**ASYMPTOTIC PROPERTIES OF A NONLINEAR DIFFUSION
PROCESS ARISING IN ARTICULAR CARTILAGE**

by

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Introduction

There has been considerable effort put forth to analyse degenerative joint diseases in terms of the principles of mechanics and hydrodynamics [8]. This effort has led to mathematical models which are systems of nonlinear partial differential equations (see e.g., [6] and [7]). The mathematical models are based on assumptions concerning the nature of the physical system which are not a priori known. Therefore laboratory experiments are performed with an eye towards validating the mathematical models.

This paper deals with a particular mathematical model, a nonlinear diffusion equation, which has been proposed by Mow and Monsour [4], [5] to describe the stress relaxation of articular cartilage. More precisely, the stress relaxation function $f(t)$ is related to a solution $u(x,t)$ of a nonlinear PDE problem (see (1.1)-(1.4) below). In this paper we analytically determine the behavior of $f(t)$ (Theorem 1). The consistency of these results to already existing theory and experimental findings is discussed in [1].

We consider the nonlinear PDE problem:

$$u_t = K(u_x)u_{xxx}, \quad 0 < x < 1, \quad t > 0 \quad (1.1)$$

$$u(0, t) = U(t), \quad 0 \leq t \leq T, \quad u(0, t) = U(T), \quad t \geq T \quad (1.2)$$

$$u(1, t) = 0, \quad 0 < t < \infty \quad (1.3)$$

$$u(x, 0) = 0, \quad 0 \leq x \leq 1. \quad (1.4)$$

The functions K and U have the following properties:

(A) U is continuously differentiable in $[0, T]$, U' is nondecreasing in $[0, T]$, and

$$U(0) = 0, \quad U'(t) > 0, \quad 0 < t < T, \quad (1.5)$$

(B) K is defined in an interval $[a, b]$ which contains zero, K is continuously differentiable, and

$$K(e) > 0, \quad K'(e) > 0, \quad a \leq e \leq b, \quad (1.6)$$

(C) The PDE problem (1.1)-(1.4) admits a solution u such that

(i) u and u_x are continuous in $D = [0, 1] \times [0, \infty)$, (ii) u_t is continuous in $D - \{0, T\}$, (iii) all partial derivatives of u up to and including the third order are continuous in $(0, 1) \times (0, \infty)$,

(iv) $u_{xt}(0, t) = u_{tx}(0, t)$ for $0 < t < T$ and (v) for each $t_0 > T$ there exists a positive constant B such that

$$u_{tx}(x, t) \geq -B, \quad 0 < x < 1, \quad T < t < t_0. \quad (1.7)$$

Our main concern is the function

$$f(t) = -u_x(0, t), \quad t \geq 0. \quad (1.8)$$

We are particularly interested in studying the asymptotic behavior of f as $t \rightarrow +\infty$. For this purpose we define a relaxation constant (see the remark below)

$$\tau(f) = -\lim_{t \rightarrow +\infty} t[\ln(f(t) - f(\infty))]^{-1} \quad (1.9)$$

where we denote,

$$f(\infty) = \lim_{t \rightarrow +\infty} f(t). \quad (1.10)$$

Of course the relaxation constant is defined only when the limit in (1.10) exists, $f(t) > f(\infty)$ for large t , and the limit in (1.9) exists.

We will verify the following results.

THEOREM 1. Under conditions (A), (B) and (C) (above). The function $f(t)$, defined in (1.8), has the following properties.

- (i) $f(0) = 0$, $f(t) \geq U(t)$, $0 \leq t \leq T$.
- (ii) f is nondecreasing in $[0, T]$.
- (iii) If f' and U'' exist, in $(0, T)$ and u_{xx} is continuous in a neighborhood of $(0, 1) \times \{0\}$ then $f'(t) \geq U'(t)$ in $(0, T)$.
- (iv) Let $K_0 = K(-f(T))$, $K_1 = K(0)$ and let $u^{(i)}$ be the solution to the linear PDE problems

$$u_t^{(i)} = K_i u_{xxx}^{(i)}, \quad 0 < x < 1, \quad t > 0 \quad (1.11)$$

having the same boundary conditions and initial conditions as u for $i = 0, 1$. Let

$$f^{(i)}(t) = -u_x^{(i)}(0, t), \quad t \geq 0, \quad i = 0, 1. \quad (1.12)$$

Then

$$f^{(1)}(t) \leq f(t) \leq f^{(0)}(t), \quad t \geq 0, \quad (1.13)$$

(v) f is nonincreasing for $t > T$ and

$$f(\infty) = U(T). \quad (1.14)$$

(vi) The relaxation constant $\tau(f)$ is defined for f and

$$\tau(f) = [\pi^2 K(-U(T))]^{-1}. \quad (1.15)$$

REMARK. Equations (1.14) and (1.15) may be expressed in the form:

$$f(t) = U(T) - a(t) \exp[-n^2 K(-U(T))t] \quad (1.16)$$

where $a(t)$ is slowly varying as $t \rightarrow +\infty$, i.e.

$$\lim_{t \rightarrow +\infty} t^{-1} \ln a(t) = 0. \quad (1.17)$$

This expresses mathematically the physically observed observation [3] that the long time behavior of $f(t)$ is essentially determined by the value of U at T . Note that $U(T)$ is the average of $-u_x(x, t)$ in the interval $[0, 1]$ for each fixed $t \geq T$. The formula shows that $\tau = \tau(f)$ decreases with increasing $U(T)$ as observed in laboratory

experiments [3]. A linear model, i.e., $K \equiv \text{constant}$, can not provide this dependence of τ on $U(T)$.

2. Proof of Theorem 1

The proof of Theorem 1 will be presented in steps. Some of the results obtained along the way are not stated in Theorem 1 but they may be of interest. The main analytical tool is the well known maximum principle which has many forms (see e.g. Chapter 2 in [2]). For our purposes it is convenient to use the following version:

MAXIMUM PRINCIPLE. Let $x_2 > x_1$, $t_2 > t_1$, $R = [x_1, x_2] \times [t_1, t_2]$, $R_0 = (x_1, x_2) \times (t_1, t_2]$, and let $\partial R = R - R_0$. Let $u(x, t)$ be continuous in R with partial derivatives $u_t(x, t)$, $u_x(x, t)$ and $u_{xx}(x, t)$ in R_0 . Let $p(x, t)$ and $q(x, t)$ be defined in R_0 with $p(x, t) \geq 0$. Let $F(x, t, u) \equiv u_t(x, t) - p(x, t)u_{xx}(x, t) - q(x, t)u_x(x, t)$. Then: (i) if $F \leq 0$ in R_0 there is at least one point in ∂R at which u takes its maximum value, (ii) if $F \geq 0$ in R_0 there is at least one point in ∂R at which u takes its minimum value, (iii) if $F \equiv 0$ in R_0 then u takes both its maximum and minimum values in ∂R , (iv) if $F < 0$ in R_0 then u can not achieve its maximum value in R_0 , (v) if $F > 0$ in R_0 then u can not achieve its minimum value in R_0 .

Lemma 1. For the solution $u(x, t)$ of the nonlinear PDE problem we have for arbitrary $t_0 > 0$ the inequalities

$$0 \leq u(x, t) \leq U(t_0) \leq U(T), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq t_0. \quad (2.1)$$

Proof. This is an immediate consequence of the maximum principle where $p(x, t) = K(u_x(x, t)) > 0$, $q(x, t) = 0$, and $R = [0, 1] \times [0, t]$. Then $F \equiv 0$ and u attains its maximum, $U(t_0)$, and its minimum, zero, in ∂R .

Lemma 2. For the solution $u(x, t)$ of the nonlinear PDE problem we have the inequalities

$$u_x(x, t) \leq 0, \quad 0 \leq x \leq 1, \quad t \geq 0. \quad (2.2)$$

Proof. Differentiating with respect to x in (1.1) we obtain

$F(x, t, u_x) = (u_x)_t - K(u_x)(u_x)_{xxx} - K'(u_x)u_{xxx}(u_x)_x \equiv 0$, in $(0, 1) \times (0, t_0)$ for each $t_0 > 0$. Since t_0 is arbitrary it suffices to show in view of the maximum principle, that $u_x(x, t)$ is nonpositive on the boundary of the PDE region. Clearly $u_x(x, 0) \equiv 0$ since $u(x, 0) \equiv 0$. Now, by (2.1), for each fixed value of t , $u(x, t)$ takes on its maximum value at $x = 0$ and its minimum value at $x = 1$. Thus $u_x(0, t) \leq 0$ and $u_x(1, t) \leq 0$.

Lemma 3. For the solution $u(x, t)$ of the nonlinear PDE problem we have the inequalities

$$0 \leq u_t(x, t) \leq U'(t), \quad 0 \leq x \leq 1, \quad 0 \leq t < T. \quad (2.3)$$

Proof. Differentiating in (1.1) with respect to t we obtain

$$F(x, t, u_t) = (u_t)_t - K(u_x)(u_t)_{xxx} - K'(u_x)u_{xxx}(u_t)_x \equiv 0 \quad (2.4)$$

in $(0, 1) \times (0, \infty)$. Let $0 < t_0 < T$ and apply the maximum principle in $R = [0, 1] \times [0, t_0]$. First we show that $u_t(x, t) \geq 0$ in R . Certainly $u_t(0, t) = U'(t) \geq 0$ and $u_t(1, t) \equiv 0$. Since by, Lemma 1,

$u(x, t) \geq 0 = u(x, 0)$, we know that $u_t(x, 0) \geq 0$. Thus, by the maximum principle, $u_t(x, t) \geq 0$ in R .

To obtain the second inequality in (2.3) we show that the maximum of $u_t(x, t)$ in ∂R occurs at $x = 0$, $t = t_0$. Because $u_t(1, t) \equiv 0$ and $u_t(0, t) = U'(t) > 0$ for $t > 0$, the maximum can not occur when $x = 1$. Since $U'(t)$ is increasing the maximum of $u_t(0, t)$ occurs at $t = t_0$. We conclude the proof of (2.3) by showing that

$$u_t(x, 0) \leq U'(0), \quad 0 \leq x \leq 1. \quad (2.5)$$

Recall by (2.1) that $u(x, t) \leq U(t)$. Thus for fixed x , $u(x, t)/t \leq U(t)/t$ for small $t > 0$. Taking limits as $t \rightarrow 0$ (recall $u(x, 0) = U(0) = 0$) gives the inequality (2.5).

Lemma 4. The solution $u(x, t)$ of the nonlinear PDE satisfies the inequality

$$u(x, t) \leq U(t)(1 - x), \quad 0 \leq x \leq 1, \quad t \geq 0. \quad (2.6)$$

Proof. We set $v(x, t) = u(x, t) - U(t)(1 - x)$ and show $v \leq 0$. This is an immediate consequence of the maximum principle since $F(x, t, v) = v_t - K(u_x)v_{xx} = -U'(t)(1 - x) \leq 0$ and since $v(x, t) \equiv 0$ for $x = 0, 1$ or for $t = 0$.

Lemma 5. Conclusion (i) of Theorem 1 is satisfied.

Proof. The fact that $f(0) = 0$ is obvious since $u(x, 0) \equiv 0$. Now, from (2.6), $(u(x, t) - u(0, t))/x \leq -U(t)$ for $x > 0$. Taking limits as $x \rightarrow 0$ gives $u_x(0, t) \leq -U(t)$ or $f(t) \geq U(t)$.

Lemma 6. Conclusion (ii) of Theorem 1 is satisfied.

Proof. Let $t < T$. From (2.3) we have $u_t(x, t) \leq U'(t) = u_t(0, t)$ for $0 \leq x \leq 1$. Thus $f'(t) = -u_{xt}(0, t) = -u_{tx}(0, t) \geq 0$.

Lemma 7. For the solution $u(x, t)$ of the nonlinear PDE problem we have

$$u_t(x, t) \geq 0, \quad u_{xxx}(x, t) \geq 0, \quad 0 < x < 1, \quad t \geq 0. \quad (2.7)$$

Proof. Since $K(u_x) > 0$, either inequality implies the other. We show $u_t \geq 0$.

Suppose there exists a point (x_1, t_1) in $(0, 1) \times (0, \infty)$ such that $u_t(x_1, t_1) < 0$. Let $m = -u_t(x_1, t_1) > 0$. We apply the maximum principle in the region $R = [h, 1] \times [0, t_0]$ where $t_0 > t_1$ is fixed and $0 < h < x_1$. Let $F(x, t, u_t)$ be defined as in (2.4). Since $F \equiv 0$ in R_0 we know that u_t takes on its minimum value in ∂R at some point (x_2, t_2) in ∂R . Thus $u_t(x_2, t_2) \leq -m$. We have already shown (Lemma 3) that $u_t \geq 0$ in $[h, 1] \times [0, T]$. Since u_t is continuous in R we conclude further that $u_t \geq 0$ in $[h, 1] \times [0, T]$. Thus $t_2 > T_1$. Moreover, $u_t(1, t) \equiv 0$ so that $x_2 \neq 1$ and hence $x_2 = h$. Now since $u_t(0, t_2) = 0$ it follows from the mean value theorem that there exists a point (x_3, t_2) in $(0, 1) \times (T, t_0)$ such that $u_{tx}(x_3, t_2) \leq -m/h$. But h may be chosen arbitrarily small and so condition (1.7) is contradicted. We conclude that $u_t \geq 0$ in $(0, 1) \times [0, \infty)$.

Lemma 8. Suppose $U''(t)$ exists in $(0, T)$ and u_{xxx} is continuous in a neighborhood of $(0, 1) \times \{0\}$ then

$$u_t(x, t) \leq U'(t)(1 - x), \quad 0 < x < 1, \quad 0 < t < T. \quad (2.8)$$

Proof. We apply the maximum principle in the region $R = [0, 1] \times [0, t_0]$, where t_0 is arbitrary in $(0, T]$, to the function $v = u_t - U'(t)(1 - x)$. It follows from (2.4) that $F(x, t, v) = v_t - K(u_x)v_{xx} - K'(u_x)u_{xx}v_x = -U''(t)(1 - x) - K'(u_x)u_{xx}U'(t) \leq 0$ in R_0 . (Recall $U'' \geq 0$ since U' is nondecreasing, $K' \geq 0$, and $u_{xx} \geq 0$ by (2.7)). Thus v takes its maximum value in ∂R . By (2.3) we know that $v(0, t) \leq 0$ and clearly, $v(1, t) \equiv 0$. We conclude by showing that

$$u_t(x, 0) \equiv 0, \quad 0 < x < 1 \quad (2.9)$$

and hence $v(x, 0) = -U'(0)(1 - x) \leq 0$.

Since u_t , u_x , u_{xx} are all continuous in a neighborhood of $(0, 1) \times \{0\}$ and since K is also continuous we have that (i.1) is satisfied in $(0, 1) \times \{0\}$. Since $u(x, 0) \equiv 0$ we have $u_{xx}(x, 0) \equiv 0$ and thus also $u_t(x, t) \equiv 0$ in $(0, 1)$.

Lemma 9. Conclusion (iii) of Theorem 1 is satisfied.

Proof. Let $v(x, t) = U(t)(1 - x) - u(t, x)$. It suffices to show that $v_{tx}(0, t) \geq 0$. But this is an immediate consequence of Lemma 8 since $v_t(0, t) = 0$ and $v_t(x, t) \geq 0$ for $x \geq 0$.

Lemma 10. Conclusion (iv) of Theorem 1 is satisfied.

Proof. From (2.7) we have that $u_t(x, t) \geq 0$. Let $t > T$. For $h > 0$, $u(x, t + h) \geq u(x, t)$. Since $u(0, t + h) = u(0, t)$ we have $u(x, t + h) - u(0, t + h) \geq u(x, t) - u(0, t)$. Dividing by x and taking limits as

$x \rightarrow 0$ we obtain $u_x(0, t+h) \geq u_x(0, t)$ or $f(t+h) \leq f(t)$.

Lemma 11. For the solution $u(x, t)$ of the nonlinear PDE we have that $u_x(1, t)$ is nonincreasing in $[0, \infty)$.

Proof. From (2.7) we know that $u_t(x, t) \geq 0$. Thus for $h > 0$, $u(x, t+h) \geq u(x, t)$. Since $u(1, t) = u(1, t+h) (\neq 0)$ the above inequality is equivalent to $u(x, t+h) - u(1, t+h) \geq u(x, t) - u(1, t)$. Dividing by $x - 1$ (< 0) and taking limits as $x \rightarrow 1$ gives the desired result.

Lemma 12. For the solution u of the nonlinear PDE we have for each $t_0 \geq T$:

$$-f(T) \leq u_x(x, t) \leq 0, \quad 0 \leq x \leq 1, \quad t \leq 0, \quad (2.10)$$

$$-f(t_0) \leq u_x(x, t) \leq u_x(1, t_0), \quad 0 \leq x \leq 1, \quad t \geq t_0 \geq T. \quad (2.11)$$

Proof. We have already established the inequality $u_x(x, t) \leq 0$ in Lemma 2. Let $F(x, t, u_x) \equiv 0$ be defined as in Lemma 2 and we proceed in the same manner only this time we seek the minimum of u_x on the boundary of the PDE region. Since $u_x(x, 0) \equiv 0$ and since $u_{xx}(x, t) \geq 0$ for $t > 0$ (Lemma 7) the minimum must occur when $x = 0$. However the minimum of $u_x(0, t)$ is the negative of the maximum of $f(t)$ which is, by Lemmas 6 and 10, $-f(T)$. This establishes (2.10).

Let $t_i > t_0 \geq T$ and apply the maximum principle to u_t on $R = [0, 1] \times [t_0, t_1]$ with the same choice of $F(x, t, u_t) \equiv 0$. Since $u_{xx} \geq 0$ the maximum value of u_x must occur on $\{1\} \times [t_0, t_1]$ while

the minimum value must occur on $\{0\} \times [t_0, t_2]$. Since $u_x(1, t)$ is nonincreasing (Lemma 11) and since $u_x(0, t)$ is nondecreasing (Lemma 10) the maximum is $u(1, t_0)$ while the minimum is $u(0, t_0)$. Since $t_1 > t_0$ is arbitrary (2.11) is established.

Lemma 13. Define the constants

$$K_0 = K(-f(T)), \quad K_1 = K(0) \quad (2.12)$$

and for each $t_0 \geq T$ define

$$J_0(t_0) = K(-f(t_0)), \quad J_1(t_0) = K(u_x(1, t_0)). \quad (2.13)$$

Then for the solution u of the nonlinear PDE we have the bounds

$$K_0 \leq K(u(x, t)) \leq K_1 \quad 0 \leq x \leq 1, \quad t \geq 0, \quad (2.14)$$

$$J_0(t_0) \leq K(u(x, t)) \leq J_1(t_0), \quad 0 \leq x \leq 1, \quad t \geq t_0 \geq T. \quad (2.15)$$

Proof. This follows immediately from Lemma 12 and the fact that K is nondecreasing.

Lemma 14. Let u be the solution of the nonlinear PDE and let $u^{(i)}$, $i = 0, 1$ be as defined in Theorem 1, (iv). Then

$$u^{(0)}(x, t) \leq u(x, t) \leq u^{(1)}(x, t), \quad 0 \leq x \leq 1, \quad t \geq 0. \quad (2.16)$$

Proof. Let $m(x, t) = u(x, t) - u^{(0)}(x, t)$. Then $m_t - K_0 m_{xx} = (K(u_x) - K(u_x^{(0)})) u_{xxx} > 0$.

Since $m \equiv 0$ on the boundary of the PDE region the maximum principle

implies that $m \geq 0$ inside the region. Thus the first inequality

$u^{(0)}(x, t) \leq u(x, t)$ is satisfied. The second inequality is obtained in a completely similar fashion.

Lemma 15. Conclusion (iv) of Theorem 1 is satisfied.

Proof. Since $u^{(0)}(0, t) = u(0, t) = u^{(1)}(0, t)$, we have from (2.16) that $u^{(0)}(x, t) - u^{(0)}(0, t) \leq u(x, t) - u(0, t) \leq u^{(1)}(x, t) - u^{(1)}(0, t)$. Dividing by x and taking limits as $x \rightarrow 0$ gives the desired result.

Lemma 16. Let $u, u^{(0)}, u^{(1)}$ be as defined in Theorem 1, (iv). Then

$$u_x^{(1)}(1, t) \leq u_x(1, t) \leq u_x^{(0)}(1, t), \quad t \geq 0. \quad (2.17)$$

Proof. The proof is essentially the same as for Lemma 15'.

Lemma 17. Let u be the solution of the nonlinear PDE. Then $u_x(x, t)$ converges uniformly in $x \in [0, 1]$ to $-U(T)$ as $t \rightarrow +\infty$. In particular conclusion (v) of Theorem 1 is satisfied.

Proof. The solution $u^{(i)}$ of (1.11) may be determined by Fourier series techniques and one obtains

$$u_x^{(i)}(x, t) = -U(T) - \sum_{n=1}^{\infty} a_n^{(i)} \exp(-t/\tau_n^{(i)}) \cos(n\pi x), \quad t > T, \quad (2.18)$$

where

$$a_n^{(i)} = 2 \int_0^T U'(\tau) \exp(K_i^2 n^2 \pi^2 \tau) d\tau, \quad n = 1, 2, \dots, \quad (2.19)$$

$$\tau_n^{(i)} = [K_i^2 n^2 \pi^2]^{-1}, \quad n = 1, 2, \dots \quad (2.20)$$

Thus $u_x^{(i)}(x, t) \rightarrow -U(T)$ as $t \rightarrow +\infty$ for each $x \in [0, 1]$. In particular, $u_x^{(i)}(0, t)$ and $u_x^{(i)}(1, t)$, $i = 0, 1$ all converge to $-U(T)$. In view of Lemmas 15 and 16 we see that $u_x(0, t) (= -f(t))$ and $u_x(1, t)$ both converge to $-U(T)$ and hence, by (2.11), $u_x(x, t)$ converges uniformly to $-U(T)$.

Lemma 18. Let $t_0 > T$ be specified. Let $u^{(i)}(x, t; t_0)$ be solutions of the linear PDE

$$u_{tt}^{(i)} = J_i(t_0) u_{xx}^{(i)} \quad (2.12)$$

in $R = [0, 1] \times (t_0, \infty)$ such that

$$u^{(i)}(x, t; t_0) = u(x, t), \quad (x, t) \in \partial R, \quad (2.22)$$

where the constants $J_i(t_0)$, $i = 0, 1$, are defined in (2.13). Let

$$f^{(i)}(t; t_0) \equiv u_x^{(i)}(0, t; t_0), \quad t \geq t_0, \quad i = 0, 1. \quad (2.23)$$

Then

$$f^{(1)}(t; t_0) \leq f(t) \leq f^{(0)}(t; t_0), \quad t \geq t_0. \quad (2.24)$$

Proof. First of all it follows from (2.15) and the maximum principle (as in Lemma 14) that

$$u^{(0)}(x, t; t_0) \leq u(x, t) \leq u^{(1)}(x, t; t_0), \quad 0 \leq x \leq 1, \quad t \geq t_0. \quad (2.25)$$

The conclusion then follows from the argument used in Lemma 15.

Lemma 19. Conclusion (vi) of Theorem 1 is satisfied.

Proof. First we observe from (1.5), (2.18), (2.19), and Lemma 15 that

$$f(t) - U(T) > 0, \quad t > T, \quad (2.26)$$

and hence $\ln(f(t) - U(T))$ is defined for $t > T$.

Next let $f^{(i)}(t, t_0)$ be as defined in Lemma 18. One finds that

$$f^{(i)}(t; t_0) - U(T) = \sum_1^\infty c_n^{(i)} \exp(-J_i(t_0) n^2 \pi^2 t), \quad t > t_0, \quad (2.27)$$

where

$$c_n^{(i)} = 2n\pi \exp(J_i(t_0)n^2\pi^2 t_0) \int_0^1 [U(T)(1-x) - u(x, t_0)] \sin(n\pi x) dx. \quad (2.28)$$

Observe that if $c_1^{(i)} > 0$ then

$$\lim_{t \rightarrow +\infty} t^{-1} \ln[f^{(i)}(t; t_0) - U(T)] = -J_i(t_0)\pi^2. \quad (2.29)$$

We know from Lemma 5 that $U(T)(1-x) - u(x, t_0) \geq 0$ and hence

$c_1^{(i)} \geq 0$ and $c_1^{(i)} = 0$ only if $u(x, t_0) - U(T)(1-x) \equiv 0$. But the last equation would imply that $f(t_0) = U(T)$ which contradicts (2.26).

This establishes (2.29).

From (2.24) and (2.8) we have, for $t_0 > T$,

$$\begin{aligned} -J_1(t_0)\pi^2 &\leq \liminf_{t \rightarrow +\infty} t^{-1} \ln(f(t) - U(T)) \leq \limsup_{t \rightarrow +\infty} t^{-1} \ln(f(t) - U(T)) \\ &\leq -J_0(t_0)\pi^2. \end{aligned} \quad (2.30)$$

However, from Lemma 17 and (2.13), both $J_0(t_0)$ and $J_1(t_0)$ converge to $K(-U(T))$ as $t_0 \rightarrow +\infty$. We conclude that $\lim_{t \rightarrow +\infty} t^{-1} \ln(f(t) - U(T))$ exists and equals $-K(-U(T))\pi^2$.

Proof of Theorem 1. The proof is contained in Lemmas 5, 6, 9, 10, 15, and 19.

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