MONOTONE METHOD FOR SECOND ORDER
PERIODIC BOUNDARY VALUE PROBLEMS

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1. Introduction.

In recent years, there has been an extensive study of the existence of periodic solutions [1,8-11,14,15]. In [8,11], the existence of solutions of first and second order PBVP (periodic boundary value problems) has been obtained by a novel approach of combining the classical method of lower and upper solutions and the method of alternative problems (Lyapunov-Schmidt method), which provide conditions that are easily verifiable and which covers previous known results of other authors. As a constructive method for obtaining extremal solutions of initial and boundary value problems, the monotone iterative procedure has been employed by several researchers [5-7,11-13,15]. The objective of this paper is to employ this useful technique for second order PBVP to obtain the minimal and maximal solutions as limits of monotone iterates. Our method can be used to study semilinear parabolic initial boundary value problems and other problems at resonance.

2. An Abstract Existence Result.

Let H be a real Hilbert Space with norm $\|\cdot\|$. Consider the nonlinear operator equation

$$(2.1) \quad Lx = Nx$$

where $L: D(L) \subset H \rightarrow H$ is a linear operator and $N: D(N) \subset H \rightarrow H$ is a nonlinear operator with $D(L) \cap D(N)$ nonempty. Let $H_0 = N(L)$ (null space of L) be of finite dimension and $H = H_0 \oplus H_1$. Suppose that $P: H \rightarrow H_0$ and $Q: H_1 \rightarrow H_1$ are the idempotent projection operator and the compact partial inverse of L on H_1 , respectively. Then it is a well known fact [3] that the equation (2.1) is equivalent to the following coupled system of operator equations:

$$\begin{cases} x_1 = Q(1-P)N(x_0+x_1), \\ 0 = PN(x_0+x_1) \end{cases}$$

where $x = x_0 + x_1$ with $x_0 \in H_0$ and $x_1 \in H_1$.

Concerning the existence of a solution for the problem (2.1), we have the following result [4].

Theorem 2.1. Assume that

(1) $\|Nx\| \leq J_0, x \in D(N);$

(11) There exist $r_0, R_0 > 0$ such that $\langle N(x_0+x_1), x_0 \rangle \geq 0$ (or ≤ 0) whenever

$$\|x_0\| = R_0 \text{ and } \|x_1\| \leq r_0.$$

Then, the equation (2.1) has at least one solution.

This theorem is employed as a basic tool in our discussion.

3. Existence of Solutions for 2nd Order Periodic Boundary Value Problems.

Let us consider the periodic boundary value problem (PBVP for short)

$$(3.1) \quad -u'' = f(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $f \in C([0, 2\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Relative to the lower and upper solutions α, β of the problem (3.1), we shall list the following assumptions:

$$(A_0) \quad \alpha, \beta \in C^2([0, 2\pi], \mathbb{R}), \quad \alpha(t) \leq \beta(t), \quad t \in [0, 2\pi], \quad \alpha(0) = \alpha(2\pi), \quad \beta(0) = \beta(2\pi)$$

and for $0 < t < 2\pi$,

$$-\alpha'' \leq f(t, \alpha, \alpha') \quad \text{and} \quad -\beta'' \geq f(t, \beta, \beta');$$

(A₁) f satisfies a Nagumo condition relative to α, β ; see [2].

$$(A_2) \quad (i) \quad \alpha'(0) \geq \alpha'(2\pi) \quad \text{and for } i = 0, 2\pi, \quad -\alpha''(i) \leq f(i, \alpha(i), \alpha'(i));$$

$$(ii) \quad \beta'(0) \leq \beta'(2\pi) \quad \text{and for } i = 0, 2\pi, \quad -\beta''(i) \geq f(i, \beta(i), \beta'(i));$$

In [8], the following existence result is proved.

Theorem 3.1. For the PBVP (3.1), let (A_0) , (A_1) and (A_2) be satisfied. Then there exists a solution u of (3.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ and $|u'(t)| \leq N$ on $[0, 2\pi]$ where the Nagumo constant N depends only on α, β and the Nagumo function.

The proof of Theorem 3.1 combines the two basic techniques namely, the method of lower and upper solutions and the Lyapunov-Schmidt method. The various steps involved in this interesting approach are (i) appropriate modifications of f in terms of α and β , (ii) application of the abstract existence result (Theorem 2.1) to the modified problem, (iii) use of generalized maximum principle to show the uniqueness of solutions for the modified linear problem and (iv) homotopy arguments [8].

4. Monotone Method for Second Order PBVP.

In order to develop the monotone method for obtaining the extremal periodic solutions of the problem (3.1), we shall assume that f satisfies the following condition.

(A_3) For $t \in [0, 2\pi]$, $\alpha(t) \leq \tilde{u} \leq u \leq \beta(t)$ and $|u'| \leq d$, $d > \max\{N, \max|\alpha'(t)|, \max|\beta'(t)|\}$, $f(t, u, u') - f(t, \tilde{u}, u') \geq -M(u - \tilde{u})$, for some $M > 0$.

For any $\eta \in [\alpha, \beta] = \{\eta \in C[[0, 2\pi], \mathbb{R}]: \alpha(t) \leq \eta(t) \leq \beta(t), t \in [0, 2\pi]\}$ consider the quasi-linear PBVP

$$(4.1) \quad -u'' = G(t, u, u'), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where $G(t, u, u') \equiv f(t, \eta(t), q(u')) - M(u - \eta(t))$ and $q(u') = \max[-d, \min(u', d)]$.

In view of the definition of $q(u')$, it is easy to see that G is defined on

$[0, 2\pi] \times [\alpha, \beta] \times \mathbb{R}$ and (A_3) is equivalent to

$$(A_3^*) \quad f(t, u, q(u')) - f(t, \tilde{u}, q(u')) \geq -M(u - \tilde{u}), \quad M > 0,$$

for $t \in [0, 2\pi]$, $\alpha(t) \leq \tilde{u} \leq u \leq \beta(t)$ and $u' \in \mathbb{R}$.

Relative to the PBVP (4.1), we prove the following lemmas.

Lemma 4.1. Let the assumptions $(A_0)-(A_2)$ hold. Then all the conditions $(A_0)-(A_2)$ are also verified with respect to the PBVP (4.1) (i.e., $(A_0)-(A_2)$ hold with f replaced by G).

Proof. When (A_0) and (A_3) are satisfied, we have on $[0, 2\pi]$,

$$\begin{aligned} G(t, \alpha, \alpha') - f(t, \alpha, \alpha') &= f(t, \eta, q(\alpha')) - M(\alpha - \eta) - f(t, \alpha, \alpha') \\ &\geq -M(\eta - \alpha) - M(\alpha - \eta) = 0, \end{aligned}$$

since $|\alpha'| \leq d$ and $q(\alpha') = \alpha'$. Similarly, using (A_0) and (A_3) , and the fact that $q(\beta') = \beta'$, one can show that $G(t, \beta, \beta') \leq f(t, \beta, \beta')$. This, in view of (A_0) , implies

$$-\alpha'' \leq G(t, \alpha, \alpha'), \quad \beta'' \geq G(t, \beta, \beta'), \quad \text{for } t \in (0, 2\pi)$$

and the boundary conditions are the same as in (A_0) . Since $f(t, \alpha, \alpha') \leq G(t, \alpha, \alpha')$ and $f(t, \beta, \beta') \geq G(t, \beta, \beta')$, it is easy to see that (A_2) is satisfied with G replacing f .

When (A_1) holds, we have

$$(4.2) \quad |f(t, u, u')| \leq h(|u'|), \quad t \in [0, 2\pi], \quad \alpha(t) \leq u \leq \beta(t) \quad \text{and} \quad u' \in \mathbb{R},$$

where $h \in C[\mathbb{R}_+, (0, \infty)]$ and $\int_0^\infty \frac{s}{h(s)} ds = \infty$. This implies that there exists a

positive constant N depending only on α, β and h such that any solution $u \in C^2[[0, 2\pi], \mathbb{R}]$ of $-u'' = f(t, u, u')$ with $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 2\pi]$ satisfies $|u'(t)| \leq N$, $t \in [0, 2\pi]$. Because of (4.2) and the definition of G , we get

$$|G(t, u, u')| \leq h(|q(u')|) + Mr,$$

for $t \in [0, 2\pi]$, $\alpha(t) \leq u \leq \beta(t)$ and $u' \in \mathbb{R}$. Observe that

$$\int_0^\infty \frac{s}{h_0(s)} ds = \int_0^d \frac{s}{h(s)} ds + \int_d^\infty \frac{s}{h(d)} ds = \infty,$$

where $h_0(|u'|) = h(|q(u')|)$. Since $\liminf_{s \rightarrow \infty} h_0(s)$ is positive, there exists $K > 0$ such that $\frac{\phi(s)}{h_0(s)} \leq K$ for all s , where $\phi(s) \equiv Mr$ and $\frac{\phi(s) + h_0(s)}{s} \leq (1+K) \frac{h_0(s)}{s}$. As a consequence,

$$\int_0^\infty \frac{s}{h_0(s) + Mr} ds \geq \frac{1}{1+K} \int_0^\infty \frac{s}{h_0(s)} ds = \infty,$$

proving that G also satisfies a Nagumo condition. The proof of the lemma is complete.

Lemma 4.2. Let (A_0) , (A_1) , (A_2) and (A_3) hold. Then there exists a solution u of (4.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ and $|u'(t)| \leq \tilde{N}$ on $[0, 2\pi]$.

Furthermore, the solution $u(t)$ is unique.

Proof. By Lemma 4.1, we have all the assumptions (A_0) to (A_2) satisfied relative to PBVP (4.1). Now, applying Theorem 3.1, we obtain the existence of a solution $u = u(t)$ of (4.1) with $\alpha(t) \leq u(t) \leq \beta(t)$ and $|u'(t)| \leq \tilde{N}$ on $[0, 2\pi]$. It remains to show that $u(t)$ is unique.

Let us suppose that for any $\eta \in [\alpha, \beta]$, the PBVP (4.1) admits two solutions $u_1(t)$, $u_2(t)$ with $u_1, u_2 \in [\alpha, \beta]$. We shall prove that $u_1(t) \equiv u_2(t)$ by showing (i) $u_1(t) \leq u_2(t)$ and (ii) $u_2(t) \leq u_1(t)$. If (i) is not true, there must exist

a $t_0 \in [0, 2\pi]$ and a $\varepsilon > 0$ such that

$$u_1(t_0) = u_2(t_0) + \varepsilon \quad \text{and} \quad u_1(t) \leq u_2(t) + \varepsilon, \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, we have $u_1'(t_0) = u_2'(t_0)$ and $u_1''(t_0) \leq u_2''(t_0)$. Then, at t_0 , we get

$$\begin{aligned} 0 &\geq u_1''(t_0) - u_2''(t_0) = -f(t_0, \eta(t_0), q(u_1'(t_0))) + M(u_1(t_0) - \eta(t_0)) \\ &\quad + f(t_0, \eta(t_0), q(u_2'(t_0))) - M(u_2(t_0) - \eta(t_0)) = M\varepsilon > 0 \end{aligned}$$

which is a contradiction

If $t_0 = 0$ or 2π , $u_1(2\pi) = u_1(0) = u_2(0) + \varepsilon = u_2(2\pi) + \varepsilon$, $u_1'(0) \leq u_2'(0)$ and $u_1'(2\pi) \geq u_2'(2\pi)$. In view of the boundary conditions of PBVP (4.1), we get

$$u_2'(2\pi) \leq u_1'(2\pi) = u_1'(0) \leq u_2'(0) = u_1'(2\pi)$$

which implies $u_2'(2\pi) = u_1'(0) = u_2'(0) = u_1'(2\pi)$. Then for $i = 0$ or 2π , we obtain

$$\begin{aligned} u_1''(i) - u_2''(i) &= -f(i, \eta(i), q(u_1'(i))) + M(u_1(i) - \eta(i)) \\ &\quad + f(i, \eta(i), q(u_2'(i))) - M(u_2(i) - \eta(i)) \\ &= M(u_1(i) - u_2(i)) = M\varepsilon > 0, \end{aligned}$$

which is a contradiction to the fact that the maximum of $u_1(t) - u_2(t)$ is ε at $t = 0$ or 2π . Thus, $u_1(t) \leq u_2(t)$ on $[0, 2\pi]$.

Repeating the above arguments for $u_2(t) - u_1(t)$, one can show that $u_2(t) \leq u_1(t)$ on $[0, 2\pi]$. Therefore, $u_1(t) \equiv u_2(t)$ and the proof of the lemma is complete.

Since, for every $\eta \in [\alpha, \beta]$, the PBVP (4.1) admits a unique solution u , we can define the mapping A by

$$(4.3) \quad A\eta = u.$$

The properties of A that enable us to generate the monotone iterates that converge to the minimal and the maximal solutions of PBVP (3.1) are given in the next result.

Lemma 4.3. Under the assumptions of Lemma 4.2, the mapping A defined by (4.3) satisfies

- (i) $\alpha \leq A\alpha$, and $\beta \geq A\beta$; (meaning A maps $[\alpha, \beta]$ into itself.)
- (ii) for $\eta_1, \eta_2 \in [\alpha, \beta]$, $\eta_1 \leq \eta_2$ implies $A\eta_1 \leq A\eta_2$ (i.e., A is a monotone operator on the segment $[\alpha, \beta]$).

Proof. Let $A\alpha = \alpha_1$ and $A\beta = \beta_1$, where α_1, β_1 are the unique solutions of PBVP (4.1) corresponding to $\eta = \alpha$ and $\eta = \beta$ respectively. We shall only prove that $\alpha \leq \alpha_1$ and similar arguments can be used to show $\beta \geq \beta_1$.

Suppose $\alpha(t) \leq \alpha_1(t)$, $t \in [0, 2\pi]$ is not true. Then there exists a $t_0 \in [0, 2\pi]$ and a $\epsilon > 0$ such that

$$\alpha(t_0) = \alpha_1(t_0) + \epsilon, \quad \alpha(t) \leq \alpha_1(t) + \epsilon, \quad t \in [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, we also have $\alpha'(t_0) = \alpha_1'(t_0)$ and $\alpha''(t_0) \leq \alpha_1''(t_0)$. Because of (A_0) , (4.1) and the fact that $q(\alpha_1'(t_0)) = q(\alpha'(t_0)) = \alpha'(t_0)$, we get

$$\begin{aligned} 0 \geq \alpha''(t_0) - \alpha_1''(t_0) &\geq -f(t_0, \alpha(t_0), \alpha'(t_0)) + f(t_0, \alpha(t_0), q(\alpha_1'(t_0))) \\ &\quad - M(\alpha_1(t_0) - \alpha(t_0)) = M\epsilon > 0 \end{aligned}$$

which is a contradiction.

If $t_0 = 0$ or 2π , using the boundary conditions in (A_0) and (4.1), we have

$$\alpha(2\pi) = \alpha(0) = \alpha_1(0) + \epsilon = \alpha_1(2\pi) + \epsilon$$

and

$$(4.4) \quad \alpha'(0) \leq \alpha_1'(0) = \alpha_1'(2\pi) \leq \alpha'(2\pi),$$

which, in view of (A_2) (i), implies that $\alpha'(0) = \alpha_1'(0) = \alpha_1'(2\pi) = \alpha'(2\pi)$.

Thus, when (A_0) and (A_2) (i) hold, for $i = 0, 2\pi$, we obtain

$$\begin{aligned} \alpha''(i) - \alpha_1''(i) &\geq -f(i, \alpha(i), \alpha'(i)) + f(i, \alpha(i), q(\alpha_1'(i))) \\ &= M(\alpha_1(i) - \alpha(i)) = M\varepsilon > 0 \end{aligned}$$

which is again a contradiction. This proves $\alpha \leq \alpha_1$.

To prove that A is a monotone operator on $[\alpha, \beta]$, let us set $A\eta_1 = u_1$ and $A\eta_2 = u_2$, where $\eta_1, \eta_2 \in [\alpha, \beta]$, $\eta_1 \leq \eta_2$ and u_1, u_2 are unique solutions of PBVP (4.1) with $\eta = \eta_1$ and $\eta = \eta_2$ respectively. We claim that $u_1(t) \leq u_2(t)$ whenever $\eta_1(t) \leq \eta_2(t)$, $t \in [0, 2\pi]$. If this is not true, there exists a $t_0 \in [0, 2\pi]$ and a $\varepsilon > 0$ such that

$$u_1(t_0) = u_2(t_0) + \varepsilon \quad \text{and} \quad u_1(t) \leq u_2(t) + \varepsilon \quad \text{on} \quad [0, 2\pi].$$

If $t_0 \in (0, 2\pi)$, $u_1'(t_0) = u_2'(t_0)$, $u_1''(t_0) \leq u_2''(t_0)$ and this yields a contradiction

$$\begin{aligned} 0 &\geq u_1''(t_0) - u_2''(t_0) = -f(t_0, \eta_1(t_0), q(u_1'(t_0))) + M(u_1(t_0) - \eta_1(t_0)) \\ &\quad + f(t_0, \eta_2(t_0), q(u_2'(t_0))) - M(u_2(t_0) - \eta_2(t_0)) \\ &\geq -M(\eta_2(t_0) - \eta_1(t_0)) + M(u_1(t_0) - u_2(t_0)) + M(\eta_2(t_0) - \eta_1(t_0)) = M\varepsilon > 0, \end{aligned}$$

by using (A_3^*) . On the other hand, if $t_0 = 0$ or 2π , $u_1(2\pi) = u_1(0) =$

$u_2(0) + \varepsilon = u_2(2\pi) + \varepsilon$ and $u_1'(2\pi) = u_1'(0) \leq u_2'(0) = u_2'(2\pi) \leq u_1'(2\pi)$, taking the boundary conditions in (4.1) into account. Thus, for $i = 0, 2\pi$, $u_1'(2\pi) = u_1'(0) = u_2'(0) = u_2'(2\pi)$ and it can be shown that $u_1''(i) - u_2''(i) \geq M\varepsilon > 0$, by using (A_3^*) , which again contradicts the fact that $u_1(t) - u_2(t)$ attains its maximum

value ε at 0 or 2π . Hence the claim $u_1(t) \leq u_2(t)$ on $[0, 2\pi]$ is proved.

The proof of the lemma is complete.

We can now state our main result.

Theorem 4.1. Let (A_0) , (A_1) , (A_2) and (A_3) hold. Then there exist monotone sequences $\{\alpha_n(t)\}_{n=0}^{\infty}$, $\{\beta_n(t)\}_{n=0}^{\infty}$ with $\alpha_0 \equiv \alpha$, $\beta_0 \equiv \beta$, such that $\lim_{n \rightarrow \infty} \alpha_n(t) = \rho(t)$ and $\lim_{n \rightarrow \infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$, where $\rho(t)$ and $r(t)$ are the minimal and the maximal solution of PBVP (3.1) respectively.

Proof. By Lemma 4.2, we know that for any $\eta \in [\alpha, \beta]$, the PBVP (4.1) has a unique solution $u(t)$ such that $\alpha(t) \leq u(t) \leq \beta(t)$ and $|u'(t)| \leq \tilde{N}$, $t \in [0, 2\pi]$, $\tilde{N}$ being the Nagumo constant that can be obtained for the function G . By Lemma 4.3, we see that the mapping A defined by $A\eta = u$ generates monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ where $A\alpha_n = \alpha_{n+1}$, $A\beta_n = \beta_{n+1}$. In fact, we have on $[0, 2\pi]$, $\alpha = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n \leq \dots \leq \beta_n \leq \dots \leq \beta_2 \leq \beta_1 \leq \beta_0 = \beta$, and each $\alpha_n(t)$, $\beta_n(t)$ satisfy

$$(4.5a) \quad -\alpha_n'' = f(t, \alpha_{n-1}, q(\alpha_n')) - M(\alpha_n - \alpha_{n-1}), \quad \alpha_n(0) = \alpha_n(2\pi), \quad \alpha_n'(0) = \alpha_n'(2\pi)$$

$$(4.5b) \quad -\beta_n'' = f(t, \beta_{n-1}, q(\beta_n')) - M(\beta_n - \beta_{n-1}), \quad \beta_n(0) = \beta_n(2\pi), \quad \beta_n'(0) = \beta_n'(2\pi),$$

and $|\alpha_n'(t)|, |\beta_n'(t)| \leq \tilde{N}$.

The equations (4.5a), (4.5b) can be identified as

$$(5.6a) \quad L\alpha_n = N(\alpha_n, \alpha_{n-1}),$$

$$(5.6b) \quad L\beta_n = N(\beta_n, \beta_{n-1}),$$

where $Lu = -u''$ with $D(L) = \{u \in H: u, u' \text{ are absolutely continuous, } u'' \in H \text{ and } u(0) = u(2\pi), u'(0) = u'(2\pi)\}$, $H = L_2[0, 2\pi]$, and N the Nemytskii

operator generated by G , $\alpha_n = \alpha_{n1} + \alpha_{n0}$, $\beta_n = \beta_{n1} + \beta_{n0}$ with $\alpha_{n0}, \beta_{n0} \in H_0$, $\dim H_0 = 1$ and $\alpha_{n1}, \beta_{n1} \in H_1$, the class of all functions whose average is zero. In view of (2.2) the equation (5.6a) is equivalent to the coupled equations

$$\begin{cases} -\alpha_{n1}'' = G(t, \alpha_n(t), \alpha_n'(t)) - \frac{1}{2\pi} \int_0^{2\pi} G(s, \alpha_n(s), \alpha_n'(s)) ds, \\ 0 = \int_0^{2\pi} G(s, \alpha_n(s), \alpha_n'(s)) ds \end{cases}$$

and similar coupled equations are valid for (5.6b). Now, by classical arguments it can be shown that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly and monotonically to $\rho(t)$, $r(t)$ respectively which are solutions of the BVP

$$-u'' = f(t, u, q(u')), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

It then follows by using a continuation argument, see [2; p.32] that ρ, r are actually solutions of BVP (3.1). It can be proved by induction argument that for any solution u of (4.1) with $\alpha \leq u \leq \beta$, we have $\alpha \leq \alpha_n \leq u \leq \beta_n \leq \beta$ on $[0, 2\pi]$ and this shows that $\rho(t)$, $r(t)$ are the extremal solutions of the PBVP (3.1).

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