COMPARISON RESULTS FOR PARABOLIC
DIFFERENTIAL EQUATIONS AT RESONANCE[†]

by

G. R. Shendge^{*}
Department of Mathematics
The University of Texas
Arlington, Texas 76019

A. S. Vatsala
Department of Mathematics
Bishop College
3837 Simpson-Stuart Road
Dallas, Texas 75241

Technical Report No. 188
July 1982

[†] Research partially supported by U. S. Army Research Grant #DAAG29-81-G-0008.

^{*} On study leave from the Department of Mathematics and Statistics, Marathwada University, Aurangabad 431004, Maharashtra, India.

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1. It is very well known that comparison principles for initial and boundary value problems for nonresonance cases have been very much used in the existence of solutions and the development of the monotone method [1,2,3,4,5,9]. These comparison techniques do not cover the resonance cases. Hence it is of practical interest to look at such results for resonance cases. With this view, different comparison results were recently developed for first and second order periodic boundary value problems [10]. Some special cases of these have been used in [6,7] and in developing the monotone method for first order periodic systems in [11]. In this paper we develop systematically general comparison techniques for semilinear parabolic equations with periodic and homogeneous Neumann boundary conditions, since special cases of such equations occur in many physical situations as reaction diffusion equations [5,8].

2. Consider the periodic boundary value problem (PBVP for short).

$$(2.1) \quad u_t - u_{xx} = g(t, x, u, u_x), \quad 0 \leq t \leq 2\pi, \quad 0 \leq x \leq 1,$$

$$(2.2) \quad \begin{cases} u(0, x) = u(2\pi, x), \quad x \in I, \quad u(t, 0) = u(t, 1), \\ u_x(t, 0) = u_x(t, 1), \quad t \in J \end{cases}$$

where $g \in C[J \times J \times R \times R, R]$, $J = [0, 2\pi]$, $I = (0, 1]$. For convenience we list the following conditions for functions $m, \beta, z \in C^{1,2}[J \times I, R]$.

$$(A_0) \quad m(0, x) \leq m(2\pi, x), \quad m(t, 0) = m(t, 1), \quad m_x(t, 0) \geq m_x(t, 1)$$

$$\text{and } m_t - m_{xx} \leq g(t, x, m, m_x), \quad 0 < t \leq 2\pi, \quad 0 \leq x \leq 1.$$

$$(A_1) \quad \beta(0, x) \geq \beta(2\pi, x), \quad \beta(t, 0) = \beta(t, 1), \quad \beta_x(t, 0) \leq \beta_x(t, 1)$$

$$\text{and } \beta_t - \beta_{xx} \geq g(t, x, \beta, \beta_x), \quad 0 < t \leq 2\pi, \quad 0 \leq x \leq 1.$$

(A₂) For any $u, v \in C^{1,2}[J \times I, \mathbb{R}]$, $g_u(t, x, u, v)$, $g_v(t, x, u, v)$ are continuous,
 $g_u(t, x, u, v) \leq 0$ and for any fixed t_0 $g_u(t, x, u, v) \neq 0$.

(A₃) $z(t, x) > 0$ on $J \times I$, $z(0, x) \geq z(2\pi, x)$, $z(t, 0) = z(t, 1)$, $z_x(t, 0) \leq z_x(t, 1)$
 and for every $\lambda > 0$, either

(i) $\lambda(z_t - z_{xx}) > g(t, x, \beta + \lambda z, \beta_x + \lambda z_x) - g(t, x, \beta, \beta_x)$ on $J \times I$; or

(ii) $\lambda(z_t - z_{xx}) > g(t, x, m - \lambda z, m_x - \lambda z_x) - g(t, x, m, m_x)$ on $J \times I$.

Let $v \in C^{1,2}[J \times I, \mathbb{R}]$. Then a function $G(t, x, u, w)$ is said to be a modified function relative to v if

$$G(t, x, u, w) = g(t, x, p(t, x, u), w) + \frac{P(t, x, u) - u}{1 + u^2}$$

where $p(t, x, u) = \max(v(t, x), u)$.

Let us begin by proving the following theorem.

Theorem 2.1. Consider the PBVP (2.1), (2.2). Assume that

(i) the modified PBVP

$$(2.3) \quad u_t - u_{xx} = G(t, x, u, u_x), \quad 0 \leq t \leq 2\pi, \quad 0 \leq x \leq 1,$$

$$(2.4) \quad \begin{cases} u(0, x) = u(2\pi, x), & x \in I, & u(t, 0) = u(t, 1), \\ u_x(t, 0) = u_x(t, 1), & t \in J \end{cases}$$

admits a solution u for every lower solution v of (2.1);

(ii) $r(t, x)$ is the maximal solution of the PBVP (2.1), (2.2) on $J \times I$.

Then (A₀) implies $m(t, x) \leq r(t, x)$ on $J \times I$.

Proof. Let $u(t, x)$ be a solution of the PBVP (2.3), (2.4), where the modified function G is defined relative to m . It is enough to prove that $m(t, x) \leq u(t, x)$ on $J \times I$: Suppose this is not true. Then there exists an $M > 0$ and a $(t_0, x_0) \in J \times I$ such that

$$(2.5) \quad m(t_0, x_0) = u(t_0, x_0) + M, \quad m(t, x) \leq u(t, x) + M \quad \text{on } J \times I.$$

If $(t_0, x_0) \in (0, 2\pi) \times (0, 1)$, then we have $m_t = u_t$, $m_x = u_x$ and $m_{xx} \leq u_{xx}$ at (t_0, x_0) . Moreover, since $m(t_0, x_0) > u(t_0, x_0)$, we also have $p(t_0, x_0, u) = m(t_0, x_0)$. Consequently, by (2.3) and the definition of G , we have

$$\begin{aligned} m_t(t_0, x_0) - m_{xx}(t_0, x_0) &\geq u_t(t_0, x_0) - u_{xx}(t_0, x_0) \\ &= G(t_0, x_0, u(t_0, x_0), u_x(t_0, x_0)) \\ &= g(t_0, x_0, m(t_0, x_0), m_x(t_0, x_0)) + \frac{m(t_0, x_0) - u(t_0, x_0)}{1 + u^2(t_0, x_0)} \\ &> g(t_0, x_0, m(t_0, x_0), m_x(t_0, x_0)) \end{aligned}$$

contradicting (A_0) .

If $t_0 = 2\pi$ and $x_0 \in (0, 1)$, then from (2.5), we get

$$(2.6) \quad m_t \geq u_t \quad \text{and} \quad m_x = u_x \quad \text{at } (2\pi, x_0).$$

Also if $t_0 = 0$ and $x_0 \in (0, 1)$, we get

$$m(2\pi, x_0) \geq m(0, x_0) = u(0, x_0) + M = u(2\pi, x_0) + M$$

by (2.5) and (A_0) . As a result we again get (2.6) from (2.5). Hence if $t_0 = 0$ or 2π and $x_0 \in (0, 1)$, we have $m_t \geq u_t$, $m_x = u_x$ and $m_{xx} \leq u_{xx}$ at $(2\pi, x_0)$. These relations lead to a contradiction to (A_0) as before.

Suppose next that $t_0 \in (0, 2\pi]$ and $x_0 = 1$, $i = 0, 1$. Then from (2.5) we obtain

$$(2.7) \quad m_t(t_0, 1) \geq u_t(t_0, 1), \quad m_x(t_0, 0) \leq u_x(t_0, 0), \quad m_x(t_0, 1) \geq u_x(t_0, 1).$$

Also, if $t_0 = 0$ and $x_0 = 1$, we get

$$m(2\pi, 1) \geq m(0, 1) = u(0, 1) + M = u(2\pi, 1) + M$$

by (2.5) and (A_0) . Consequently we obtain

$$(2.8) \quad m_t(2\pi, 1) \geq u_t(2\pi, 1), \quad m_x(2\pi, 0) \leq u_x(2\pi, 0), \quad m_x(2\pi, 1) \geq u_x(2\pi, 1)$$

by (2.5). Now if (A_0) holds, then (2.4) and (2.7) yield

$$(2.9) \quad m_x(t_0, 1) = u_x(t_0, 1).$$

Hence if $t_0 \in (0, 2\pi]$ and $x_0 = 1$, we have $m_t \geq u_t$, $m_x = u_x$ and $m_{xx} \leq u_{xx}$ at $(t_0, 1)$. These relations lead to a contradiction as before. A similar contradiction results from (2.8) if $t_0 = 0$ and $x_0 = 1$. Thus if (A_0) holds, we have $m(t, x) \leq u(t, x)$ on $J \times I$. But then it follows from the definition of G that $u(t, x)$ is actually a solution of the PBVP (2.1), (2.2), and since $u(t, x) \leq r(t, x)$ on $J \times I$, the proof of the theorem is complete.

We shall prove next a theorem which deals with the upper and lower solutions of the PBVP (2.1), (2.2).

Theorem 2.2. Let (A_1) and (A_2) hold. Then (A_0) implies $m(t, x) \leq \beta(t, x)$ on $J \times I$.

Proof. We show first that if one of the inequalities in (A_1) and (A_0) is strict, then $m(t, x) < \beta(t, x)$ on $J \times I$. If not, there exists a $(t_0, x_0) \in J \times I$ and an $M \geq 0$ such that

$$(2.10) \quad m(t_0, x_0) - \beta(t_0, x_0) = M \quad \text{and} \quad m(t, x) - \beta(t, x) \leq M \quad \text{on} \quad J \times I.$$

If $t_0 \in (0, 2\pi)$ and $x_0 \in (0, 1)$, then we have $m_t = \beta_t$, $m_x = \beta_x$ and $m_{xx} \leq \beta_{xx}$ at (t_0, x_0) . Consequently

$$\begin{aligned} m_{xx} - \beta_{xx} &> m_t - g(t_0, x_0, m, m_x) - \beta_t + g(t_0, x_0, \beta, \beta_x) \\ &= g_u(t_0, x_0, \xi, \beta_x)(-M) \geq 0 \end{aligned}$$

where ξ is a point between $m(t_0, x_0)$ and $\beta(t_0, x_0)$. Thus we get

$$(2.11) \quad m_{xx} - \beta_{xx} > 0 \quad \text{at} \quad (t_0, x_0)$$

which is a contradiction to the definition of (t_0, x_0) .

Suppose next that $t_0 = 2\pi$ and $x_0 \in (0, 1)$. Then from (2.10) we obtain

$$(2.12) \quad m_t(2\pi, x_0) \geq \beta_t(2\pi, x_0) \quad \text{and} \quad m_x(2\pi, x_0) = \beta_x(2\pi, x_0).$$

If, on the other hand $t_0 = 0$ and $x_0 \in (0, 1)$, we have by (2.10) and (A_0)

$$m(2\pi, x_0) \geq m(0, x_0) = \beta(0, x_0) + M \geq \beta(2\pi, x_0) + M.$$

As a result, we again get (2.12). Hence if $t_0 = 0$ or 2π and $x_0 \in (0, 1)$, we are led to the inequality (2.11) at $(2\pi, x_0)$, which is a contradiction.

Suppose next that $t_0 \in (0, 2\pi]$ and $x_0 = 1$, $i = 0, 1$. Then from (2.10)

we get

$$(2.13) \quad \begin{cases} m_t(t_0, 1) \geq \beta_t(t_0, 1), \quad m_x(t_0, 0) \leq \beta_x(t_0, 0) \\ \text{and} \quad m_x(t_0, 1) \geq \beta_x(t_0, 1). \end{cases}$$

If, on the other hand, $t_0 = 0$ and $x_0 = 1$, $i = 0, 1$, we have

$$m(2\pi, 1) \geq m(0, 1) = \beta(0, 1) + M \geq \beta(2\pi, 1) + M.$$

As a result we get

$$(2.14) \quad \begin{cases} m_t(2\pi, 1) \geq \beta_t(2\pi, 1), \quad m_x(2\pi, 0) \leq \beta_x(2\pi, 0) \\ \text{and} \quad m_x(2\pi, 1) \geq \beta_x(2\pi, 1) \end{cases}$$

by (2.10). Hence if (A_0) holds, we get

$$(2.15) \quad m_x(t_0, 1) = \beta_x(t_0, 1), \quad i = 0, 1$$

by (2.13) and (A_1) . Consequently we are led to the inequality (2.11) at $(t_0, 1)$, which is a contradiction. If $t_0 = 0$ and $x_0 = 1$, a similar contradiction follows from (2.14) at $(2\pi, 1)$. This proves that $m(t, x) < \beta(t, x)$ on $J \times I$ if (A_0) holds and one of the inequalities in (A_0) and (A_1) is strict.

Let us now suppose that none of the inequalities in (A_0) and (A_1) is strict. We shall establish $m(t, x) \leq \beta(t, x)$ on $J \times I$. If this is not true, then there exists a point $(t_0, x_0) \in J \times I$ and a constant $M > 0$ such that (2.10) holds. The existence of such a (t_0, x_0) in $J \times I$ and $M > 0$ implies that $m(t_0, x) - \beta(t_0, x) \equiv M$ for all $x \in I$. If not, there exists an $x^* \in [0, 1]$ such that $m(t_0, x^*) - \beta(t_0, x^*) < M$. Suppose that $t_0 \in (0, 2\pi]$ and $x_0 \in (0, 1)$. Let $x^* \in (x_0, 1]$. Define $z(x) = e^{\alpha(x-x_0)} - 1$, $x \in I$. Note that $z(x) < 0$ for $x \in [0, x_0)$, $z(x_0) = 0$ and $z(x) > 0$ for $x \in (x_0, 1]$. Consider now a function

$$(2.16) \quad W(x) = m(t_0, x) - \beta(t_0, x) + \varepsilon z(x), \quad x \in I,$$

where we choose $\varepsilon > 0$ so that

$$0 < \varepsilon < \frac{M - [m(t_0, x^*) - \beta(t_0, x^*)]}{z(x^*)}.$$

Clearly such a choice is possible because $z(x^*) > 0$ and $M > m(t_0, x^*) - \beta(t_0, x^*)$.

Using (A_0) and (A_1) , we obtain for $x \in I$

$$\begin{aligned} W''(x) &= m_{xx}(t_0, x) - \beta_{xx}(t_0, x) + \varepsilon z''(x), \quad (' \equiv \frac{d}{dx}), \\ &\geq g_u(t_0, x, \xi, \eta) [\beta(t_0, x) - m(t_0, x)] \\ &\quad + g_{u_x}(t_0, x, \xi, \eta) [\beta_x(t_0, x) - m_x(t_0, x)] \\ &\quad + [m_t(t_0, x) - \beta_t(t_0, x)] + \varepsilon z''(x), \end{aligned}$$

where ξ is a point between $m(t_0, x)$ and $\beta(t_0, x)$ and η is a point between $m_x(t_0, x)$ and $\beta_x(t_0, x)$. By (2.16) this implies that for $x \geq x_0$

$$\begin{aligned}
W''(x) &\geq -g_u(t_0, x, \xi, \eta)W(x) - g_{u_x}(t_0, x, \xi, \eta)W'(x) \\
&\quad - \varepsilon g_u(t_0, x, \xi, \eta) \\
&\quad + \{\varepsilon[g_u(t_0, x, \xi, \eta) + \alpha g_{u_x}(t_0, x, \xi, \eta) + \alpha^2]e^{\alpha(x-x_0)} \\
&\quad + [m_t(t_0, x) - \beta_t(t_0, x)]\}.
\end{aligned}$$

We now choose $\alpha > 0$ so that

$$[g_u(t_0, x, \xi, \eta) + \alpha g_{u_x}(t_0, x, \xi, \eta) + \alpha^2] > 0.$$

Such a choice is possible because $g_u(t_0, x, \xi, \eta)$ and $g_{u_x}(t_0, x, \xi, \eta)$ are bounded on $J \times I$. Hence using the facts that $g_u \leq 0$ and $m_t(t_0, x) \geq \beta_t(t_0, x)$, we get

$$(2.17) \quad W''(x) > -g_u(t_0, x, \xi, \eta)W(x) - g_{u_x}(t_0, x, \xi, \eta)W'(x)$$

for $x \in [x_0, 1]$. However, from the relations $W(x_0) = M$, $W(x^*) < M$ and $z(x) > 0$ for $x \in (x_0, 1]$, it follows that there exists a point $\tilde{x} \in [x_0, x^*)$ at which W has a maximum $N \geq M > 0$. Then $W(\tilde{x}) = N \geq M > 0$, and since $\tilde{x} \in (0, 1)$, we have $W'(\tilde{x}) = 0$ and $W''(\tilde{x}) \leq 0$. But since $g_u(t_0, x, \xi, \eta) \leq 0$, (2.17) implies $W''(\tilde{x}) > 0$, which is a contradiction. Suppose next that $t_0 \in (0, 2\pi]$ and $x_0 = 0$. Then from (2.10) we get

$$(2.18) \quad m_x(t_0, 0) \leq \beta_x(t_0, 0) \quad \text{and} \quad m_x(t_0, 1) \geq \beta_x(t_0, 1).$$

If (A_0) holds, then (2.18) together with $\beta_x(t_0, 0) \leq \beta_x(t_0, 1)$ implies

$$(2.19) \quad m_x(t_0, i) = \beta_x(t_0, i), \quad i = 0, 1.$$

Consequently $W'(0) = m_x(t_0, 0) - \beta_x(t_0, 0) + \varepsilon \alpha = \varepsilon \alpha > 0$. Now the relations $W(0) = M$, $W(x^*) < M$ where $x^* \in (0, 1]$ and $W'(0) > 0$ show that there exists a point $\tilde{x} \in (0, x^*)$ at which W has a maximum $N \geq M > 0$, contradicting (2.17),

which now holds on $[0,1]$. If $t_0 \in (0, 2\pi]$ and $x_0 = 1$, taking the auxiliary function $z(x) = e^{-\alpha(x-x_0)} - 1$, where $\alpha > 0$ is chosen properly and following the above argument we again arrive at a contradiction. If $x^* \in [0, x_0)$, we take the auxiliary function $z(x) = e^{\alpha(x_0-x)} - 1$ and follow similar argument. Thus in all possible cases we have shown that $m(t_0, x) - \beta(t_0, x) \equiv M$ on I whenever we have (2.10). Consequently we get for all $x \in I$

$$0 \equiv m_{xx}(t_0, x) - \beta_{xx}(t_0, x) \geq g_u(t_0, x, \xi, m_x)(-M) \geq 0$$

by (A_0) and (A_1) , where $\xi = \xi(t_0, x)$ is a point between $m(t_0, x)$ and $\beta(t_0, x)$. Since $M > 0$, it follows from the above inequality that $g_u(t_0, x, \xi(t_0, x), m_x(t_0, x)) \equiv 0$ for all $x \in I$, which contradicts the hypothesis (A_2) . This proves that we have $m(t_0, x) - \beta(t_0, x) \equiv M$ on I , which then leads to a contradiction as before. Therefore, $m(t, x) \leq \beta(t, x)$ on $J \times I$ and the proof of the theorem is complete.

Theorem 2.3. The assertion of Theorem 2.2 remains valid if (A_2) is replaced by $(A_3)(1)$.

Proof. Suppose that $m(t, x) \leq \beta(t, x)$, $(t, x) \in J \times I$ is not true. Then we can find a minimal $\lambda > 0$ and a $(t_0, x_0) \in J \times I$ such that

$$(2.20) \quad \begin{cases} m(t_0, x_0) = \beta(t_0, x_0) + \lambda z(t_0, x_0), \text{ and} \\ m(t, x) \leq \beta(t, x) + \lambda z(t, x), (t, x) \in J \times I. \end{cases}$$

If $(t_0, x_0) \in (0, 2\pi) \times (0, 1)$, we have

$$m_t = \beta_t + \lambda z_t, m_x = \beta_x + \lambda z_x \text{ and } m_{xx} \leq \beta_{xx} + \lambda z_{xx}$$

at (t_0, x_0) . Hence by (A_1) and $(A_3)(1)$, we get at (t_0, x_0)

$$\begin{aligned}
m_t - m_{xx} &\geq \beta_t - \beta_{xx} + \lambda(z_t - z_{xx}) \\
&> g(t_0, x_0, \beta, \beta_x) \\
&\quad + g(t_0, x_0, \beta + \lambda z, \beta_x + \lambda z_x) \\
&\quad - g(t_0, x_0, \beta, \beta_x).
\end{aligned}$$

That is

$$m_t - m_{xx} > g(t_0, x_0, m, m_x) \quad \text{at } (t_0, x_0)$$

contradicting (A_0) .

Suppose next that $t_0 = 2\pi$ and $x_0 \in (0, 1)$. Then from (2.20) we get

$$(2.21) \quad \begin{cases} m_t(2\pi, x_0) \geq \beta_t(2\pi, x_0) + \lambda z_t(2\pi, x_0), \quad \text{and} \\ m_x(2\pi, x_0) = \beta_x(2\pi, x_0) + \lambda z_x(2\pi, x_0). \end{cases}$$

Also, if $t_0 = 0$ and $x_0 \in (0, 1)$ we obtain, using (2.20) the inequality

$$m(2\pi, x_0) \geq m(0, x_0) = \beta(0, x_0) + \lambda z(0, x_0) \geq \beta(2\pi, x_0) + \lambda z(2\pi, x_0).$$

As a result, we again get (2.21) from (2.20). Hence if $t_0 = 0$ or 2π and $x_0 \in (0, 1)$, we get

$$m_t - m_{xx} > g(2\pi, x_0, m, m_x)$$

at $(2\pi, x_0)$, which is again a contradiction. Suppose next that $t_0 \in (0, 2\pi]$ and $x_0 = i$, $i = 0, 1$. Then from (2.20) we get

$$(2.22) \quad \begin{cases} m_t(t_0, i) \geq \beta_t(t_0, i) + \lambda z_t(t_0, i), \\ m_x(t_0, 0) \leq \beta_x(t_0, 0) + \lambda z_x(t_0, 0), \quad \text{and} \\ m_x(t_0, 1) \geq \beta_x(t_0, 1) + \lambda z_x(t_0, 1). \end{cases}$$

Also if $t_0 = 0$ and $x_0 = i$, we obtain from (2.20)

$$m(2\pi, i) \geq m(0, i) = \beta(0, i) + \lambda z(0, i) \geq \beta(2\pi, i) + \lambda z(2\pi, i).$$

Consequently, using (2.20), we get

$$(2.23) \quad \begin{cases} m_t(2\pi, 1) \geq \beta_t(2\pi, 1) + \lambda z_t(2\pi, 1), \\ m_x(2\pi, 0) \leq \beta_x(2\pi, 0) + \lambda z_x(2\pi, 0), \quad \text{and} \\ m_x(2\pi, 1) \geq \beta_x(2\pi, 1) + \lambda z_x(2\pi, 1). \end{cases}$$

Now if (A_0) holds; (2.22), (A_1) and $(A_3)(i)$ imply

$$(2.24) \quad m_x(t_0, i) = \beta_x(t_0, i) + \lambda z_x(t_0, i), \quad i = 0, 1,$$

which leads to

$$m_t - m_{xx} > g(t_0, i, m, m_x) \quad \text{at } (t_0, i),$$

which is a contradiction. A similar contradiction results from (2.23) if

$t_0 = 0$ and $x_0 = 1$. The proof of the theorem is complete.

Theorem 2.4. The assertion of Theorem 2.3 remains valid if $(A_3)(i)$ is replaced by $(A_3)(ii)$.

The proof follows by arguments similar to those in Theorem 2.3 and hence omitted.

We shall next consider equation (2.1) with the homogeneous Neumann boundary conditions

$$(2.2 N) \quad u(0, x) = u(2\pi, x), \quad x \in I, \quad u_x(t, 0) = u_x(t, 1) = 0, \quad t \in J.$$

Comparison results analogous to those for the PBVP (2.1), (2.2) can be proved for the BVP (2.1), (2.2 N). But to avoid monotony, we shall only state the analogous theorems and omit the proofs.

We need the following conditions for functions $m, \beta, z \in C^{1,2}[J \times I, \mathbb{R}]$.

$$(B_0) \quad m(0, x) \leq m(2\pi, x), \quad m_x(t, 0) \geq 0, \quad m_x(t, 1) \leq 0 \quad \text{and} \\ m_t - m_{xx} \leq g(t, x, m, m_x) \quad \text{on } (0, 2\pi] \times [0, 1].$$

$$(B_1) \quad \beta(0, x) \geq \beta(2\pi, x), \quad \beta_x(t, 0) \leq 0, \quad \beta_x(t, 1) \geq 0 \quad \text{and} \\ \beta_t - \beta_{xx} \geq g(t, x, \beta, \beta_x) \quad \text{on } (0, 2\pi] \times [0, 1].$$

$$(B_2) \quad \text{Same as } (A_2).$$

$$(B_3) \quad z(t, x) > 0 \quad \text{on } J \times I, \quad z(0, x) \geq z(2\pi, x), \\ z_x(t, 0) \leq 0, \quad z_x(t, 1) \geq 0 \quad \text{and for every } \lambda > 0 \text{ satisfies}$$

$$(i) \quad \lambda(z_t - z_{xx}) > g(t, x, \beta + \lambda z, \beta_x + \lambda z_x) \\ - g(t, x, \beta, \beta_x) \quad \text{on } J \times I,$$

$$(ii) \quad \lambda(z_t - z_{xx}) > g(t, x, m, m_x) \\ - g(t, x, m - \lambda z, m_x - \lambda z_x) \quad \text{on } J \times I.$$

We shall only state the results related to (2.1), (2.2 N) and the proofs follow by arguments similar to those for (2.1), (2.2).

Theorem 2.5. Consider the BVP (2.1), (2.2 N). Assume that

(i) the modified BVP (2.3), (2.2 N) admits a solution u for every lower solution v of (2.1), (2.2 N);

(ii) $r(t, x)$ is the maximal solution of the BVP (2.1), (2.2 N) on $J \times I$.

Then (B_0) implies $m(t, x) \leq r(t, x)$ on $J \times I$.

Theorem 2.6. Let (B_1) and (B_2) hold. Then (B_0) implies that $m(t, x) \leq \beta(t, x)$ on $J \times I$.

Theorem 2.7. The assertion of Theorem 2.6 remains valid if (B_2) is replaced by $(B_3)(i)$ or $(B_3)(ii)$.

ACKNOWLEDGEMENT

The authors are grateful to Professor V. Lakshmikantham for many stimulating discussions and helpful suggestions during this work.

REFERENCES

- [1] Bernfeld, S. R. and Lakshmikantham, V., "An Introduction to Nonlinear Boundary Value Problems," Academic Press, New York, 1974.
- [2] Chandra, J., Lakshmikantham, V. and Leela, S., Comparison Principle and Theory of Nonlinear Boundary Value Problems, To appear in "The Proceedings of the International Conference on Nonlinear Phenomena in Mathematical Sciences," Academic Press, New York, 1980.
- [3] Chandra, J., Lakshmikantham, V. and Leela, S., A Monotone Method for Infinite Systems of Nonlinear Boundary Value Problems, Arch. Rat. Mech. Anal., Vol. 68 (2), 1978, 179-190.
- [4] Lakshmikantham, V. and Leela, S., "Differential and Integral Inequalities," Vol. I and II, Academic Press, New York, 1969.
- [5] Lakshmikantham, V., Comparison Results for Reaction-Diffusion Equations in a Banach Space, Lecture notes of talks delivered at the conference on, "A Survey of Theoretical and Numerical Trends in Nonlinear Analysis," Bari, Itali, 1979, 121-150.
- [6] Lakshmikantham, V. and Leela, S., Existence and Monotone Method for Periodic Solutions of First Order Differential Equations, Jour. Math. Appl. (to appear)
- [7] Nieto-Roig, J. J., Problems at Resonance for First and Second Order Differential Equations via Lyapunov-like functions, The Univ. of Texas at Arlington, Technical Report No. 181, April, 1982.
- [8] Pao, C. V., Asymptotic Behaviour and Nonexistence of Global Solutions for a Class of Nonlinear Boundary Value Problems of Parabolic type, Jour. Math. Anal. Appl. 65 (1978), 616-637.
- [9] Schroder, Johann, "Operator Inequalities," Academic Press, New York, 1980.
- [10] Shendge, G. R. and Vatsala, A. S., Comparison Results for First and Second Order Boundary Value Problems at Resonance, to appear in Applied Math. and Computation.
- [11] Vatsala, A. S., Existence of Coupled Minimal and Maximal Periodic Quasi-Solutions for Systems of First Order Periodic Boundary Value Problems via Quasi-Solutions, to appear in The Proceedings of the Vth International Conference on "Trends in Theory and Practice of Nonlinear Differential Equations," held at The University of Texas at Arlington, in June 1982.