EFFICIENCY CONTOURS FOR ESTIMATORS
OF RELIABILITY IN A 2-PARAMETER
EXPONENTIAL FAILURE MODEL

by

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Summary & Conclusions

When there are available several point estimators of some parametric function, it would seem desirable to make a comparison among the estimators based on some measure of closeness to the true value. Along these lines, the concept of Pitman-closeness (PC) efficiency is introduced. Essentially, when comparing two estimators, PC efficiency gives the odds in favor of one of the estimators being closer to the true value in a given situation than is the other. The traditional method of comparison, i.e., mean squared (MS) efficiency is also considered. This paper presents graphical results based on simulation techniques which depict PC efficiencies as well as MS efficiencies of the following estimators of the reliability function $R(\tau)$ of a 2-parameter exponential failure model: (i) the maximum likelihood estimator ($\hat{R}_{MLE}(\tau)$), (ii) the minimum variance unbiased estimator ($\hat{R}_{MVUE}(\tau)$), and (iii) a Bayesian/structural estimator ($\hat{R}_{SE}(\tau)$). Based on the graphs, $\hat{R}_{SE}(\tau)$ is, in general, preferred (in the sense of having the best chance of being closest to the true value of $R(\tau)$) except (a) when $R(\tau)$ is very high, in which case $\hat{R}_{MLE}(\tau)$ is preferred, and (b) when $R(\tau)$ is moderate and the sample size is small to moderate, in which case $\hat{R}_{MVUE}(\tau)$ is preferred.

1. Introduction

The 2-parameter exponential distribution with Sf given by

$$\begin{aligned} Sf\{x;\mu,\beta\} &= \exp[-(x-\mu)/\beta], \quad x > \mu \\ &(\mu \geq 0, \beta > 0) \\ &= 1, \quad \text{otherwise} \end{aligned} \quad (1)$$

is often used as a failure model in reliability theory. Basically, component lifetime is guaranteed during the interval $[0, \mu]$, i.e., the probability of failure during this interval is zero. On the other hand, the failure rate is constant (non-zero) over the interval (μ, ∞) .

There are several point estimators of the reliability function $R(\tau) \equiv Sf\{\tau;\mu,\beta\}$, $\tau > 0$, based on a complete life test of size N from (1) where both μ and β are unknown. In order to judge which estimator is preferred for a given situation, they should be compared through some measure of closeness to the true value of $R(\tau)$. A common approach is mean squared (MS) efficiency. Specifically, let $\hat{R}_i(\tau)$, $i = 1, 2$, be two estimators of $R(\tau)$ with mean squared errors denoted by $MSE\{\hat{R}_i(\tau)\} \equiv E\{[\hat{R}_i(\tau) - R(\tau)]^2\}$, $i = 1, 2$. The MS efficiency of $\hat{R}_1(\tau)$ relative to $\hat{R}_2(\tau)$ is

$$rel. eff._{MS}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} \equiv MSE\{\hat{R}_2(\tau)\} / MSE\{\hat{R}_1(\tau)\}. \quad (2)$$

In general, $rel. eff._{MS}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\}$ depends on μ , β , N , τ . Furthermore, MS efficiency compares the *average* performance of an estimator relative to another; consequently, its interpretation is not always clear

especially when a single estimate of $R(\tau)$ is to be made. In addition, as will be discussed later, the MS efficiency value can be misleading.

We, therefore, propose another measure of efficiency motivated by the following definition due to Pitman [5]. As before, $\hat{R}_1(\tau)$ and $\hat{R}_2(\tau)$ are two estimators of $R(\tau)$ based on a complete life test of N components. Define $\psi\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} \equiv \Pr\{|\hat{R}_1(\tau) - R(\tau)| < |\hat{R}_2(\tau) - R(\tau)|\}$. In general, $\psi\{\hat{R}_1(\tau), \hat{R}_2(\tau)\}$ depends on μ, β, N, τ . If $\psi\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} \geq 1/2$ for all (μ, β, N, τ) with strict inequality for some $(\mu_0, \beta_0, N_0, \tau_0)$, then $\hat{R}_1(\tau)$ is called a Pitman-closer estimator of $R(\tau)$ than is $\hat{R}_2(\tau)$. We define the Pitman-closeness (PC) efficiency of $\hat{R}_1(\tau)$ relative to $\hat{R}_2(\tau)$ as

$$rel. eff._{PC}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} \equiv \psi\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} / \psi\{\hat{R}_2(\tau), \hat{R}_1(\tau)\}. \quad (3)$$

The probability statement $\Pr\{|\hat{R}_1(\tau) - R(\tau)| < |\hat{R}_2(\tau) - R(\tau)|\}$ is made in the classical sense and, consequently, will have a relative-frequency interpretation. Thus, $rel. eff._{PC}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\}$ gives the odds in favor of the estimator $\hat{R}_1(\tau)$ being closer to the true value of $R(\tau)$ than is $\hat{R}_2(\tau)$. If, for example, $rel. eff._{PC}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} = 1.5$, then the odds are 3 to 2 in favor of $\hat{R}_1(\tau)$ being closer to $R(\tau)$ than is $\hat{R}_2(\tau)$, i.e., on the average, 3 out of 5 estimates made by $\hat{R}_1(\tau)$ are closer to the true value of $R(\tau)$. Whenever $rel. eff._{PC}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\} > 1$ for some $(\mu_0, \beta_0, N_0, \tau_0)$, we shall say that $\hat{R}_1(\tau)$ is more PC efficient than $\hat{R}_2(\tau)$ at $(\mu_0, \beta_0, N_0, \tau_0)$.

In a recent paper dealing with the 2-parameter exponential distribution given by (1), Dyer [3] has derived closed form expressions for PC

efficiencies within a class of estimators of (a) the threshold parameter μ , and (b) the MTTF = $\mu + \beta$.

The primary purpose of this paper is to present graphical results based on simulation techniques depicting PC efficiencies as well as MS efficiencies of the following estimators of $R(\tau)$: (a) the maximum likelihood estimator, (b) the minimum variance unbiased estimator, and (c) a non-classical estimator based on the mean of the structural pdf of $R(\tau)$. The PC efficiency (perhaps, at times, in conjunction with MS efficiency) will provide additional insight and information with regard to the preference of estimators.

2. Comparisons of Estimators of $R(\tau)$

Let $X_1, \dots, X_N$ be a random sample of size N from (1) where both μ and β are unknown. Let $T \equiv (\tau - X_{(1)})/N(\bar{X} - X_{(1)})$, where $X_{(1)} \equiv \min\{X_1, \dots, X_N\}$ is the time to first failure and $\bar{X} \equiv (1/N) \sum_{i=1}^N X_i$ is the average time on test. By the invariance property of the method of maximum likelihood, the maximum likelihood estimator (MLE) of $R(\tau)$ is

$$\hat{R}_{MLE}(\tau) = \begin{cases} \exp(-NT), & T > 0 \\ 1, & \text{otherwise.} \end{cases} \quad (4)$$

The minimum variance unbiased estimator (MVUE) of $R(\tau)$ is [1]

$$\hat{R}_{MVUE}(\tau) = \begin{cases} 0, & T \geq 1 \\ (1 - 1/N)(1 - T)^{N-2}, & 0 < T < 1 \\ 1, & \text{otherwise.} \end{cases} \quad (5)$$

We also consider a non-classical estimator of $R(\tau)$ based on Fraser's [4] theory of structural inference - a group theoretic approach (using invariant Haar measures) to Fisher's fiducial theory. In the structural approach, as in the Bayesian approach, the given data induce a probability measure called the structural posterior distribution (SPD) on the parameter space. Unlike the Bayesian approach, however, this is achieved by assuming the existence of a structural model rather than a prior probability measure. The structural approach does not recognize the existence of a probability measure on the parameter space *apart from the data*. When sampling from a given population, the SPD is unique, whereas the Bayesian posterior distribution (BPD) is only unique relative to the selected prior. For the 2-parameter exponential failure model, it can be shown [4, p.64] that the SPD of (μ, β) is equivalent in form to the BPD of (μ, β) under Jeffreys' noninformative prior for location-scale parameters [2, pp. 53-58]. Under this particular BPD, the Bayesian estimator of $R(\tau)$ with respect to a squared-error loss function has been derived by Varde [6]. The mean of the SPD of $R(\tau)$, which is clearly equivalent in form to this Bayesian estimator, is given by

$$\hat{R}_{SE}(\tau) = \begin{cases} A[N/(N+1)] \left[(1+\tau)^{-(N-1)} - (S+\tau)^{-(N-1)} \right], & \tau > 0 \\ A \left\{ 1 - [1/(N+1)] \left[(1-N\tau)^{-(N-1)} - N(S+\tau)^{-(N-1)} \right] \right\}, & \text{otherwise} \end{cases} \quad (6)$$

where $A \equiv \left[1 - (1 - x_{(1)}/\bar{x})^{N-1} \right]^{-1}$, $S \equiv (N\bar{x} + x_{(1)})/N(\bar{x} - x_{(1)})$. We prefer to regard (6) as a structural estimator (rather than a Bayesian estimator) because of its independence under structural inference of

the choice of priors.

The distributional properties of the estimators are difficult to ascertain analytically; consequently, efficiency comparisons were made numerically under simulated conditions. Complete samples of size $N = 5, 15, 25, 40$ were used. Because of the form of each estimator, for fixed $R(\tau)$, $rel. eff._{MS}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\}$ and $rel. eff._{PC}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\}$ are invariant under scalar multiples of (μ, β) . For this reason only the value of the ratio of the parameters (and not the value of each parameter) need be considered. The range of values considered was $\beta/\mu = .2, 1, 2, 5, 10, 20$. [Note: All samples were generated using $\mu = 10$]. For fixed β/μ , different values of τ were selected so that there was also a common range of values of $R(\tau)$, i.e., $R(\tau) = .60, .70, .80, .85, .90, .95, .99$. For each choice of $(\beta/\mu, R(\tau))$, 1000 complete samples of size N (fixed) were randomly generated from (1). From each of these 1000 samples:

- (a) estimates of $R(\tau)$ based on each of the three estimators were calculated; then
- (b) a pairwise comparison of the three estimates was made. Within each pair of estimates, a difference in the values of less than .003 was taken as no essential difference between estimates, and a tie was recorded. Otherwise, the estimate which was closer to the known value of $R(\tau)$ was noted, and the estimator which gave this estimate was recorded. Ties counted one-half for each estimator.

Sample efficiencies relative to the 1000 samples were computed as follows:

- (i) the sample PC efficiency of $\hat{R}_1(\tau)$ relative to $\hat{R}_2(\tau)$ was taken to be the ratio of the number of times $\hat{R}_1(\tau)$ was closer to the known value of $R(\tau)$ to the number of times $\hat{R}_2(\tau)$ was closer;
- (ii) the sample MSE (sample variance plus the square of the bias of the sample) based on the 1000 estimates made from each method of estimation was calculated. Correspondingly, the sample MS efficiency of $\hat{R}_1(\tau)$ relative to $\hat{R}_2(\tau)$ was taken to be the ratio of the sample MSE of $\hat{R}_2(\tau)$ to the sample MSE of $\hat{R}_1(\tau)$.

For fixed N , we define a PC efficiency contour for $\hat{R}_1(\tau)$ relative to $\hat{R}_2(\tau)$ to be a curve joining points in the $(\beta/\mu, R(\tau))$ -plane for which $rel. eff._{PC}\{\hat{R}_1(\tau), \hat{R}_2(\tau)\}$ is equal. In a similar way, we define a MS efficiency contour. Figures 1-5 are based on these definitions and the sample efficiencies obtained through simulation.

3. Discussion of Results

We offer the following comments relative to the values of N , β/μ , $R(\tau)$ considered in the Figures.

1. For fixed N , when comparing $\hat{R}_{MVUE}(\tau)$ to $\hat{R}_{MLE}(\tau)$ the efficiency contours are lines parallel to the β/μ -axis. In other words, for fixed N , $rel. eff._{PC}\{\hat{R}_{MVUE}(\tau), \hat{R}_{MLE}(\tau)\}$ and $rel. eff._{MS}\{\hat{R}_{MVUE}(\tau), \hat{R}_{MLE}(\tau)\}$ depend only on the value of $R(\tau)$. Consequently, Figure 1 depicts simultaneously efficiency versus reliability for each value of N considered.

Generally speaking, $\hat{R}_{MLE}(\tau)$ is preferred (in the sense of having higher PC as well as MS efficiency) over $\hat{R}_{MVUE}(\tau)$ if and only if reliability is high.

2. To many reliability practitioners an estimate of one for $R(\tau)$ is unacceptable. In this case, $\hat{R}_{MLE}(\tau)$ is preferred (in the sense of having higher PC as well as MS efficiency) over $\hat{R}_{SE}(\tau)$ if and only if reliability is very high and $\tau > X_{(1)}$.

In addition, note that from Figures 2 and 3 and particularly when $N = 5$ or 15 , there are small regions in the $(\beta/\mu, R(\tau))$ -plane where $\hat{R}_{SE}(\tau)$ is more PC efficient yet less MS efficient than $\hat{R}_{MLE}(\tau)$. The reason for this occurrence can, generally speaking, be explained as follows. Based on 1000 pairs of estimates made when $N = 15$, $\beta/\mu = 20$, and τ was such that $R(\tau) = .99$, the value of $\hat{R}_{MLE}(\tau)$ was one in 865 of the pairs. Of these pairs, the value of $\hat{R}_{SE}(\tau)$ was closer to the true value $R(\tau) = .99$ a total of 478 times. In the remaining pairs, there were 271 ties, and thus the value of $\hat{R}_{MLE}(\tau)$, namely one, was closer to $R(\tau) = .99$ a total of 116 times. In each of the 135 pairs in which the value of $\hat{R}_{MLE}(\tau)$ was not one, this value was always closer to $R(\tau) = .99$. Since the value of $\hat{R}_{MLE}(\tau)$ was usually one or, otherwise, was very near one; the sample variance was very small, and the square of the sample bias was negligible. In this example, the sample $rel. eff._{MS}\{\hat{R}_{SE}(\tau), \hat{R}_{MLE}(\tau)\} = .74$, a value which is misleading since the odds are slightly better than 3 to 2 in favor of $\hat{R}_{SE}(\tau)$ being closer to the true value $R(\tau) = .99$.

3. From Figures 4 and 5, both efficiencies are one or very near one when $.60 \leq R(\tau) \leq .85$ and $N = 25, 40$. The reason is that in this particular situation there is very little difference between the corresponding estimates obtained from $\hat{R}_{SE}(\tau)$ and $\hat{R}_{MVUE}(\tau)$. In fact, based on simulation: when $N = 25$, more than 90% of the time the values within a pair of estimates differed in absolute value by less than .003; when $N = 40$, the difference in absolute value was less than .003 more than 99% of the time.

4. Pairwise comparisons of the estimators based on either measure of efficiency can be made from Figures 1-5. From these comparisons, for given N , one can determine the region in the $(\beta/\mu, R(\tau))$ -plane over which each of the three estimators is the most efficient and the extent of that efficiency. Unfortunately, β/μ and $R(\tau)$ are both unknown. Hence, unless some idea of an approximate value of $R(\tau)$ is available (we may take $\beta/\mu \approx (\bar{X}/X_{(1)}) - 1$), it may not be clear which one of the three estimators should be used in a given situation. But even though the simultaneous implementation of the figures is, at times, restricted; nevertheless, there is a great deal of information to be gained from the individual graphs, and it is for this reason that they are presented.

5. In conclusion, *generally speaking*, we prefer the estimator $\hat{R}_{SE}(\tau)$ over the estimators $\hat{R}_{MLE}(\tau)$ and $\hat{R}_{MVUE}(\tau)$ with the following exceptions:

- (a) $\hat{R}_{MLE}(\tau)$ is preferred regardless of sample size when reliability is very high (say, $R(\tau) \approx .99$) and $\tau > X_{(1)}$.
- (b) $\hat{R}_{MVUE}(\tau)$ is preferred when $R(\tau) \leq .70$ and $N = 5$, and when $.60 \leq R(\tau) \leq .85$ and $N = 15$.

Fig. 1. Efficiency of $\hat{R}_{MVUE}(\tau)$ Relative to $\hat{R}_{MLE}(\tau)$.

Fig. 2. Pitman-Closeness Efficiency Contours for $\hat{R}_{SE}(\tau)$ Relative to $\hat{R}_{MLE}(\tau)$.

Fig. 3. Mean Squared Efficiency Contours for $\hat{R}_{SI}(\tau)$ Relative to $\hat{R}_{MLE}(\tau)$.

Fig. 4. Pitman-Closeness Efficiency Contours for $\hat{R}_{SE}(\tau)$ Relative to $\hat{R}_{MVUE}(\tau)$.

Fig. 5. Mean Squared Efficiency Contours for $\hat{R}_{SE}(\tau)$ Relative to $\hat{R}_{MVUE}(\tau)$.

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