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no.338

**ON THE SEMIVALUES AND THE POWER
CORE OF COOPERATIVE TU GAMES**

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TR # 338**

September, 1999

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On the Semivalues and the Power Core of cooperative TU games

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The Semivalues were introduced axiomatically by P.Dubey, A.Neyman and R.J.Weber (1981) as an important class of values for cooperative TU games. This class contains the Shapley value, the Banzhaf value, and many other values. For the Shapley value characterizations of games for which the Shapley value is coalitionally rational are due to Inarra and Usategui (1993), Izawa and Takahashi (1998), and Marin-Solano and Rafels (1999). In this paper the same problem of coalitional rationality is discussed for Semivalues, by using special formulas for the computation of Semivalues. The characterization shows that this is a prosperity property as defined by Van Gellekom, Potters and Reijnierse (1999) and a threshold for the property can be computed by using averages per capita. A characterization in terms of the Potential game is also given, by using concepts of average convexity and weak average convexity.

Key words: Core, Semivalue, Power Game, Potential Game, average convexity

Runing head: Power Core and Semivalues

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Introduction.

The problem of characterizing the cooperative TU games for which the Shapley value lies in the core was solved in several papers by Inarra and Usategui (1993), Izawa and Takahashi (1998), and Marin-Solano and Rafels (1999). All these papers were discussing exclusively the Shapley value, which is an efficient value. No similar theory has been developed for some nonefficient value, because such a value does not belong to the core. Recent papers by Dragan (1996), Calvo and Santos (1997) and Sanchez (1997) have introduced what we call the Power Game of a given game (or the auxiliary game). For any value Ψ , the Power Game of a game (N, v) relative to Ψ is the game (N, π^Ψ) defined by $\pi^\Psi(S, v) = \sum_{i \in S} \Psi_i(S, v)$, for all coalitions $S \subseteq N$. The Power Core of the game (N, v) relative to Ψ is the core of the game (N, π^Ψ) . As the players outcomes are $\Psi_i(N, v)$, $\forall i \in N$, the Power Core relative to various values, including the efficient ones, plays the same role relative to the outcomes as the usual core for efficient values. Therefore, it will be interesting to consider the more general problem of characterizing the games for which some values, not necessarily efficient, lie in the Power Core relative to those values. In other words, we intend to characterize the games for which some values have the property of coalitional rationality in the Power Game, and this is our motivation for the present study. In this paper, we discuss the case of the Semivalues introduced by Dubey, Neyman and Weber (1981), a class of values which contains the Shapley value, the Banzhaf value and many other interesting values. In an earlier paper, we have shown a formula which gives the Power Core of a game relative to a Semivalue in terms of the given game. So, in this case we can write explicitly the inequalities determining the Power Core, that is the coalitional rationality conditions, and check whether or not the Semivalue lies in the Power Core. However, we may transform these conditions so that we may check directly whether or not they are satisfied, or to get a characterization of games possessing such properties. The new conditions will be derived from some computational formulas, similar to the formulas proved earlier by Dragan (1992) for the Shapley value and called the Average per Capita formulas. So, we start our present paper by proving Average per Capita formulas for Semivalues (Theorem 1). Further, they will be helpful in obtaining a characterization of games for which a Semivalue lies in the Power Core relative to this Semivalue (Theorem 3); from this characterization we derive easily the Inarra/Usategui characterization of games for which the Shapley value lies in the core, taking into account that the Shapley value is the unique efficient Semivalue. However, we may get also other characterizations, for example the characterization of games for which the Banzhaf value is in the Power Core relative to

the Banzhaf value. Corollary 4 shows a very interesting fact, namely that the appartenance of a Semivalue to the Power Core is almost a prosperity property in the sense of Van Gellekom, Potters and Reijnierse (1999). Further, we consider the potential approach used in the same type of problems for the Shapley value by Marin-Solano and Rafels (1999). They use the dividend form of a game, from which the characterization of games for which the Shapley values of the game and all its subgames lie in the corresponding cores, is derived by means of the potential game of the given game. As the Semivalues have a potential, as shown already by Calvo and Santos (1997), a similar approach can be used. We start by giving a new proof for Marin-Solano and Rafels result, then we introduce the concept of weak average convex game and prove that the Shapley value of a game lies in the core if and only if the game is weak average convex, by using only the coalitional form (Theorem 6); obviously, for such games it is possible that the Shapley values of some subgames do not lie in the core. The class of weak average convex games is strictly included in the class of average convex games discussed by Marin-Solano and Rafels. Then, we show that the Semivalues of a game and its subgames are all in the corresponding Power Cores if and only if the Potential Game relative to the Semivalue is average convex (Theorem 7). Certainly, this is generalizing the result of Marin-Solano and Rafels, who consider only the unique efficient Semivalue, the Shapley value. The proof shows that every game for which the Potential Game relative to a Semivalue is weak average convex has that Semivalue in the Power Core, and viceversa (Corollary 8). In a recent paper by Dragan (1999), it was shown that the existence of a potential for a value is equivalent to the Balanced Contributions property of a value due to Myerson (1977), and to other two properties called the Recursion formula and the Shapley blueprint properties. All values possessing a potential have all four properties and otherwise have none of them. It has been proposed that all these values are called Myerson values, because the first published among these four properties are due to Myerson. The equivalence between the Balanced Contributions property and the existence of a potential has been proved by Ortman (1998); his theorem is stated (as Theorem 9) to motivate once more the above proposal. Then, we show that the proof of the characterization of games having Semivalues in the Power Core holds also for all Myerson values (Theorem 10 and Corollary 11), hence the Marin-Solano and Rafels characterization has been extended to a more general class of values than the Semivalues.

1. Average per capita formulas for Semivalues.

The Semivalues were axiomatically introduced by Dubey, Neyman and Weber (1981), as values Ψ defined on the set G of the cooperative TU games, with value in R^n for the games (N, v) in the space G^N of the cooperative TU games with the set of players N , where $n = |N|$. The axioms were linearity, symmetry, monotonicity and projection on the space AG of additive games. As the first two axioms are standard axioms in the theory of the Shapley value, we confine ourselves to explain the last two axioms. A game (N, v) is monotonic if $S \subseteq T$ implies $v(S) \leq v(T)$; a value is monotonic if for any monotonic game, $S \subseteq T$ implies that the outcome given by the value to the players in S is not larger than the outcome given by the value to the players

in T . A value satisfies the projection axiom if for any additive game the outcome given by the value to a player equals the worth of the player. If N is a finite set of players, Dubey/Neyman/Weber have shown that a Semivalue Ψ is defined for any game (N, v) in G^N by

$$\Psi_i(N, v) = \sum_{S: i \in S \subseteq N} p_s^n [v(S) - v(S - \{i\})], \quad \forall i \in N, \quad (1)$$

where $p^n = (p_1^n, p_2^n, \dots, p_n^n)^T$ is a vector of nonnegative weights satisfying

$$\sum_{s=1}^{s=n} \binom{n-1}{s-1} p_s^n = 1. \quad (2)$$

Note that the normalization condition (2) is needed to satisfy the projection axiom, because for any additive game the brace in (1) equals $v(\{i\})$, a common factor to all terms, and after factoring out we are left in (1) with the sum shown in the left hand side of (2) multiplying $v(\{i\})$. Note also that to define a value on the set G of all TU games we need in fact a sequence of weight vectors $p^1, p^2, \dots, p^n, \dots$ because N is arbitrary in (1); a weight vector p^t will be used to express the Semivalue for a game with the set of players T . To make sure that a game (N, v) can be extended to a game $(N \cup d, v)$, where d is a new player, and in the new game any player i of the original game gets by Ψ the same outcome, the weight vectors should satisfy Pascal triangle type of relations

$$p_s^n = p_s^{n+1} + p_{s+1}^{n+1}, \quad s = 1, 2, \dots, n, \quad (3)$$

for whatever n . Obviously, from (3) we get that (2) written for an integer t holds for $t < n$, whenever holds for $t = n$; for example, we have for different t the equalities $1 = p_1^1 = p_1^2 + p_2^2 = p_1^3 + 2p_2^3 + p_3^3$, a.s.o.

It is worthwhile to mention that among the Semivalues, we get the Shapley value for $p_s^n = (n!)^{-1}(s-1)!(n-s)!$, $s = 1, 2, \dots, n$, then the Banzhaf value for $p_s^n = 2^{1-n}$, $s = 1, 2, \dots, n$, and the Marginal contribution value for $p_n^n = 1$, $p_s^n = 0$, $s < n$. Clearly, the axioms used in defining these values were weaker than those used in defining the three mentioned values, as a lot of arbitrariness is left. However, the Semivalues have properties very similar to the above values. They were discussed by Calvo and Santos (1997), Dragan (1998, 1999), and Sanchez (1997).

In an earlier paper, (Dragan, 1992), it has been proved a formula for the Shapley value which was called the Average per Capita formula, from which an algorithm for computing this value has been derived. In this section, we prove a generalization of that formula for Semivalues and this formula shows that the computation of a Semivalue has the same difficulty as the computation of the Shapley value; further this formula will be used to derive a characterization of games having coalitionally rational Semivalues. Recall the notations: for any game (N, v) , and $k = 1, 2, \dots, n-1$, denote

$$v_k = \binom{n}{k}^{-1} \sum_{|S|=k} v(S), \quad v_k^i = \binom{n-1}{k}^{-1} \sum_{|S|=k, i \notin S} v(S), \quad \forall i \in N, \quad (4)$$

and for convenience accept also that v_n is also given by the first formula for $k = n$, while $v_n^i = 0$, $\forall i \in N$. The interpretations of these expressions are: v_k is the average worth of coalitions of size k , while v_k^i is the average worth of coalitions of size k which do not contain the player i . The Average per Capita formula for the Shapley value is:

$$SH_i(N, v) = \sum_{k=1}^{k=n} k^{-1}(v_k - v_k^i), \quad \forall i \in N. \quad (5)$$

To prove a similar formula for the Semivalues, we shall rearrange the terms in (1). Rewrite (1) as

$$\Psi_i(N, v) = p_n^n v(N) + \sum_{S: i \in S \subseteq N} p_s^n v(S) - \sum_{S: i \in S \subseteq N} p_s^n v(S - \{i\}), \quad \forall i \in N, \quad (6)$$

and consider the two sums separately. For the first, we can write

$$\sum_{S: i \in S \subseteq N} p_s^n v(S) = \sum_{s=1}^{s=n-1} p_s^n \left(\sum_{|S|=s, i \in S} v(S) \right) = \sum_{s=1}^{s=n-1} p_s^n \left(\sum_{|S|=s} v(S) - \sum_{|S|=s, i \notin S} v(S) \right), \quad (7)$$

and for the second, we write

$$\sum_{S: i \in S \subseteq N} p_s^n v(S - \{i\}) = \sum_{s=1}^{s=n-1} p_{s+1}^n \left(\sum_{|S|=s, i \notin S} v(S) \right). \quad (8)$$

Now, from (6), (7) and (8) we obtain

$$\Psi_i(N, v) = p_n^n v(N) + \sum_{s=1}^{s=n-1} [p_s^n \binom{n}{s} v_s - p_s^{n-1} \binom{n-1}{s} v_s^i], \quad (9)$$

where we have used the relations (3) and we have introduced the averages (4). If we denote for all $t = 1, 2, \dots, n$ and $s \leq t$ the weights

$$q_s^t = s p_s^t \binom{t}{s}, \quad (10)$$

we get from (9) that

$$\Psi_i(N, v) = q_n^n n^{-1} v_n + \sum_{s=1}^{s=n-1} s^{-1} (q_s^n v_s - q_s^{n-1} v_s^i), \quad \forall i \in N. \quad (11)$$

Note that the new weights were obtained by dividing the original weights to the weights for the Shapley value. Therefore, for the Shapley value the new weights are equal to one, and in this case from (11) we obtain the Average per Capita formula. So, (11) is a generalization for Semivalues of the Average per Capita formula. Clearly, the weights q_s^t may also be considered as data to define a Semivalue; however, now the normalization condition (2) becomes

$$\sum_{s=1}^{s=n} q_s^n = n \quad (12)$$

and the Pascal triangle conditions (3) become

$$q_s^{t-1} = (1 - st^{-1}) q_s^t + st^{-1} q_{s+1}^t, \quad s = 1, 2, \dots, t-1, \quad (13)$$

for all integers t . We proved:

Theorem 1: Let Ψ be a Semivalue associated with a nonnegative vector p^n satisfying (2); let q^n be the nonnegative weight vector defined by (10) and satisfying (12). Then, the Semivalue is given by the Weighted Average per Capita formula (11).

Obviously, formula (11) provides an average per capita formula for the Banzhaf value for $q_s^n = 2^{1-n} s \binom{n}{s}$, $s = 1, 2, \dots, n$, a formula for the Marginal Contribution value a.s.o. Returning to the algorithm for the computation of the Shapley value based upon the Average per Capita formula, we notice that as soon as we have the weight vector q^n , we should simply multiply the averages by the weights before making the differences. Hence, the algorithm is very little changed by the existence of weights. Note also that if we insist to express the Semivalue in terms of the original weights, an easy computation based upon formula (11) gives for all $i \in N$:

$$\Psi_i(N, v) = p_n^n v_n + \sum_{s=1}^{s=n-1} s^{-1} \binom{n-1}{s-1} [np_s^n v_s - (n-s)p_s^{n-1} v_s^i], \quad (14)$$

2. The Semivalues and the Power Core.

As the unique efficient Semivalue is the Shapley value, a property which follows easily from (11) or (14), in all the other cases the Semivalues do not belong to the core. Therefore, the relationships between the Semivalues and the core are not interesting anymore. Instead, taking into account that a Semivalue is providing each player with its "power" in the game, the relationship between a Semivalue and the core of its Power Game becomes interesting. Recall that Dragan (1996a) has introduced for the Banzhaf value the Power Game of a given game. For a game (N, v) , the Power Game relative to a Semivalue Ψ is the game (N, π^Ψ) defined by

$$\pi^\Psi(T, v) = \sum_{i \in T} \Psi_i(T, v), \quad \forall T \subseteq N. \quad (15)$$

This game has also been used by Calvo and Santos (1997) and Sanchez (1997), under the name of auxiliary game of (N, v) , to derive general results on values possessing a potential. Obviously, the Power Game may be defined by (15) for whatever value. But, recall that for a Semivalue the Power Game can be expressed in terms of the given game by

$$\pi^\Psi(T, v) = \sum_{S: S \subseteq T} [sp_s^t - (t-s)p_{s+1}^t] v(S), \quad \forall T \subseteq N. \quad (16)$$

(see Dragan (1998) and (1999)). This fact led us to discuss the coalitional rationality mainly for Semivalues.

Definition 2: The Power Core of a game (N, v) relative to a Semivalue Ψ associated with a nonnegative weight vector p^n is the set of n - dimensional vectors

$$PC^\Psi(N, v) = \{ x \in R^n : x(T) \geq \pi^\Psi(T, v), \forall T \subset N, x(N) = \pi^\Psi(N, v) \}, \quad (17)$$

where as usual $x(T) = \sum_{i \in T} x_i$, for all $T \subseteq N$.

In words, the Power Core of a game is the core of the Power Game relative to that value. Obviously, in general, for different values the Power Core is different, and for efficient values the Power Core coincides with the core. The Power Core, like the core, is a set derived from the idea that any coalition can not accept less than the power available to itself in case that the coalition forms. However, a Semivalue may not belong to the Power Core relative to that Semivalue, as it is clear from the case of the Shapley value, where the Power Core is the core, because there are games for which the Shapley value is not in the core. Notice that a Semivalue Ψ belongs to the Power Core relative to Ψ if and only if we have

$$\sum_{i \in T} \Psi_i(N, v) \geq \pi^\Psi(T, v) = \sum_{i \in T} \Psi_i(T, v), \quad \forall T \subseteq N, \quad (18)$$

This system of inequalities defines what we call the coalitional rationality of Ψ , because (18) says that the members of any coalition can not get together more in that coalition than what they get in N . It is important enough in the formation of the grand coalition to motivate us in investigating the games for which (18) holds for a fixed Semivalue Ψ . Obviously, for the efficient Semivalues we shall obtain the results of Inarra/Usategui, and

Marin-Solano/Rafels. To get a characterization for the games with the property that the Semivalue Ψ is in the Power Core, we have to express (18) in terms of the given game and the given Semivalue. It is clear that (18) makes sense for any value, but for Semivalues we can get another expression for (18); the main tool will be the Weighted Average per Capita formulas derived in the precedent section. Formula (16) gives us an expression for the right hand side in (18), so that we have only to compute the left hand side in (18) by using those formulas. Denote

$$\sigma_T^\Psi(N, v) = \sum_{i \in T} \Psi_i(N, v), \quad \forall T \subseteq N, \quad (19)$$

and use (11) in the right hand side. We get

$$\sigma_T^\Psi(N, v) = tn^{-1}q_n^n v_n + \sum_{s=1}^{s=n-1} s^{-1} \left(tq_s^n v_s - q_s^{n-1} \sum_{i \in T} v_s^i \right), \quad \forall T \subseteq N, \quad (20)$$

where the averages occur. This equality will be used in the left hand side of (18). Notice that we do not need formula (16) for the right hand side in (18), because an alternate expression can be obtained from (20), where we should replace N by T , and on the other hand the averages are computed in the subgame (T, v) . Therefore, to coexist in the same formula with the averages in (N, v) , the averages in the subgame (T, v) should get a different notation; we shall denote the last ones by $v_s(T)$ and $v_s^i(T)$. Hence, with these changes, we get for the right hand side in (18):

$$\pi^\Psi(T, v) = \sigma_T^\Psi(T, v) = q_t^t v_t(T) + \sum_{s=1}^{s=t-1} s^{-1} \left(tq_s^t v_s(T) - q_s^{t-1} \sum_{i \in T} v_s^i(T) \right), \quad \forall T \subset N, \quad (21)$$

where the sum can still be simplified. As the sum in the parenthesis (21) makes $tv_s(T)$ and by (13), we have $q_s^t - q_s^{t-1} = st^{-1}(q_s^t - q_{s+1}^t)$, we obtain

$$\pi^\Psi(T, v) = q_t^t v_t(T) + \sum_{s=1}^{s=t-1} (q_s^t - q_{s+1}^t) v_s(T), \quad \forall T \subset N. \quad (22)$$

Now, by using (20) and (22), we proved the following main result of this section:

Theorem 3: Let Ψ be a Semivalue associated with a weight vector p^n . Let q^n be the weight vector derived from p^n by (10), and q^t , for $t < n$, be the weight vectors derived from q^n by means of (13). Then the Semivalue lies in the Power Core if and only if we have for all coalitions $T \subset N$ the inequalities

$$\begin{aligned} n^{-1}q_n^n v_n + \sum_{s=1}^{s=n-1} s^{-1} (q_s^n v_s - t^{-1}q_s^{n-1} \sum_{i \in T} v_s^i) &\geq \\ &\geq t^{-1} [q_t^t v_t(T) + \sum_{s=1}^{s=t-1} (q_s^t - q_{s+1}^t) v_s(T)], \end{aligned} \quad (23)$$

where v_s and v_s^i , $i \in N$, for $s = 1, 2, \dots, n$, are the averages in the game (N, v) , while $v_s(T)$, for $s = 1, 2, \dots, t$, are the averages in the subgame (T, v) , for $T \subset N$.

Note that from Theorem 3 we may get an interesting system of conditions to be satisfied in order to have the Banzhaf value in the Power Core; it is enough to take in (23) the weights

$$q_s^t = 2^{1-t} s \binom{t}{s}, \quad s \leq t, \quad t = 1, 2, \dots, n. \quad (24)$$

In the general case, we have the very interesting

Corollary 4: In the space of games G^N , for a given vector q^n , let

$$H = n(q^n)^{-1} \text{Max} \left(t^{-1} [q_t^t v_t(T) + \sum_{s=1}^{s=t-1} (q_s^t - q_{s+1}^t) v_s(T) + \sum_{s=1}^{s=n-1} s^{-1} q_s^{n-1} \sum_{i \in T} v_s^i] \right), \quad (25)$$

where the maximum is taken over all coalitions $T \subset N$. Then, there is a threshold

$$\alpha = H - n(q^n)^{-1} \cdot \sum_{s=1}^{s=n-1} s^{-1} q_s^n v_s \quad (26)$$

depending on the worth of coalitions $T \subset N$ and on the weight vector q^n , as well as on the weight vectors q^t derived from q^n via (13), so that a game (N, v) has the Semivalue defined by q^n in the Power Core relative to the Semivalue, if and only if

$$v(N) \geq \alpha. \quad (27)$$

Obviously, there are a lot of games in G^N satisfying this condition, hence having the Semivalue in the Power Core, and this is true for whatever set of players N and whatever vector of weights. Note that for the Shapley value α does not depend on the averages in the subgames.

Recently, Van Gellekom, Potters and Reijnierse (1999) have introduced the concept of Prosperity property as follows: let (N, v^0) be an incomplete game, that is only the value $v^0(N)$ in v^0 is not defined; a property P is called a (strong) prosperity property if for every incomplet game (N, v^0) there exists a number $\alpha_P(v^0) \geq \sum_{i \in N} v^0(\{i\})$ such that the game (N, v) with $v(T) = v^0(T)$, $\forall T \subset N$, has the property P if and only if $v(N) \geq \alpha_P(v^0)$. Notice the relationship between our Corollary 4 and this new concept. Denote by $\alpha_P(v)$ the maximum among the number found in the right hand side of (27) and the number $\sum_{i \in N} v(\{i\})$. If $v(N) \geq \alpha_P(v)$, then the game has the property that the Semivalue defined by the weight vector q^n , lies in the Power Core, but it is possible that the property holds even for $v(N) < \alpha_P(v)$; precisely, this is the case when the number in the right hand side of (27) is smaller than $\alpha_P(v)$. Hence, the idea of prosperity property should be somehow refined to include the property of lying in the Power Core.

As mentioned above, the characterization of games with the Shapley value in the core by means of linear inequalities has been first obtained by Inarra and Usategui (1993). Let us retrieve that characterization, from the more general characterization offered by the above Theorem 3. The weights for the Shapley value are $q_s^t = 1$ for all $s \leq t$ and all $t = 1, 2, \dots, n$, hence the inequalities of Theorem 3 become

$$n^{-1} v_n + \sum_{s=1}^{s=n-1} s^{-1} (v_s - t^{-1} \sum_{i \in T} v_s^i) \geq t^{-1} v_t(T). \quad (28)$$

Taking into account the notations (4) and the fact that for any $S \subset N$ and $T \subset N$ we have

$$\sum_{i \in T} v_s^i = \binom{n-1}{s}^{-1} \sum_{|S|=s} (t - |S \cap T|) v(S), \quad (29)$$

the paranthesis in (28) becomes in coalitional form

$$v_s - t^{-1} \sum_{i \in T} v_s^i = \binom{n}{s}^{-1} \sum_{|S|=s} v(S) - t^{-1} \binom{n-1}{s}^{-1} \sum_{|S|=s} (t - |S \cap T|) v(S). \quad (30)$$

Hence, (28) can be rewritten as

$$n^{-1} v(N) + \sum_{s=1}^{s=n-1} (st \binom{n}{s})^{-1} \sum_{|S|=s} (n-s)^{-1} (|S \cap T| n - st) v(S) \geq t^{-1} v(T). \quad (31)$$

If we use the Inarra/Usategui notations, that is we denote

$$\begin{aligned} \gamma_n(S) &= (n!)^{-1} (n-s)! (s-1)! \quad (\text{the Shapley value coefficients}) \\ \alpha_T(S) &= (n-s)^{-1} (|S \cap T| n - st), \quad \forall S, T \subseteq N, S \neq N, \\ \alpha_T(N) &= t, \quad \forall T \subseteq N, \end{aligned} \quad (32)$$

then, after multiplying (31) by t and using the last equality (32), we get

$$\sum_{\emptyset \neq S \subseteq N} \gamma_n(S) \alpha_T(S) v(S) \geq v(T), \quad \forall T \subseteq N, \quad (33)$$

which is the condition derived from the Inarra/Usategui condition (A.1) shown in the mentioned paper, where the right hand side is also somehow transformed.

We believe that the form (23) of the characterization conditions is easy to use while we apply the algorithm for computing the averages given in the paper by Dragan (1992). As we do not intend to use the concept of average convex game in the manner of the Inarra/Usategui paper, to show that for such games the Semivalues have also some properties, we confine ourselves to the form (23) of the characterization. Instead, we shall discuss in the next section the case of average convex games by using a potential approach similar to the one used by Marin-Solano and Rafels.

3. The average convex games, the weak average convex games and the potential game.

The concept of average convex game has been introduced by Sprumont (1990) and Inarra and Usategui (1993); such games are called by Sprumont quasiconvex games. They proved that for such games the Shapley value lies in the core. A game (N, v) is average convex if for all coalitions S and T with $S \subset T \subseteq N$ we have

$$\sum_{j \in S} [v(S) - v(S - \{j\})] \leq \sum_{j \in S} [v(T) - v(T - \{j\})], \quad (34)$$

or

$$v(T) \geq v(S) + s^{-1} \sum_{j \in S} [v(T - \{j\}) - v(S - \{j\})]. \quad (35)$$

Conditions (35) explain better the name chosen for this class of games, while from (34) we can derive immediately the result that any convex game is also average convex. The relationship between these two classes of games is a strict inclusion (see Marin-Solano and Rafels paper for an example of an average convex game which is not convex). Izawa and Takahashi (1998) have introduced the concept of totally convex game, which they show to be equivalent to the appartenance to the Shapley value to the core. They proved that the class of average convex games is strictly included in the class of totally convex games; in their paper it is shown an example of a totally convex game which is not average convex. The result by Marin-Solano and Rafels to be discussed below use the average convexity of the Potential game of the given game, so that they are obtained by using a potential approach. In this section we consider a similar approach for the Semivalues, to clarify the role of average convexity in the absence of efficiency. However, we shall not use the dividend form in the proofs as Marin-Solano/Rafels.

The potential of the Shapley value introduced by Hart and Mas Colell (1988, 1989), is a functional $P : G \rightarrow R$ such that for any game (N, v) we have for any coalition $T \subseteq N$ and the subgames of (N, v)

$$\begin{aligned} P(T, v) &= v(T), \text{ if } |T| = 1, \\ \sum_{i \in T} [P(T, v) - P(T - \{i\}, v)] &= v(T), \forall i \in T, \text{ if } |T| > 1. \end{aligned} \quad (36)$$

As the definition asserts, the potential is associating a number to any game (N, v) and to each of its subgames (T, v) for $T \subseteq N$. Hence, for a game (N, v) the potential is associating another game, denoted P_v , and called the Potential game of (N, v) , defined by $P_v(T) = P(T, v)$, for all $T \subseteq N$. Marin-Solano and Rafels show that a game and its subgames have all their Shapley values in the corresponding cores if and only if the Potential game is average convex; to prove this they use the dividend form of the game. We show a more direct proof, which does not use the dividend form: the average convexity of the Potential game can be written as

$$\sum_{j \in S} [P(S, v) - P(S - \{j\}, v)] \leq \sum_{j \in S} [P(T, v) - P(T - \{j\}, v)], \quad (37)$$

for all $S \subseteq T \subseteq N$. Now, from the Hart and Mas Colell result that the marginal contribution to the potential is the Shapley value of the corresponding player, and the efficiency of the Shapley value, we can rewrite (37) as

$$v(S) \leq \sum_{j \in S} SH_j(T, v), \quad \forall S \subseteq T \subseteq N, \quad (38)$$

which says that the Shapley value of the subgame (T, v) lies in the corresponding core, for all $T \subseteq N$. Obviously, from (38) we get back (37) and the result holds. This proof leads us to a new concept in which the set of players N is arbitrary but fixed:

Definition 5: A cooperative TU game (N, v) is weak average convex, if for all coalitions $S \subseteq N$ we have

$$\sum_{j \in S} [v(S) - v(S - \{j\})] \leq \sum_{j \in S} [v(N) - v(N - \{j\})]. \quad (39)$$

Clearly, an average convex game is weak average convex, because in (39) we require only the condition (34) for $T = N$ to be satisfied. Instead, a weak average convex game may not be average convex, as it is shown by the following example: consider the three person game given by $v(1) = 1, v(2) = 2, v(3) = 3, v(1, 2) = 5, v(1, 3) = 6, v(2, 3) = 4$, and $v(1, 2, 3) = 9$; it easy to check that (39) are satisfied, hence the game is weak average convex, but (34) does not hold for $T = \{2, 3\}$, hence it is not average convex. The same example shows that a subgame of a weak average convex game may not be weak average convex. In fact a game is average convex if and only if the game and all its subgames are weak average convex.

Note that from the above proof of the Marin-Solano and Rafels theorem, where the coalition $T \subseteq N$ is arbitrary, by taking $T = N$ we obtain the result:

Theorem 6: A game has the Shapley value in the core if and only if its Potential game is weak average convex.

Note that if the given game is average convex, then by Inarra/Usategui (1993) the game and all its subgames have the Shapley value in the core, hence the Potential game will be weak average convex, but the converse is not true in general.

Note also that the above proof suggests the fact that some similar result should still hold for any other value possessing a potential. We develop now this remark.

Let us return to the Semivalues and recall that a Semivalue Ψ associated with a nonnegative weight vector p^n has a potential, that is a functional Q satisfying conditions similar to (36); namely, the potential for a game (N, v) and its subgames is given by

$$Q(T, v) = \sum_{S \subseteq T} p_s^t v(S), \quad \forall T \subseteq N. \quad (40)$$

This potential can be found in the paper by Calvo and Santos (1997). In consequence, Q generates also a Potential game relative to Ψ , say Q_v^Ψ , defined for all $T \subseteq N$ by $Q_v^\Psi(T) = Q(T, v)$. Now, the average convexity conditions for the last potential game are

$$\sum_{j \in S} [Q(S, v) - Q(S - \{j\}, v)] \leq \sum_{j \in S} [Q(T, v) - Q(T - \{j\}, v)], \quad (41)$$

for all $S \subseteq T$. However, now if we use the definition of the potential given by (36), then from (41) we obtain

$$\pi^\Psi(S, v) \leq \sum_{j \in S} \Psi_j(T, v), \quad \forall S \subseteq T. \quad (42)$$

This says that the Semivalue Ψ for the subgame (T, v) of the game (N, v) is in the Power Core of this game. Conversely, from (42) we get (41), hence we proved:

Theorem 7: A Semivalue Ψ for the subgame (T, v) of the game (N, v) lies in the Power Core of this game, for all $T \subseteq N$, if and only if the potential game relative to this Semivalue is average convex.

As in the case of the Shapley value, from the above proof we get the result:

Corollary 8: A game (N, v) has a Semivalue Ψ in its Power Core if and only if its potential game Q_v^Ψ is weak average convex.

Clearly, the Marin-Solano and Rafels theorem and Theorem 6 above are particular cases of the Theorem 7 and Corollary 8 just proved, because the Shapley value is one of the Semivalues. As we have already mentioned above, these last results are valid for a more general class of values than the Semivalues, a fact explained below.

4. Myerson values and the average convexity.

A characterization of the Shapley value due to Myerson (1977) proves that the Shapley value is the unique value which is efficient and satisfies the axiom of Balanced Contributions. A value Ψ defined on G with real values in R^n for each game (N, v) in G^N satisfies the Balanced Contributions axiom if we have for each pair of players i and j the equalities

$$\Psi_i(S, v) - \Psi_i(S - \{j\}, v) = \Psi_j(S, v) - \Psi_j(S - \{i\}, v), \quad (43)$$

for all coalitions $S \subseteq N$ containing these players. In words, (43) says that the marginal contribution of j to the outcome of i in S , equals the marginal contribution of i to the outcome of j in S . Now, it is easy to prove:

Theorem 9, (Ortman, 1998): For any value Ψ , the following two statements are equivalent:

- i) Ψ has a potential;
- ii) Ψ satisfies the Balanced Contributions conditions.

Note that in Theorem 9 the value is not supposed to be efficient and no efficiency condition has been used in the proof. In a recent paper by Dragan (1999), it was given a short proof of the fact that i) and ii) are equivalent to other two properties of values of cooperative TU games, which were called the Recursion formula and the Shapley blueprint property. The paper by Sanchez (1997) is also discussing the topic. A value Ψ on G has the Recursion formula property if for all games (N, v) and all $S \subseteq N$ with $|S| > 1$ we have for all $i \in S$ the equality

$$\Psi_i(S, v) = s^{-1} [\pi^\Psi(S, v) - \pi^\Psi(S - \{i\}, v) + \sum_{j \in S - \{i\}} \Psi_j(S - \{j\}, v)], \quad (44)$$

where (N, π^Ψ) is the Power Game of (N, v) relative to Ψ and $\Psi_k(\{k\}, v) = v(\{k\})$, for all $k \in N$. As for a Semivalue from formula (16) one could derive the marginal contribution in the Power Game relative to Ψ in terms of the coalitional form to be used in (44), this is a recursion formula for the Semivalue, and this justifies the name of the property. However, even if the marginal contribution in the Power Game can not be expressed in terms of the coalitional form, the equivalence mentioned above holds true for whatever value Ψ (see Dragan (1998)). A value Ψ on G has the Shapley blueprint property if we have for any game (N, v) the equality

$$\Psi(N, v) = SH(N, \pi^\Psi), \quad (45)$$

where (N, π^Ψ) is the Power Game of (N, v) . Calvo and Santos have proved that (45) is equivalent to the existence of a potential for the value Ψ . Clearly, the equivalence of these four properties is a sufficiently rich asset for a value to be able to introduce a name for the class of all values possessing these properties. In the above mentioned paper by Dragan (1999), it has been shown that all Semivalues associated with vectors of weights $p^n \geq 0$ satisfy all four properties; examples of values which do not satisfy any of the four properties are also given. As the Balanced Contributions is the first published property among the four properties and Myerson is the author of the paper, we call these values Myerson values. We believe that no confusion can occur between this class of values and the concept of value introduced by Myerson for games with communication structure; for a discussion of the last concept see, for example, the recent paper by Hamiache (1999).

Now, any Myerson value Ψ , which by definition has a potential, say R , defines for any game (N, v) a Potential game (N, R_v^Ψ) relative to Ψ by $R_v^\Psi(T) = R(T, v)$, for all $T \subseteq N$. Also, the Power Core of (N, v) is the core of the Power Game (N, π^Ψ) . The proofs of Theorem 7 and Corollary 8 may be repeated for any Myerson value, hence the following more general results holds:

Theorem 10: A Myerson value Ψ lies in the Power Core of a game relative to this value and in the Power Cores of all its subgames, if and only if the Potential game is average convex.

Corollary 11: A Myerson value Ψ lies in the Power Core of a game if and only if the Potential game is weak average convex.

Note that the results obtained in this section show that the potential approach used by Marin-Solano and Rafels is an excellent theoretical tool, as we were able to extend their results in connection with more general values, not necessarily efficient, but possessing a potential. Computationally, we believe that the best approach to check whether a

Semivalue lies in the Power Core is the method of averages per capita, that is to use the inequalities (23) proved in Theorem 3. The computation of averages for different coalition sizes can be done in parallel saving memory and time.

Acknowledgement: The research results presented in this paper have been obtained while Professor Juan Enrique Martinez Legaz was visiting the University of Texas at Arlington, Department of Mathematics, in January 1997, supported by Direccio General de Investigacion Cientifica y Tecnica (DGICYT) under Project PB 95-0679, and in May 1998, supported by Comissionat per Universitats i Recerca de la Generalitat de Catalunya under Grant 1998 SGR-00062.

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