COMPARISON OF POINT ESTIMATORS
OF NORMAL PERCENTILES

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ABSTRACT

There are available several point estimators of the percentiles of a normal distribution with both mean and variance unknown. Consequently, it would seem appropriate to make a comparison among the estimators through some "closeness to the true value" criteria. Along these lines, the concept of Pitman-closeness efficiency is introduced. Essentially, when comparing two estimators, the Pitman-closeness efficiency gives the "odds" in favor of one of the estimators being closer to the true value than is the other in a given situation. Through the use of Pitman-closeness efficiency, this paper compares (a) the maximum likelihood estimator, (b) the minimum variance unbiased estimator, (c) the best invariant estimator, and (d) the median unbiased estimator within a class of estimators which includes (a), (b), and (c). Mean squared efficiency is also discussed.

1. INTRODUCTION

In fatigue life determination, the distribution of component strength (at some fixed number of cycles), say X , is frequently taken to be normal with mean μ and variance σ^2 . Let x_p be the stress level (at some fixed number of cycles) beyond which a specified proportion p of the population of components can endure. Analytically, $\text{Prob}\{X > x_p\} = p$; thus $x_p = \mu + z_{1-p}\sigma$, where z_{1-p} is the $100(1-p)^{\text{th}}$ percentile of the standard normal distribution. Obviously, the stress level that a component can endure must be taken into account whenever a maximum operating load is recommended.

There are several methods which give point estimates of x_p based on a random sample of size n from a normal distribution when both μ and σ^2 are unknown. (Confidence limits on x_p are discussed in Owen (1968).) Accuracy in estimation of x_p can be of great concern to the fatigue analyst. Underestimating x_p clearly devaluates the true strength of the component. Perhaps even more serious is that when x_p is overestimated, there is a distinct possibility that a significantly smaller proportion than specified will endure beyond the estimated x_p . Such an occurrence could seriously jeopardize the application of the component. In addition, such factors as expensive test apparatus and lengthy testing procedures frequently dictate small sample sizes in fatigue life determination. In fact, the Federal Aviation Administration (FAA) allows for the qualification of components based on samples of size 6 or less for fatigue tests. In order to judge which estimate of x_p might be preferred for a given situation, a comparison based on some "closeness to the true value" criteria should be made. The following approach is commonly used. Consider two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of $\theta = h(\mu, \sigma)$ having mean squared errors $\text{MSE}\{\hat{\theta}_1; \mu, \sigma, n\}$ and $\text{MSE}\{\hat{\theta}_2; \mu, \sigma, n\}$, respectively, where $\text{MSE}\{\hat{\theta}_1; \mu, \sigma, n\} = E[(\hat{\theta}_1 - \theta)^2]$, $i = 1, 2$. The mean squared efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is given by

$$\text{rel. eff.}_{\text{MS}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} = \text{MSE}\{\hat{\theta}_2; \mu, \sigma, n\} / \text{MSE}\{\hat{\theta}_1; \mu, \sigma, n\}. \quad (1.1)$$

When $\text{rel. eff.}_{\text{MS}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} < 1$ for some (μ_0, σ_0, n_0) , we say that $\hat{\theta}_2$ is more mean squared efficient than $\hat{\theta}_1$ at (μ_0, σ_0, n_0) . If $\text{rel. eff.}_{\text{MS}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} \leq 1$ for all (μ, σ, n) with a strict inequality for some (μ_0, σ_0, n_0) , then $\hat{\theta}_1$ is mean squared inadmissible relative to $\hat{\theta}_2$. However, mean squared efficiency is based on theoretical averages, thus its usefulness is not always clear especially when the sample size is small. In addition, as will be discussed later, the efficiency value can be misleading and even unreliable (see, e.g., Box and Tiao (1973) p. 307).

Consequently, we propose another measure of efficiency motivated by the following definition due to Pitman (1939). Let $\psi\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} = \text{Prob}\{|\hat{\theta}_1 - \theta| < |\hat{\theta}_2 - \theta|\}$. Then $\hat{\theta}_1$ is called a Pitman-closer estimator of θ than is $\hat{\theta}_2$ if $\psi\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} > .5$ for all (μ, σ, n) with strict inequality for some (μ_0, σ_0, n_0) . [Note: In Pitman's original definition, $\text{Prob}\{|\hat{\theta}_1 - \theta| < |\hat{\theta}_2 - \theta|\}$ was a fiducial probability statement. In this paper, $\text{Prob}\{|\hat{\theta}_1 - \theta| < |\hat{\theta}_2 - \theta|\}$ is determined in the classical sense and, therefore, will have a relative-frequency interpretation.] We define the Pitman-closeness efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ as

$$\text{rel. eff.}_{\text{PC}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} = \psi\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} / \psi\{\hat{\theta}_2, \hat{\theta}_1; \mu, \sigma, n\}. \quad (1.2)$$

Notice that $\text{rel. eff.}_{\text{PC}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\}$ gives the "odds" in favor of the estimator $\hat{\theta}_1$ being closer to the true value of θ than is $\hat{\theta}_2$. For example, if $\text{rel. eff.}_{\text{PC}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} = 1.5$, then the "odds" are 3 to 2 in favor of $\hat{\theta}_1$ being closer to θ than is $\hat{\theta}_2$. When $\text{rel. eff.}_{\text{PC}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} < 1$ for some (μ_0, σ_0, n_0) , we shall say that $\hat{\theta}_2$ is more Pitman-closeness efficient than $\hat{\theta}_1$ at (μ_0, σ_0, n_0) . If $\text{rel. eff.}_{\text{PC}}\{\hat{\theta}_1, \hat{\theta}_2; \mu, \sigma, n\} \leq 1$ for all (μ, σ, n) with a strict inequality for some (μ_0, σ_0, n_0) , then $\hat{\theta}_1$ is said to be Pitman-closeness inadmissible relative to $\hat{\theta}_2$.

The primary intent of this paper is to compare, through the use of Pitman-closeness efficiency, several point estimators of the normal percentile $x_p = \mu + z_{1-p}\sigma$ based on a random sample of size

n from a normal distribution with unknown parameters. Mean squared efficiency will also be discussed. The estimators considered are (a) maximum likelihood (MLE), (b) minimum variance unbiased (MVUE), (c) Mann's (1969) best invariant (BIE), and (d) median unbiased (MUE) within a class of estimators which includes MLE, MVUE, and BIE. While fatigue testing would seem to place emphasis on values of the proportion p such that $p \geq .75$, we will consider the general problem, i.e., p is such that $0 < p < 1$. Hopefully, Pitman-closeness efficiency (perhaps, at times, in conjunction with mean squared efficiency) will provide additional insight and information with regard to the preference of estimators, particularly for small sample sizes. For the interested reader, Pitman-closeness efficiency has recently been used to determine preferences among estimators of certain reliability characteristics (earliest time to failure, mean time to failure, and the reliability function) of a two-parameter exponential distribution (see Dyer (1977), Dyer and Lu (1977)).

2. ESTIMATION OF x_p

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a normal distribution with unknown mean μ and variance σ^2 . The maximum likelihood estimator of x_p is

$$\hat{x}_p(\text{MLE}) = \bar{X} + z_{1-p}(1 - n^{-1})^{1/2}S, \quad (2.1)$$

where $\bar{X} = (1/n) \sum_{i=1}^n X_i$ and $S = \left[\sum_{i=1}^n (X_i - \bar{X})^2 / (n - 1) \right]^{1/2}$.

Since $E(S) = m\sigma$, where

$$m = \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]} \left(\frac{2}{n-1} \right)^{1/2}, \quad (2.2)$$

and $(\bar{X}, S)$ is a complete sufficient statistic for (μ, σ) , then the minimum variance unbiased estimator of x_p is

$$\hat{x}_p(\text{MVUE}) = \bar{X} + (z_{1-p}/m)S. \quad (2.3)$$

The estimators $\hat{x}_p(\text{MLE})$ and $\hat{x}_p(\text{MVUE})$ are both members of the class of estimators $Q = \{\bar{X} + \alpha S\}$, where $\alpha = \alpha(p, n)$ is a real number for fixed p and n . Furthermore,

$$\begin{aligned} \text{MSE}\{\bar{X} + \alpha S; \mu, \sigma, p, n\} \\ = [(\alpha - z_{1-p} m)^2 + (1/n) + (1 - m^2) z_{1-p}^2] \sigma^2. \end{aligned} \quad (2.4)$$

Clearly, the estimator within the class Q with smallest mean squared error (which coincides, in this case, with Mann's (1969) best invariant estimator) is

$$\hat{x}_p(\text{BIE}) = \bar{X} + z_{1-p} m S. \quad (2.5)$$

It can be shown numerically that

$$(1 - n^{-1})^{1/2} < m < 1 < m^{-1},$$

and that

$$0 < |m - (1 - n^{-1})^{1/2}| < |1 - m| < |m^{-1} - 1|.$$

It follows that $\text{rel. eff.}_{\text{MS}}\{\hat{x}_p(\text{MLE}), \hat{x}_p(\text{MVUE}); \mu, \sigma, p, n\} > 1$ provided $p \neq .5$ (when $p = .5$, the two estimators are equivalent), hence $\hat{x}_p(\text{MVUE})$ is mean squared inadmissible relative to $\hat{x}_p(\text{MLE})$. Furthermore, $\hat{x}_p(\text{BIE})$ has the smallest mean squared error within the class Q . Consequently,

Theorem 1. $\hat{x}_p(\text{MVUE})$ is mean squared inadmissible relative to $\hat{x}_p(\text{MLE})$; however, both $\hat{x}_p(\text{MLE})$ and $\hat{x}_p(\text{MVUE})$ are mean squared inadmissible relative to $\hat{x}_p(\text{BIE})$.

We now consider Pitman-closeness efficiency. The following well known result (see, e.g., Owen (1968)) will be needed.

Lemma. The probability that an estimator within the class Q overestimates x_p is given by

$$\text{Prob}\{\bar{X} + \alpha S > x_p\} = T(\sqrt{n} \alpha; n - 1, \sqrt{n} z_{1-p}), \quad (2.6)$$

where $T(\cdot; f, \delta)$ is the noncentral t distribution function with f degrees of freedom and noncentrality parameter δ .

As will soon become apparent, a comparison between any two estimators in Q based on Pitman-closeness efficiency is inherently dependent upon the probability that the estimators overestimate (or underestimate) x_p . Table I is based on equation (2.6) and gives the probability that $\hat{x}_p(\text{MVUE})$, $\hat{x}_p(\text{BIE})$, and $\hat{x}_p(\text{MLE})$ overestimate

TABLE I

Probability of $\hat{x}_p(\cdot)$ Overestimating x_p

n	p	MVUE	BIE	MLE	n	p	MVUE	BIE	MLE
2	.750	.514	.628	.653	6	.750	.503	.558	.580
	.900	.540	.705	.743		.900	.509	.593	.627
	.950	.551	.732	.775		.950	.512	.607	.645
	.990	.562	.759	.806		.990	.516	.622	.665
	.999	.568	.772	.822		.999	.519	.630	.676
3	.750	.506	.593	.619	10	.750	.502	.543	.561
	.900	.520	.649	.689		.900	.506	.569	.597
	.950	.526	.669	.714		.950	.508	.579	.611
	.990	.533	.691	.741		.990	.511	.591	.626
	.999	.538	.702	.755		.999	.513	.597	.635
4	.750	.504	.575	.600	15	.750	.501	.534	.549
	.900	.513	.621	.660		.900	.504	.555	.578
	.950	.518	.638	.681		.950	.506	.563	.590
	.990	.524	.657	.705		.990	.509	.573	.602
	.999	.527	.667	.718		.999	.510	.578	.610
5	.750	.503	.565	.588	20	.750	.501	.529	.542
	.900	.510	.605	.641		.900	.504	.547	.568
	.950	.514	.620	.660		.950	.505	.554	.577
	.990	.519	.636	.682		.990	.507	.562	.588
	.999	.522	.646	.694		.999	.509	.567	.595

x_p , where $p = .75, .9, .95, .99, .999$. Since $T(\sqrt{n}\alpha; n-1, \sqrt{n}z_{1-p}) = 1 - T(-\sqrt{n}\alpha; n-1, -\sqrt{n}z_{1-p}) = 1 - T(-\sqrt{n}\alpha; n-1, \sqrt{n}z_p)$, one minus the entry in the table for a given p is the probability of overestimating x_{1-p} .

We prove the following

Theorem 2. $\hat{x}_p(\text{MLE})$ is Pitman-closeness inadmissible relative to $\hat{x}_p(\text{BIE})$; however, both $\hat{x}_p(\text{MLE})$ and $\hat{x}_p(\text{BIE})$ are Pitman-closeness inadmissible relative to $\hat{x}_p(\text{MVUE})$.

Proof. Suppose $p > .5$. Then $\text{Prob}\{\hat{x}_p(\text{MVUE}) < \hat{x}_p(\text{BIE}) < \hat{x}_p(\text{MLE})\} = 1$, and by the lemma

$$\begin{aligned}
 & \psi(\hat{x}_p(\text{MVUE}), \hat{x}_p(\text{BIE}); \mu, \sigma, p, n) \\
 &= \text{Prob}\{[\hat{x}_p(\text{MVUE}) + \hat{x}_p(\text{BIE})]/2 > x_p\} \\
 &= T(c_1; n-1, \sqrt{n}z_{1-p}), \quad (2.7)
 \end{aligned}$$

where $c_1 = \sqrt{n}(m^{-1} + m)z_{1-p}/2$. In a similar way,

$$\psi\{\hat{x}_p(\text{MVUE}), \hat{x}_p(\text{MLE}); \mu, \sigma, p, n\} = T(c_2; n-1, \sqrt{n} z_{1-p}), \quad (2.8)$$

where $c_2 = \sqrt{n}[m^{-1} + (1 - n^{-1})^{1/2}]z_{1-p}/2$; and

$$\psi\{\hat{x}_p(\text{BIE}), \hat{x}_p(\text{MLE}); \mu, \sigma, p, n\} = T(c_3; n-1, \sqrt{n} z_{1-p}), \quad (2.9)$$

where $c_3 = \sqrt{n}[m + (1 - n^{-1})^{1/2}]z_{1-p}/2$. Since $(1 - n^{-1})^{1/2} < m < m^{-1}$, then $c_1 < c_2 < c_3$ for fixed p , provided $p > .5$. Hence

$$\begin{aligned} \psi\{\hat{x}_p(\text{MVUE}), \hat{x}_p(\text{BIE}); \mu, \sigma, p, n\} &< \psi\{\hat{x}_p(\text{MVUE}), \hat{x}_p(\text{MLE}); \mu, \sigma, p, n\} \\ &< \psi\{\hat{x}_p(\text{BIE}), \hat{x}_p(\text{MLE}); \mu, \sigma, p, n\}. \end{aligned}$$

However, it can be shown that (also see Table I) $\text{Prob}\{\hat{x}_p(\text{MVUE}) > x_p\} = T(\sqrt{n} z_{1-p}/m; n-1, \sqrt{n} z_{1-p})$ is an increasing function of p when n is fixed. Thus when $p > .5$, $\text{Prob}\{\hat{x}_p(\text{MVUE}) > x_p\} > \text{Prob}\{\hat{x}_{.5}(\text{MVUE}) > x_{.5}\} = T(0; n-1, 0) = .5$. The theorem for the case $p > .5$ follows since $\psi\{\hat{x}_p(\text{MVUE}), \hat{x}_p(\text{BIE}); \mu, \sigma, p, n\} > \text{Prob}\{\hat{x}_p(\text{MVUE}) > x_p\}$. The proof when $p < .5$ is treated in a similar way. $\square$

While Theorem 2 says that $\hat{x}_p(\text{MVUE})$ is preferred over $\hat{x}_p(\text{MLE})$ and $\hat{x}_p(\text{BIE})$ from a Pitman-closeness efficiency point of view, it would be of interest to examine the extent of this efficiency. Table II is based on equation (1.2) along with equations (2.7) - (2.9) and their counterparts when $p < .5$ evaluated when p (or equivalently, $1 - p$) = .75, .9, .95, .99, .999. The mean squared efficiencies (from equation (2.4)) have also been included for comparison purposes.

There are some interesting features within Table II which we now discuss. First of all, $\hat{x}_p(\text{BIE})$ is more efficient (in the sense of Pitman-closeness as well as mean squared) than $\hat{x}_p(\text{MLE})$. But whereas the mean squared efficiency is stable across the values of n and p , there is considerable change in the Pitman-closeness efficiency not only across the values of n for fixed p but also across the values of p for fixed n , especially when n is small. For example, when $p = .95$ both measures of efficiency decrease

TABLE II

Efficiency of Estimators of x_p

		eff. of $\hat{x}_p$ (MVUE) rel. to $\hat{x}_p$ (BIE)		eff. of $\hat{x}_p$ (MVUE) rel. to $\hat{x}_p$ (MLE)		eff. of $\hat{x}_p$ (BIE) rel. to $\hat{x}_p$ (MLE)	
n	$\frac{p}{1-p}$	PC	MS	PC	MS	PC	MS
2	.750	1.316	.876	1.380	.881	1.782	1.006
	.900	1.599	.763	1.717	.772	2.621	1.012
	.950	1.722	.726	1.865	.736	3.048	1.015
	.990	1.859	.687	2.032	.700	3.591	1.018
	.999	1.930	.667	2.120	.680	3.912	1.020
3	.750	1.215	.942	1.278	.947	1.536	1.005
	.900	1.389	.877	1.502	.887	2.018	1.012
	.950	1.464	.852	1.597	.864	2.241	1.014
	.990	1.550	.825	1.710	.839	2.518	1.018
	.999	1.600	.810	1.774	.826	2.685	1.019
4	.750	1.170	.963	1.229	.967	1.426	1.004
	.900	1.304	.919	1.405	.928	1.781	1.010
	.950	1.360	.901	1.480	.912	1.939	1.013
	.990	1.428	.880	1.570	.894	2.134	1.016
	.999	1.467	.868	1.622	.883	2.253	1.017
5	.750	1.145	.973	1.198	.977	1.362	1.004
	.900	1.255	.940	1.348	.948	1.650	1.009
	.950	1.303	.925	1.411	.936	1.776	1.011
	.990	1.359	.909	1.487	.921	1.931	1.014
	.999	1.393	.900	1.532	.913	2.026	1.015
6	.750	1.128	.979	1.178	.982	1.319	1.003
	.900	1.224	.952	1.309	.959	1.566	1.008
	.950	1.265	.941	1.365	.950	1.673	1.010
	.990	1.315	.927	1.431	.938	1.803	1.012
	.999	1.345	.919	1.471	.931	1.882	1.013
10	.750	1.092	.989	1.132	.991	1.230	1.002
	.900	1.160	.974	1.226	.979	1.397	1.005
	.950	1.189	.967	1.266	.973	1.468	1.006
	.990	1.224	.959	1.313	.967	1.553	1.008
	.999	1.246	.954	1.342	.963	1.605	1.009
20	.750	1.062	.995	1.090	.996	1.153	1.001
	.900	1.106	.988	1.152	.991	1.259	1.003
	.950	1.125	.985	1.178	.988	1.302	1.003
	.990	1.149	.981	1.210	.985	1.354	1.004
	.999	1.163	.978	1.229	.983	1.386	1.005

with increasing n . Yet while $\hat{x}_p$ (BIE) is only slightly (less than 2%) more mean squared efficient than $\hat{x}_p$ (MLE), the "odds" in favor of $\hat{x}_p$ (BIE) being closer to x_p can be quite high (e.g., approximately 3 to 1 when $n = 2$). The reason for this is as follows. When $p = .95$, the value of $\hat{x}_p$ (BIE) is usually relatively close to (at least 90% of) but always less than the value of $\hat{x}_p$ (MLE). Consequently, there is very little difference in the respective mean squared errors. On the other hand, from Table I, when $p = .95$ the probability of $\hat{x}_p$ (BIE) overestimating x_p can be quite high for small values of n (e.g., approximately 73% when $n = 2$). But whenever $\hat{x}_p$ (BIE) overestimates and $p > .5$, its value is *always* closer to x_p than is the corresponding value of $\hat{x}_p$ (MLE). Thus, in this particular case, Pitman-closeness efficiency appears to be much more sensitive than is mean squared efficiency to changes in $p(n)$ when $n(p)$ is fixed. In passing, we offer an incidental comment with regard to the interpretation of the information provided through the values of each of the two measures of efficiency. For example, when $n = 2$ and $p = .999$ the statement "the "odds" are 4 to 1 in favor of $\hat{x}_p$ (BIE) being closer to x_p than is $\hat{x}_p$ (MLE)" would seem to provide a clearer physical meaning than does the statement " $\hat{x}_p$ (BIE) is 2% more mean squared efficient than $\hat{x}_p$ (MLE)". It is not particularly easy to give a *precise* explanation of the latter statement to a layman (or, for that matter, anyone else)!

Quite a different situation occurs when comparing $\hat{x}_p$ (MVUE) with $\hat{x}_p$ (BIE). The estimator $\hat{x}_p$ (BIE) is Pitman-closeness inadmissible relative to $\hat{x}_p$ (MVUE); however, $\hat{x}_p$ (MVUE) is mean squared inadmissible relative to $\hat{x}_p$ (BIE). Notice that, for fixed p and increasing n , $\hat{x}_p$ (MVUE) becomes less Pitman-closeness efficient yet more mean squared efficient. This is, of course, not too surprising since both estimators are consistent. The most noticeable disparateness between the two measures of efficiency occurs when the sample size is small. Consider the extreme case when $n = 2$ and $p = .999$. Here, $\hat{x}_p$ (MVUE) is only 67% as mean squared

efficient as $\hat{x}_p$ (BIE), yet the "odds" are approximately 2 to 1 in favor of $\hat{x}_p$ (MVUE) being closer to x_p . Based on simulation, the reason for this somewhat paradoxical situation is as follows. For a given sample, the value of $\hat{x}_p$ (MVUE) is always less than the corresponding value of $\hat{x}_p$ (BIE). The difference between the values, for a given sample, is least pronounced whenever $\hat{x}_p$ (MVUE) overestimates x_p (approximately 57% of the time, in which case the value of $\hat{x}_p$ (MVUE) is always closer to x_p). The difference between the values, for a given sample, is most pronounced whenever $\hat{x}_p$ (BIE) underestimates x_p (approximately 23% of the time, in which case the value of $\hat{x}_p$ (BIE) is always closer to x_p). Approximately 20% of the time, $\hat{x}_p$ (MVUE) underestimates and $\hat{x}_p$ (BIE) overestimates x_p , in which case the value of $\hat{x}_p$ (MVUE), for a given sample, is closer to x_p about 50% of the time. Consequently, the variance of $\hat{x}_p$ (BIE), $2.71 \sigma^2$, is much smaller than the variance of $\hat{x}_p$ (MVUE), $5.95 \sigma^2$. And although the square of the bias of $\hat{x}_p$ (BIE), $1.26 \sigma^2$, is about half its variance, still the mean squared error of $\hat{x}_p$ (BIE), $3.97 \sigma^2$, is less than the mean squared error (variance) of $\hat{x}_p$ (MVUE). Thus, in this case, a mean squared efficiency of only 67% is misleading since the less efficient estimator is actually closer to the true value 67% of the time.

From the above discussions, we are *not* advocating that mean squared efficiency be disregarded, but we are arguing against its exclusive use when choosing between two estimators. Mean squared efficiency might be considered jointly with Pitman-closeness efficiency, especially when one of the efficiency measures is less than unity while the other is greater than unity. In such a situation, a simulated comparison similar to that described above should be carried out. That such a discrepancy can exist between the two efficiency measures is a result of the measures being based on different concepts. Mean squared efficiency essentially compares the *average* precision of one estimator *independently* to that of another. On the other hand, Pitman-closeness efficiency compares the *individual*

performance of one estimator *relative* to that of another.

Within the class Q , there exists a Pitman-closest estimator of x_p in the sense that this estimator is a Pitman-closer estimator of x_p relative to all other estimators in the class. Specifically, we prove the following

Theorem 3. Within the class Q , the median unbiased estimator, i.e.,

$$\hat{x}_p(\text{MUE}) = \bar{X} + \alpha^* S \quad (2.10)$$

where α^* is such that

$$T(\sqrt{n} \alpha^*; n - 1, \sqrt{n} z_{1-p}) = .5, \quad (2.11)$$

is the Pitman-closest estimator of x_p .

Proof. For fixed n and p , the estimators in Q are ordered. Since the median unbiased estimator is as likely to underestimate as to overestimate the true value of x_p (Lehmann (1959) p. 83), the result follows immediately. $\square$

The values of α^* which satisfy equation (2.11) have been determined numerically for $n = 2(1)20$ and p (or equivalently, $1 - p$) = .75, .9, .95, .99, .999 and appear in Table III. The median unbiased estimator is also a 50% one-sided confidence limit on x_p and, as such, has been previously dealt with by Owen (1963, 1968). Although the values of α^* can be found within a large collection of tables in Owen (1963), this source is not readily available. Thus Table III is given for the convenience of the reader as well as for the completeness of the paper.

When $p > .5$,

$$\begin{aligned} \psi\{\hat{x}_p(\text{MUE}), \hat{x}_p(*); \mu, \sigma, p, n\} &= \text{Prob}\{\hat{x}_p(\text{MUE}) + \hat{x}_p(*) > 2x_p\} \\ &= T(\sqrt{n} a; n - 1, \sqrt{n} z_{1-p}); \end{aligned}$$

thus

$$\begin{aligned} \text{rel. eff.}_{PC}\{\hat{x}_p(\text{MUE}), \hat{x}_p(*); \mu, \sigma, p, n\} \\ = \frac{T(\sqrt{n} a; n - 1, \sqrt{n} z_{1-p})}{1 - T(\sqrt{n} a; n - 1, \sqrt{n} z_{1-p})}, \end{aligned} \quad (2.12)$$

TABLE III

Coefficient, α^* , for the Median Unbiased Estimator of

$$x_p, \text{ i.e., } \hat{x}_p(\text{MUE}) = \bar{X} + \alpha^* \text{sgn}(.5 - p)S$$

$\begin{matrix} p \\ 1-p \\ n \end{matrix}$	.750	.900	.950	.990	.999
2	.8874	1.7842	2.3387	3.3760	4.5267
3	.7732	1.4985	1.9384	2.7645	3.6882
4	.7385	1.4189	1.8295	2.6008	3.4646
5	.7217	1.3818	1.7793	2.5258	3.3622
6	.7119	1.3605	1.7505	2.4828	3.3036
7	.7055	1.3466	1.7318	2.4551	3.2657
8	.7009	1.3369	1.7187	2.4357	3.2393
9	.6975	1.3296	1.7091	2.4213	3.2197
10	.6949	1.3241	1.7016	2.4103	3.2047
11	.6928	1.3197	1.6957	2.4016	3.1928
12	.6911	1.3161	1.6910	2.3945	3.1832
13	.6897	1.3132	1.6870	2.3886	3.1752
14	.6885	1.3107	1.6837	2.3837	3.1684
15	.6875	1.3085	1.6808	2.3795	3.1627
16	.6866	1.3067	1.6784	2.3759	3.1577
17	.6858	1.3051	1.6762	2.3727	3.1534
18	.6852	1.3037	1.6743	2.3699	3.1496
19	.6846	1.3024	1.6727	2.3674	3.1462
20	.6840	1.3013	1.6712	2.3652	3.1432

where $a = [\alpha^* \text{sgn}(.5 - p) + m^{-1} z_{1-p}]/2$ if $\hat{x}_p(*) = \hat{x}_p(\text{MVUE})$; $a = [\alpha^* \text{sgn}(.5 - p) + m z_{1-p}]/2$ if $\hat{x}_p(*) = \hat{x}_p(\text{BIE})$; and $a = [\alpha^* \text{sgn}(.5 - p) + (1 - n^{-1})^{1/2} z_{1-p}]/2$ if $\hat{x}_p(*) = \hat{x}_p(\text{MLE})$. Equation (2.12) also holds for $p < .5$ provided p is replaced by $1 - p$. Table IV is based on equation (2.12) evaluated when p (or equivalently, $1 - p$) = .75, .9, .95, .99, .999. In addition, the mean squared efficiencies of $\hat{x}_p(\text{MUE})$ relative to $\hat{x}_p(\text{MVUE})$, $\hat{x}_p(\text{BIE})$, and $\hat{x}_p(\text{MLE})$ have also been included.

Although $\hat{x}_p(\text{MUE})$ is preferred from a Pitman-closeness efficiency point of view, its mean squared efficiency can be low when n is extremely small. For instance, when comparing $\hat{x}_p(\text{MUE})$ with $\hat{x}_p(\text{MVUE})$, a low mean squared efficiency occurs when $n = 2$ and p

TABLE IV

Efficiency of $\hat{x}_p$ (MUE) Relative to
 $\hat{x}_p$ (MVUE), $\hat{x}_p$ (BIE), and $\hat{x}_p$ (MLE)

n	$\frac{p}{1-p}$	$\hat{x}_p$ (MVUE)		$\hat{x}_p$ (BIE)		$\hat{x}_p$ (MLE)	
		PC	MS	PC	MS	PC	MS
2	.750	1.028	.965	1.275	.845	1.335	.850
	.900	1.080	.857	1.443	.654	1.542	.662
	.950	1.102	.806	1.501	.585	1.614	.594
	.990	1.125	.751	1.560	.516	1.685	.526
	.999	1.136	.724	1.589	.483	1.721	.493
3	.750	1.012	.991	1.199	.933	1.261	.938
	.900	1.040	.957	1.329	.839	1.434	.849
	.950	1.053	.937	1.377	.798	1.500	.810
	.990	1.068	.911	1.431	.752	1.571	.765
	.999	1.077	.896	1.460	.726	1.610	.740
4	.750	1.008	.996	1.161	.959	1.219	.963
	.900	1.027	.978	1.266	.898	1.364	.907
	.950	1.037	.966	1.307	.870	1.420	.881
	.990	1.049	.950	1.352	.836	1.484	.849
	.999	1.056	.940	1.378	.816	1.520	.830
5	.750	1.006	.997	1.138	.971	1.191	.974
	.900	1.021	.985	1.228	.926	1.318	.934
	.950	1.029	.977	1.263	.905	1.367	.914
	.990	1.039	.966	1.303	.878	1.424	.890
	.999	1.045	.959	1.326	.863	1.456	.876
6	.750	1.005	.998	1.122	.977	1.172	.980
	.900	1.018	.989	1.202	.942	1.285	.949
	.950	1.024	.983	1.233	.925	1.329	.934
	.990	1.033	.975	1.269	.904	1.380	.914
	.999	1.039	.969	1.290	.890	1.410	.902
10	.750	1.003	.999	1.089	.988	1.128	.990
	.900	1.011	.995	1.147	.969	1.212	.974
	.950	1.016	.992	1.170	.960	1.245	.966
	.990	1.022	.988	1.196	.947	1.283	.955
	.999	1.027	.984	1.212	.940	1.305	.948
20	.750	1.002	.999	1.060	.995	1.088	.996
	.900	1.007	.998	1.099	.986	1.144	.988
	.950	1.010	.997	1.114	.981	1.166	.985
	.990	1.014	.995	1.132	.975	1.192	.980
	.999	1.017	.993	1.143	.972	1.207	.976

is large or small. The reason for this is basically the same as that given when comparing $\hat{x}_p$ (MVUE) with $\hat{x}_p$ (BIE). Whenever the value of $\hat{x}_p$ (MUE) for a given sample is closer to x_p than is the corresponding value of $\hat{x}_p$ (MVUE), relatively speaking it is only slightly closer. Whenever the value of $\hat{x}_p$ (MVUE) for a given sample is closer to x_p , relatively speaking it is much closer than is the corresponding value of $\hat{x}_p$ (MUE). However, in this case, the "odds" in favor of $\hat{x}_p$ (MUE) being closer to x_p are only slightly better than even; consequently, joint consideration of both efficiency measures would seem to favor $\hat{x}_p$ (MVUE).

Generally speaking, the probability of $\hat{x}_p$ (MVUE) overestimating x_p is slightly more or less than .5 (see Table I). Practically speaking, except when $n = 2$, it makes very little difference based on either efficiency measure whether $\hat{x}_p$ (MUE) or $\hat{x}_p$ (MVUE) is used. However, at least one of these estimators is preferred over any other estimator within the class Q . Strictly speaking, when $n = 2$, $\hat{x}_p$ (MVUE) is strongly preferred; when $n \geq 3$, $\hat{x}_p$ (MUE) is slightly preferred.

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