SOBOLEV TYPE DIFFERENTIAL EQUATIONS

by

*V. Lakshmikantham and M. Lord
Department of Mathematics
University of Texas at Arlington
Arlington, Texas 76019*

Technical Report No. 58

June, 1977

SOBOLEV TYPE DIFFERENTIAL EQUATIONS

V. Lakshmikantham[†] and M. Lord

1. Introduction. An imbedding method for solving linear Fredholm integral equations was introduced by Sobolev [3] which involves the solution of the following differential equation with initial value for the resolvent kernel

$$(1.1) \quad \begin{aligned} K_x(t, y, x) &= K(t, x, x)K(x, y, x) , \\ K(t, y, 0) &= \phi(t, y) , \quad 0 \leq t, y, x \leq a . \end{aligned}$$

The differential equation in (1.1) is unusual in that the solution $K(t, y, x)$ is evaluated at different combinations of the independent variables (t, y, x) . We will refer to any differential equation with this property as a Sobolev type differential equation.

We introduce a Sobolev type differential equation which generalizes (1.1) and consider conditions for existence and uniqueness. A Picard type theorem is obtained, which by way of application, is used to obtain conditions for existence and uniqueness for the system (1.1).

Some further results are obtained for Sobolev type differential equations including solution of a linear constant coefficient equation, a Bellman-Gronwall type inequality and a variation of constants formula for solutions of perturbed linear equations. These results generalize in a natural way corresponding well known results as found in [2].

An application of Sobolev's imbedding technique to nonlinear Fredholm integral equations was obtained by Kagiwada and Kalaba [1]. The result obtained there involves a system of integro-differential equations of Sobolev type. Theory relating to this result is more complex and will be developed in a subsequent paper.

[†]Supported by U.S. Army Research Office Grant DAAG29-77-G0062

2. Existence and Uniqueness. Let I denote the interval $[0, a]$. We are interested in studying solutions in R^n of the Sobolev type initial value problem

$$(2.1) \quad \begin{aligned} u_x(t, y, x) &= F(t, y, x, u(t, y, x), u(t, x, x), u(x, y, x)) , \\ u(t, y, 0) &= \phi(t, y) , \quad (t, y, x) \in I \times I \times I . \end{aligned}$$

Remark: The differential equation could be generalized so that the function F depends on the solution at (z_1, z_2, z_3) where z_i is either t, y , or x . This would introduce 27 terms of the form $u(z_1, z_2, z_3)$. The form considered in (2.1) is adequate for the application to the system (1.1) and the difficulties in considering the most general function F appear to be notational.

If $||\cdot||$ denotes a convenient norm on R^n , then the ball of radius b about the initial value is given by

$$S = \{u \in R^n \mid ||u - \phi(t, y)|| \leq b, t, y \in I\} .$$

The continuity conditions are

$$(2.2) \quad \begin{aligned} \text{i)} \quad &\phi \in C[I \times I, R^n] , \\ \text{ii)} \quad &F \in C[I \times I \times I \times S \times S \times S, R^n] \end{aligned}$$

A Lipschitz condition on F is given by

$$(2.3) \quad \begin{aligned} &||F(t, y, x, u_1, v_1, w_1) - F(t, y, x, u_2, v_2, w_2)|| \\ &\leq L_1 ||u_1 - u_2|| + L_2 ||v_1 - v_2|| + L_3 ||w_1 - w_2|| , \end{aligned}$$

whenever $(t, y, x, u_i, v_i, w_i) \in I \times I \times I \times S \times S \times S$, $i = 1, 2$.

The following theorem gives a sufficient condition for local existence and uniqueness of solutions of (2.1).

Theorem 2.1. Assume the continuity condition (2.2) and the Lipschitz condition (2.3) hold. Let

$$M = \sup ||F(t,y,x,u,v,w)|| , (t,y,x,u,v,w) \in I \times I \times I \times S \times S \times S ,$$

and $\delta = \min\{a,b/M\}$. Then there exists a unique solution of (2.1) defined for $(t,y,x) \in \Delta \times \Delta \times \Delta$, where $\Delta = [0,\delta]$.

Proof. The initial value problem (2.1) is equivalent to the integral equation

$$(2.4) \quad u(t,y,x) = \phi(t,y) + \int_0^x F(t,y,s,u(t,y,s),u(t,s,s),u(s,y,s))ds .$$

According to the method of successive approximations, let us consider the sequence $\{u_n(t,y,x)\}$, given by

$$(2.5) \quad u_{n+1}(t,y,x) = \phi(t,y) + \int_0^x F(t,y,s,u_n(t,y,s),u_n(t,s,s),u_n(s,y,s))ds , \\ u_0(t,y,x) = \phi(t,y) .$$

Clearly $u_0 \in S$ and is continuous on $\Delta \times \Delta \times \Delta$. It is desired to show that $u_n \in S$ and u_n is continuous on $\Delta \times \Delta \times \Delta$. Now for all $(t,y,x) \in \Delta \times \Delta \times \Delta$

$$||u_1(t,y,x) - \phi(t,y)|| \leq Mx \leq M\delta \leq b ,$$

and continuity of u_1 follows from (2.2). If u_n is continuous and $u_n \in S$, then $u_{n+1} \in C[\Delta \times \Delta \times \Delta, R]$ and

$$||u_{n+1}(t,y,x) - \phi(t,y)|| \leq Mx \leq M\delta \leq b , \text{ for all } t,y,x \in \Delta .$$

Therefore, the sequence of iterates u_n are continuous and stay within the ball S .

In order to show convergence of the sequence of iterates we need an appropriate estimate on successive differences of iterates. Proceeding by induction

$$\begin{aligned}
 ||u_2(t,y,x)-u_1(t,y,x)|| &\leq \int_0^x ||F(t,y,s,u_1(t,y,s),u_1(t,s,s),u_1(s,y,s)) \\
 &\quad -F(t,y,s,\phi(t,y),\phi(t,s),\phi(s,y))|| ds \\
 &\leq \int_0^x \{L_1 ||u_1(t,y,s)-\phi(t,y)|| + L_2 ||u_1(t,s,s)-\phi(t,s)|| \\
 &\quad + L_3 ||u_1(s,y,s)-\phi(s,y)||\} ds \\
 &\leq \int_0^x \{L_1 M s + L_2 M s + L_3 M s\} ds \\
 &= M(L_1+L_2+L_3) \frac{x^2}{2} \leq M(L_1+L_2+L_3) \frac{\delta^2}{2},
 \end{aligned}$$

for all $t,y,x \in \Delta$. We hypothesize

$$(2.6) \quad ||u_n(t,y,x)-u_{n-1}(t,y,x)|| \leq M(L_1+L_2+L_3)^{n-1} \frac{x^n}{n!} \leq M(L_1+L_2+L_3)^{n-1} \frac{\delta^n}{n!},$$

for all $t,y,x \in \Delta$. Then

$$\begin{aligned}
 ||u_{n+1}(t,y,x)-u_n(t,y,x)|| &\leq \int_0^x ||F(t,y,s,u_n(t,y,s),u_n(t,s,s),u_n(s,y,s)) \\
 &\quad -F(t,y,s,u_{n-1}(t,y,s),u_{n-1}(t,s,s),u_{n-1}(s,y,s))|| ds \\
 &\leq \int_0^x \{L_1 ||u_n(t,y,s)-u_{n-1}(t,y,s)|| + L_2 ||u_n(t,s,s)-u_{n-1}(t,s,s)|| \\
 &\quad + L_3 ||u_n(s,y,s)-u_{n-1}(s,y,s)||\} ds \\
 &\leq \frac{M(L_1+L_2+L_3)^n}{n!} \int_0^x s^n ds = \frac{M(L_1+L_2+L_3)^n x^{n+1}}{(n+1)!} \\
 &\leq M(L_1+L_2+L_3)^n \frac{\delta^{n+1}}{(n+1)!}, \text{ for all } t,y,x \in \Delta.
 \end{aligned}$$

By induction the estimate (2.6) is established. The uniform convergence of $u_n(t,y,x)$ on $\Delta \times \Delta \times \Delta$ follows easily by comparison with the series

$$\sum_{n=1}^{\infty} M(L_1+L_2+L_3)^n \frac{\delta^{n+1}}{(n+1)!} .$$

Let $u(t,y,x) = \lim_{n \rightarrow \infty} u_n(t,y,x)$. Since

$$\begin{aligned} & ||F(t,y,x,u_n(t,y,x),u_n(t,x,x),u_n(x,y,x)) \\ & - F(t,y,x,u(t,y,x),u(t,x,x),u(x,y,x))|| \\ & \leq L_1 ||u_n(t,y,x) - u(t,y,x)|| + L_2 ||u_n(t,x,x) - u(t,x,x)|| \\ & + L_3 ||u_n(x,y,x) - u(x,y,x)|| , \end{aligned}$$

and convergence of $u_n(t,y,x)$ to $u(t,y,x)$ is uniform,

$$\begin{aligned} & \lim_{n \rightarrow \infty} F(t,y,x,u_n(t,y,x),u_n(t,x,x),u_n(x,y,x)) \\ & = F(t,y,x,u(t,y,x),u(t,x,x),u(x,y,x)) \end{aligned}$$

uniformly on $\Delta \times \Delta \times \Delta$. Taking the limit of both sides of (2.5) shows that $u(t,y,x)$ is a solution of (2.4) and hence of (2.1). This completes the proof of local existence.

Suppose that $v(t,y,x)$ is also a solution of (2.4), then an induction argument using the Lipschitz condition (2.3) gives the estimate

$$||v(t,y,x) - u(t,y,x)|| \leq b(L_1+L_2+L_3)^n \frac{x^n}{n!} \leq b(L_1+L_2+L_3)^n \frac{\delta^n}{n!} ,$$

for $t,y,x \in \Delta$. Therefore, $\lim_{n \rightarrow \infty} u_n(t,y,x) = v(t,y,x)$, which establishes the uniqueness of solutions. This completes the proof.

As an application of Theorem 2.1 take in (2.1)

$$F(t,y,x,u(t,y,x),u(t,x,x),u(x,y,x)) = u(t,x,x)u(x,y,x) .$$

Then we obtain the system (1.1) considered by Sobolev [3]. By the continuity of the initial function ϕ , any function $u \in S$ is bounded, say $|u(t,y,x)| \leq L$ for all $t,y,x \in I$. The Lipschitz condition follows from

$$\begin{aligned} |F(t,y,x,u_1,v_1,w_1) - F(t,y,x,u_2,v_2,w_2)| &= |v_1 w_1 - v_2 w_2| \\ &\leq |w_1| |v_1 - v_2| + |v_2| |w_1 - w_2| \\ &\leq L |v_1 - v_2| + L |w_1 - w_2|, \end{aligned}$$

whenever $(t,y,x,u_i,v_i,w_i) \in I \times I \times I \times S \times S \times S$, $i = 1,2$. Theorem 2.1 then guarantees local existence and uniqueness of solutions of (1.1).

3. Linear Equations and Inequalities. The first result determines solutions of the first order linear scalar equation

$$\begin{aligned} (3.1) \quad u_x(t,y,x) &= L_1 u(t,y,x) + L_2 u(t,x,x) + L_3 u(x,y,x), \\ u(t,y,0) &= \phi(t,y), \quad t,y,x \in R. \end{aligned}$$

The case in which the coefficients are functions of x can be easily generalized.

Theorem 3.1. The solution of (3.1) is given by

$$\begin{aligned} (3.2) \quad u(t,y,x) &= [\phi(t,y) - \phi(t,0) - \phi(0,y) + \phi(0,0)] e^{L_1 x} \\ &\quad + [\phi(t,0) - \phi(0,0)] e^{(L_1 + L_2)x} + [\phi(0,y) - \phi(0,0)] e^{(L_1 + L_3)x} \\ &\quad + \phi(0,0) e^{(L_1 + L_2 + L_3)x}, \end{aligned}$$

for all $t,y,x \in R_+$.

Proof. Assume solutions of the form

$$u(t,y,x) = h(t,y,x) e^{L_1 x}, \quad h(t,y,0) = \phi(t,y).$$

Differentiate with respect to x and equate with the right hand side of (3.1) yields

$$(3.3) \quad \begin{aligned} h_x(t,y,x) &= L_2 h(t,x,x) + L_3 h(x,y,x) , \\ h(t,y,0) &= \phi(t,y) . \end{aligned}$$

If $t = y = x$ then $h(x,x,x) = \phi(0,0)e^{(L_2+L_3)x}$. Now let $y = x$ in (3.3).

Then $h_x(t,x,x) = L_2 h(t,x,x) + \phi(0,0)L_3 e^{(L_2+L_3)x}$, $h(t,0,0) = \phi(t,0)$.

Now make the assumption that $h(t,y,x) = H(t,x)e^{L_2 x}$, $H(t,0) = \phi(t,0)$.

Differentiating with respect to x , we have from (3.3) $H_x(t,x) = \phi(0,0)L_3 e^{L_3 x}$.

Integrating gives $H(t,x) = \phi(t,0) + \phi(0,0)(e^{L_3 x} - 1)$. Thus

$h(t,x,x) = \phi(t,0)e^{L_2 x} + \phi(0,0)(e^{(L_2+L_3)x} - e^{L_2 x})$. If in (3.3) we let $t = x$

a similar analysis determines

$$h(x,y,x) = \phi(0,y)e^{L_3 x} + \phi(0,0)(e^{(L_2+L_3)x} - e^{L_3 x}) .$$

Now (3.3) can be written as

$$(3.4) \quad \begin{aligned} h_x(t,y,x) &= L_2(\phi(t,0) - \phi(0,0))e^{L_2 x} + L_3(\phi(0,y) - \phi(0,0))e^{L_3 x} \\ &\quad + \phi(0,0)(e^{(L_2+L_3)x} - 1) , \quad h(t,y,0) = \phi(t,y) . \end{aligned}$$

If (3.4) is integrated from 0 to x and then solved in terms of $u(t,y,x)$, the formula (3.2) is obtained.

Remark: If $u(t,y,x) = u(x)$ and $L_2 = L_3 = 0$ then (3.2) yields the solution $u(x) = u(0)e^{L_1 x}$.

When the initial function is constant, $\phi(t,y) = c$, then the solution assumes the simplified form $u(x) = ce^{(L_1+L_2+L_3)x}$.

Consider the integral inequality

$$(3.5) \quad u(t,y,x) \leq \phi(t,y) + \int_0^x L_1 u(t,y,s) ds + \int_0^x L_2 u(t,s,s) ds + \int_0^x L_3 u(s,y,s) ds ,$$

for $(t,y,x) \in R_+ \times R_+ \times R_+ .$

$$\text{Let } m(t,y,x) = \phi(t,y) + \int_0^x L_1 u(t,y,s) ds + \int_0^x L_2 u(t,s,s) ds + \int_0^x L_3 u(s,y,s) ds .$$

Then $m(t,y,x)$ satisfies the differential inequality

$$(3.6) \quad m_x(t,y,x) \leq L_1 m(t,y,x) + L_2 m(t,x,x) + L_3 m(x,y,x) , \quad m(t,y,0) = \phi(t,y) .$$

The solution of (3.6) can be obtained by the techniques used in the proof of Theorem 3.1. The result can be stated as

Theorem 3.2. If $u(t,y,x)$ is any solution of the integral inequality

(3.5) then

$$\begin{aligned} u(t,y,x) \leq & [\phi(t,y) - \phi(t,0) - \phi(0,y) + \phi(0,0)] e^{L_1 x} + [\phi(t,0) - \phi(0,0)] e^{(L_1 + L_2)x} \\ & + [\phi(0,y) - \phi(0,0)] e^{(L_1 + L_2)x} + \phi(0,0) e^{(L_1 + L_2 + L_3)x} . \end{aligned}$$

We now consider solutions of the linear equation (3.1) under perturbations.

Let

$$(3.7) \quad \begin{aligned} u_x(t,y,x) = & L_1 u(t,y,x) + L_2 u(t,x,x) + L_3 u(x,y,x) \\ & + F(t,y,x,u(t,y,x),u(t,x,x),u(x,y,x)), \quad u(t,y,0) = \phi(t,y) , \end{aligned}$$

where $F \in C[R \times R \times R \times R \times R \times R, R]$. A variation of constants formula is given by

Theorem 3.3. Let $\phi(t,y,x)$ represent the solution of the unperturbed equation (3.1) as given in Theorem 3.1. Then the solution of (3.7) is given by

$$\begin{aligned}
(3.8) \quad u(t,y,x) = & \phi(t,y,x) + \int_0^x e^{L_1(x-s)} F(t,y,s,u(t,y,s),u(t,s,s),u(s,y,s))ds \\
& + \int_0^x e^{L_1(x-s)} (e^{L_2(x-s)} - 1) F(t,s,s,u(t,s,s),u(t,s,s),u(s,s,s))ds \\
& + \int_0^x e^{L_1(x-s)} (e^{L_3(x-s)} - 1) F(s,y,s,u(s,y,s),u(s,s,s),u(s,y,s))ds \\
& + \int_0^x e^{L_1(x-s)} (e^{(L_2+L_3)(x-s)} - e^{L_2(x-s)} - e^{L_3(x-s)} + 1) \cdot \\
& \cdot F(s,s,s,u(s,s,s),u(s,s,s),u(s,s,s))ds .
\end{aligned}$$

Remark: If $u(t,y,x) = u(x)$, $F(t,y,x,u(t,y,x),u(t,x,x),u(x,y,x)) = F(x,u(x))$,
and $L_2 = L_3 = 0$, then (3.8) yields the known result

$$u(x) = u(0)e^{L_1 x} + \int_0^x e^{L_1(x-s)} F(s,u(s))ds .$$

The proof of Theorem 3.3 is analogous to the proof of Theorem 3.1 and is omitted.

REFERENCES

- [1] H.H. Kagiwada and R.E. Kalaba, "Extension of S.L. Sobolov's Initial Value Method to Nonlinear Integral Equations", (to appear) Journal of Nonlinear Analysis.
- [2] V. Lakshmikantham and S. Leela, Differential and Integral Inequalities, Academic Press, New York, 1969.
- [3] S.L. Sobolev, Uspekhi Matematicheskikh Nauk Vol. 9, No. 3, 1954, Pp. 234-235.