}(0) = u_{1,1}(2\pi) - u_{2,1}(2\pi) = p_1(2\pi)$. Now using Lemma 4.1, we get $u_{1,1}(t) \leq u_{2,1}(t)$.

It is now easy to define the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha = \alpha_0$, $\beta = \beta_0$ such that $\alpha_n = A(\alpha_{n-1}, \beta_{n-1})$ and $\beta_n = A(\beta_{n-1}, \alpha_{n-1})$ and we conclude $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \beta_n \leq \dots \leq \beta_1 \leq \beta_0$ on $[0, 2\pi]$. It then follows using standard arguments that $\lim_{n \rightarrow \infty} \alpha_n(t) = \rho(t)$, $\lim_{n \rightarrow \infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$. This implies that

$$\begin{aligned} (4.7) \quad \rho'(t) &= f_1(t, \rho_1, [\rho]_{p_1}, [r]_{q_1}), \quad \rho(0) = \rho(2\pi), \\ r'(t) &= f_1(t, r_1, [r]_{p_1}, [\rho]_{q_1}), \quad r(0) = r(2\pi). \end{aligned}$$

We shall show that (ρ, r) are coupled minimal and maximal quasisolutions

respectively. Let (u, v) be any coupled solution of (4.1) such that $u, v \in [\alpha, \beta]$. Let us assume that for some integer $k > 0$, $\alpha_{k-1} \leq u, v \leq \beta_{k-1}$ on $[0, 2\pi]$. Then using $p_i = \alpha_{k,i} - u_i$, and employing mqmp of f and (B_4) we arrive at

$$\begin{aligned} p_i' &= f_i(t, \alpha_{k-1,i}, [\alpha_{k-1}]_{p_i}, [\beta_{k-1}]_{q_i}) - M_i(\alpha_{k,i} - \alpha_{k-1,i}) \\ &\quad - f_i(t, u_i, [u]_{p_i}, [v]_{q_i}) \\ &\leq M_i(u_i - \alpha_{k-1,i}) - M_i(\alpha_{k,i} - \alpha_{k-1,i}) = -M_i(\alpha_{k,i} - u_i) = -M_i p_i. \end{aligned}$$

Since $p_i(0) = p_i(2\pi)$, this implies $\alpha_{k,i} \leq u_i$ by Lemma 4.1.

Using similar arguments we can conclude $\alpha_k \leq u, v \leq \beta_k$ on $[0, 2\pi]$.

Since $\alpha = \alpha_0 \leq u, v \leq \beta_0 = \beta$ on $[0, 2\pi]$, it follows by induction

$\alpha_n \leq u, v \leq \beta_n$ on $[0, 2\pi]$ for all n . Hence we have $\rho \leq u, v \leq r$ on $[0, 2\pi]$ proving (ρ, r) are coupled minimal and maximal periodic quasi solutions of (4.1). Since any solution u of (4.1) such that $u \in [\alpha, \beta]$ can be considered as (u, u) coupled periodic quasi solutions of (4.1) we also have $\rho \leq u \leq r$ on $[0, 2\pi]$. This completes the proof.

Remark 4.2. If $q_i = 0$, then ρ, r are minimal and maximal periodic quasi solutions respectively of (4.1).

Theorem 4.2. Let the hypothesis of Theorem 4.1. hold. Further let f satisfy the following condition

$$(4.8) \quad \sum_{i=1}^n (u_i - v_i)(f_i(t, u_i, [u]_{p_i}, [v]_{q_i}) - f_i(t, v_i, [v]_{p_i}, [u]_{q_i})) \leq -L \|u - v\|^2, \quad L > 0$$

whenever $\alpha(t) \leq u(t), v(t) \leq \beta(t)$. Then there exists a unique periodic solution u of (4.1).

Proof. It is enough to prove $\rho \equiv r$ in 4.7. Consider $p = \sum_{i=1}^n (|\rho_i - r_i|)^2$.

We know $p(0) = p(2\pi)$. Also using (4.8) we get

$$p' = \sum_{i=1}^n 2(p_i - r_i)(p'_i - r'_i) \leq -2L \sum_{i=1}^n |p_i - r_i|^2.$$

This implies $p' \leq -2Lp$. Now using (1) of Lemma 4.1 we conclude $p \equiv 0$, which completes the proof.

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