MONOTONE METHODS FOR NONLINEAR BOUNDARY
VALUE PROBLEMS IN BANACH SPACES

by

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1. Introduction

Monotone methods have been used to generate multiple solutions of nonlinear boundary value problems for both ordinary and partial differential equations. Keller [10] and Sattinger [13], extending the chord method, considered nonlinear partial differential equations containing no gradient term. The inclusion of the gradient term was first introduced by Chandra and Davis [6] who considered the ordinary boundary value problem

$$(1.1) \quad u'' = f(t, u, u')$$

$$(1.2) \quad B^i u = \alpha_i u(i) + (-1)^{i+1} \beta_i u'(i) = b_i \quad i = 0, 1.$$

Here $f \in C[[0,1] \times R \times R, R]$, $\alpha_0, \alpha_1 \geq 0$, $\beta_0, \beta_1 > 0$. They assumed that f depends linearly on u' ; and this restriction on u' was eliminated by Bernfeld and Chandra [3] and by Amann and Crandall [1] in the case of elliptic differential equations. The extension of these

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results to R^n ($1 < n \leq \infty$) was accomplished by Chandra, Lakshmikantham, and Leela [8]. This extension is important, for parabolic equations of the form

$$u_{xx} = f(t, x, u, u_t, u_x)$$

can be approximated, using the method of lines, by n -dimensional and infinite dimensional systems of type (1.1) (See Liskovets [11] and Thompson [15]). This leads to important numerical considerations for the solution of initial-boundary value problems for nonlinear parabolic partial differential equations.

The approach used in the above mentioned papers is to ascertain the existence of a lower solution $u_0(t)$ and upper solution $v_0(t)$ of (1.1), (1.2) such that $u_0(t) \leq v_0(t)$. By assuming a Nagumo condition on $f(t, u, u')$ one is able to obtain the existence of a solution $z(t)$ such that $u_0(t) \leq z(t) \leq v_0(t)$. In order to obtain the monotone iterations one considers

$$(1.3) \quad u'' = F(t, u, u', n)$$

where $F(t, u, u', n) = f(t, n, u') + \gamma(u - n)$ where $u_0(t) \leq n(t) \leq v_0(t)$ and $|f_u(t, u, u')| \leq \gamma$ for $t \in [0, 1]$, $u_0(t) \leq u(t) \leq v_0(t)$ and $|u'| \leq N$, where N is a prescribed constant arising from the Nagumo condition. The existence of solution of (1.3), (1.2) follows as in (1.1), (1.2). Moreover by using the classical maximum principle (in R^1) or a more generalized comparison result in R^n ($2 < n \leq \infty$) [8] one obtains for each $n(t)$ such that $u_0(t) \leq n(t) \leq v_0(t)$ a unique solution $w(t)$ of (1.3), (1.2) such

that $u_0(t) \leq w(t) \leq v_0(t)$. Letting A be the mapping defined by $w = Az$, one shows (using the maximum principle or generalized comparison result) that $Au_0 \leq u_0$, $Av_0 \leq v_0$ and A is monotone, that is $z_1 \leq z_2$ implies $Az_1 \leq Az_2$. Finally by defining the sequences $u_n = Au_{n-1}$, $v_n = Av_{n-1}$ one obtains that $\{u_n\}$ and $\{v_n\}$ converge monotonically to the minimal and maximal solutions of (1.1), (1.2) lying in the segment $\langle u_0, v_0 \rangle \stackrel{\text{def.}}{=} \{z(t) : u_0(t) \leq z(t) \leq v_0(t), t \in [0,1]\}$. If (1.1), (1.2) has only one solution in $\langle u_0, v_0 \rangle$ then $u_n(t)$ and $v_n(t)$ both converge to this solution.

In this paper we study the boundary value problem (1.1), (1.2) in an arbitrary Banach Space E . The extension of (1.1), (1.2) from R^n [8] to E requires some significant changes. For example, we talk about orderings induced by cones that are generated by linear functionals. We do not require the cones to have interior as is the case in R^n ($1 \leq n \leq \infty$) [8] and as is sometimes assumed in [17]. Moreover we develop a comparison principle that leads to the monotonicity of A which reduces to the case in [8] when $E = R^n$ and the cone is generated by the positive functionals. In addition we give conditions when the lower solution $u_0(t) \leq v_0(t)$ (upper solution) by using ideas from the maximum principle combined with a result of Schroder's [14] concerning inverse monotone mappings. In many of the above papers it is assumed that $u_0(t) \leq v_0(t)$. Using an idea of Amann and Crandall [1] and a technique in [2], we weaken the growth condition in [3], [8] of $f(t, u, u')$ on u' , essentially assuming quadratic growth. Finally, we weaken the assumption of

complete continuity on f by imposing conditions on f in terms of the measure of noncompactness. (See [9]).

We conclude the paper with an indication of the application of our method to systems of partial differential equations, a problem of great importance, for example, in applications. An expository version of this paper was presented at a recent U. S. Army Conference [5].

2. Comparison Results

Let E be a real Banach space with $|\cdot|$ and let E^* denote the set of continuous linear functionals. Let K be a cone in E which induces a partial ordering $\leq$ as follows: $x \leq y$ if and only if $y - x \in K$.

A linear functional $\phi \in E^*$ is said to be a positive linear functional if $\phi(x) \geq 0$ whenever $x \in K$. Let K^* denote the set of positive linear functionals. It then follows that K is contained in the closed half space.

$$C_\phi = [x \in E : \phi(x) \geq 0, \phi \text{ a positive linear functional}].$$

Thus the positive linear functionals are support functionals and since K is a cone in E , K is the intersection of all the closed half-spaces which support it. If $S \subset K^*$ and $K = \bigcap [C_\phi : \phi \in S]$ then we say that S generates K . Let $S_u = [\phi \in S : ||\phi|| = 1]$ and $\overline{S_u}$ be the closure of S_u in the weak star topology. We denote by K° , the interior of K and if $K^\circ \neq \emptyset$, K is said to be a solid cone. It is interesting to note that $x \in K^\circ$ if and only if there exists an $\epsilon > 0$ such that $\phi(x) \geq \epsilon$ for all $\phi \in S_u$.

We consider the boundary value problem (BVP, for short) in E :

$$(2.1) \quad u'' = f(t, u, u'),$$

$$(2.2) \quad B_i^j u = \alpha_i u(i) + (-1)^{i+1} \beta_i u'(i) = b_i, \quad i = 0, 1,$$

where $f \in C[I \times E \times E, E]$, $\alpha_0, \alpha_1 \geq 0$, $\beta_0, \beta_1 > 0$ and $I = [0, 1]$. Let K be a cone generated by S_u .

As indicated in the introduction an essential feature of the monotone method is the development of a comparison theorem, which in one dimension, is the classical generalized maximum principle [12, p. 8]. This result also allows us to assert that the lower solution lies below the upper solution. We shall need the following preliminary information before presenting the comparison result.

Definition. We shall say that $f(t, x, x')$ is quasimonotone nonincreasing in x with respect to K if $x \leq y$, $\phi(x) = \phi(y)$, $\phi(x') = \phi(y')$ implies

$$\phi(f(t, x, x')) \geq \phi(f(t, y, y')) \quad \text{for all } \phi \in S_u.$$

For the case in which $B = R^n$ $1 \leq n \leq \infty$, where K is a cone generated by the positive functionals this condition on f implies $f_i(t, x, x'_i) \equiv f_i(t, x, x'_i)$. This can be seen by letting $x = y$, thus obtaining $f_i(t, x, x') = f_i(t, x, y')$ whenever $x'_i = y'_i$.

Consider a family of functions $\{z_\lambda(t) = z(\lambda, t) : t \in I, \lambda > 0\}$. We shall say that the family $\{z_\lambda(t)\}$ is admissible if $z(\lambda, \cdot) \in C^2[I, E]$ for each λ and $z(\cdot, t)$ is continuous in λ for each t and $z(0, t) \equiv 0$. The family $\{z_\lambda(t)\}$ is said to satisfy a uniformity condition at $\lambda = 0$ ($\lambda = \infty$) if for each $\phi \in \bar{S}_u$, $\phi(z_\lambda(t)) \rightarrow 0$ as $\lambda \rightarrow 0$ uniformly for $t \in I$ ($\phi(z_\lambda(t)) \rightarrow \infty$ as $\lambda \rightarrow \infty$ uniformly for $t \in I$).

We need the following lemma [17]:

Lemma 2.1. Let $S \subset U = \{\phi \in E^* : \|\phi\| \leq 1\}$. If $v(t) = \inf[\phi(u(t)) : \phi \in S]$, where $u \in C[I, E]$ then $v(t)$ is continuous on I ; (ii) If $u \in E$ and $d = \inf[\phi(u) : \phi \in S]$ then there is a $\psi \in \bar{S}$ such that $\psi(u) = d$.

We now give the comparison theorem:

Theorem 2.1. Assume that

(A₁) K is a cone in E generated by S_u ;

(A₂) $f \in C[I \times E \times E, E]$ and $f(t, x, x')$ is quasimonotone nonincreasing in x with respect to K ;

(A₃) $V, W \in C^2[I, E]$ and for $t \in I$, $i = 0, 1$,

$$V'' \geq f(t, V, V''), \quad B^i V \leq b_i,$$

$$W'' \leq f(t, W, W'), \quad B^i W \geq b_i;$$

(A₄) $\{z_\lambda(t)\}$ is an admissible family satisfying a uniformity condition at $\lambda = 0$ and $\lambda = \infty$ such that for $t \in I$ and $\phi \in \bar{S}_u$,

$$\begin{aligned} \phi(z''_\lambda(t)) &< \phi(f(t, W(t) + z_\lambda(t), W'(t) + z'_\lambda(t))) \\ &\quad - \phi(f(t, W(t), W'(t))), \end{aligned}$$

and $B^i z_\lambda > 0$, $i = 0, 1$.

Then $V(t) \leq W(t)$ on I .

Proof: Since $W(t) - V(t)$ is continuous on I and $\phi(Z_\lambda(t)) \rightarrow 0(\infty)$ $\lambda \rightarrow 0(\infty)$ uniformly in $t \in I$ for all $\phi \in \bar{S}_u$, there exists a $\lambda_\phi > 0$ such that $\phi(W(t) + z_{\lambda_\phi}(t) - V(t)) \geq 0$, $t \in I$, $\phi \in \bar{S}_u$. Let $\Omega_\phi = [\delta_\phi \geq 0 : \phi(W(t) + z_{\delta_\phi}(t) - V(t)) \geq 0, t \in I]$ and define $\eta_\phi = \inf \Omega_\phi$. If we set

$$\lambda_\phi = \begin{cases} \eta_\phi & \text{if } \eta_\phi \neq 0, \\ 1 & \text{if } \eta_\phi = 0, \end{cases}$$

then we have

$$\phi(W(t) + z_{\lambda_\phi}(t) - V(t)) \geq 0 \quad \text{for } t \in I \text{ and } \phi \in \bar{S}_u.$$

By Lemma 2.1 if $m(t) = \inf_{\phi \in \bar{S}_u} \phi(W(t) - V(t))$, then $m(t)$ is continuous on I . Now the conclusion of the theorem is equivalent to the inequality $m(t) \geq 0$ on I . If the conclusion is not true, then $m(t) < 0$ for some $t \in I$ and since $m(t)$ is continuous there exists a $t_0 \in I$ such that $m(t_0) = \min[m(t) : t \in I] < 0$. Again, applying Lemma 2.1 there exists a $\psi \in \bar{S}_u$ such that $m(t_0) = \psi(W(t_0) - V(t_0))$. Hence if we define $h(t) = \psi(W(t) + z_{\lambda_\psi}(t) - V(t))$, it is clear that $\eta_\psi \neq 0$ and by the minimal choice of $\lambda_\psi = \eta_\psi$, it follows that $h(t_0) = 0$. If $t_0 \in (0,1)$, we also have $h'(t_0) = 0$ and $h''(t_0) \geq 0$. This means that

$$\psi(W(t_0) + z_{\lambda_\psi}(t_0) - V(t_0)) = 0,$$

$$\psi(W'(t_0) + z'_{\lambda_\psi}(t_0) - V'(t_0)) = 0,$$

$$\text{and} \quad \psi(W''(t_0) + z''_{\lambda_\psi}(t_0) - V''(t_0)) \geq 0.$$

Hence using the assumptions (A_3) and (A_4) , we obtain

$$\begin{aligned} 0 &\leq \psi(W''(t_0) + z''_{\lambda_\psi}(t_0) - V''(t_0)) \\ &< \psi(f(t_0, W(t_0) + z_{\lambda_\psi}(t_0), W'(t_0) + z'_{\lambda_\psi}(t_0)) - f(t_0, V(t_0), V'(t_0))). \end{aligned}$$

By the quasimonotonicity of $f(t, x, y)$ we have

$$\psi(f(t_0, V(t_0), V'(t_0))) \geq \psi(f(t_0, W(t_0) + z_{\lambda_\psi}(t_0), W'(t_0) + z'_{\lambda_\psi}(t_0))).$$

Thus we arrive at

$$\begin{aligned} 0 &\leq \psi(W''(t_0) + z''_{\lambda_\psi}(t_0) - V''(t_0)) \\ &< \psi(f(t, V(t_0), V'(t_0)) - f(t, V(t_0), V'(t_0))) = 0, \end{aligned}$$

which is a contradiction.

If $t_0 = 0$ ($t_0 = 1$), we have

$$\psi(W'(0) + z'_{\lambda\psi}(0) - V'(0)) \geq 0 \quad (\psi(W'(1) + z'_{\lambda\psi}(1) - V'(1)) \leq 0).$$

From the boundary conditions in (A_3) , this implies

$$\psi(B^0 z_{\lambda}(0)) \leq 0 \quad (\psi(B^1 z_{\lambda}(1)) \leq 0)$$

which is again a contradiction. This proves the theorem.

Let us now apply Theorem 2.1 to the case when f is linear that is, $f(t, x, x') = Q(t)x + P(t)x'$, where $P(t)$ and $Q(t)$ are linear operators mapping E into E . Then the quasi-monotonicity of f with respect to K implies that if $x \leq y$, $\phi(x) = \phi(y)$ then $\phi(Q(t)x) \geq \phi(Q(t)y)$ for all $\phi \in \bar{S}_u$. Thus $Q(t)x$ is quasimonotone nonincreasing in x relative to K . Furthermore, we find that $\phi(P(t)x') = \phi(P(t)y')$ when $\phi(x') = \phi(y')$. If $E = R^n$ and $K = R_+^n$, this implies that $P(t)$ is a diagonal matrix; that is the system is weakly coupled. We thus arrive at the corollary.

Corollary 2.1. Assume that

- (i) K is a cone in E generated by S_u ;
- (ii) $W \in C^2[I, E]$ and for $t \in I$, $i = 0, 1$,

$$W'' \leq P(t)W' + Q(t)W, \quad B^i W \geq 0,$$

where $Q(t)x$ is quasimonotone nonincreasing in x with respect to K , $P(t)$, $Q(t)$ are linear mappings from $E \rightarrow E$ and $\phi(P(t)x') = \phi(P(t)y')$ whenever $\phi(x') = \phi(y')$ for all $\phi \in \bar{S}_u$;

(iii) $\{z_\lambda(t)\}$ is an admissible family satisfying a uniformity condition at $\lambda = 0$ and $\lambda = \infty$ such that for $t \in I$ and $\phi \in \overline{S}_u$,

$$\phi(z''_\lambda(t)) < \phi(Q(t)z_\lambda(t)) + \phi(P(t)z'_\lambda(t)),$$

and $B^i z_\lambda > 0$, $i = 0, 1$.

Then $W(t) \geq 0$ on I . If the inequalities in (ii) are reversed then $W(t) \leq 0$.

The proof of this corollary follows from Theorem 2.1 by letting $V(t) \equiv 0$.

Remark: When $E = R$, the real line, Theorem 2.1 reduces to a similar result of Schroder [[14], references therein]; and Corollary 2.1 corresponds to the classical generalized maximum principle [12, p.8].

For the scalar case Schroder [14] has given several examples for which condition (iii) is satisfied.

If E is the Banach space of bounded sequences (and thus includes R^n , $1 \leq n \leq \infty$) then Theorem 2.1 and Corollary 2.1 reduce to analogous results in [8]. We observe, moreover, that Theorem 2.1 has greater flexibility than the results in [8] for K is arbitrary whereas in [8] K is the set of vectors all of whose components are nonnegative.

As we have mentioned our results complement those of R. Thompson [15], [17]. In particular, the differences can most easily be observed in the linear case, Corollary 2.1 with $E = R^\infty$. When K is a solid cone (an assumption we do not require), Thompson has essentially assumed that $Q(t)x$ is quasimonotone nonincreasing in x , the diagonal ele-

ments $\{q_{ii}(t)\}$ are non-negative and the matrix is diagonally dominant; that is,

$$(2.3) \quad q_{ii}(t) \geq |q_{i1}(t)| + |q_{i2}(t)| + \dots + |q_{ii-1}(t)| + |q_{i,i+1}(t)| + \dots$$

(recall $q_{ij}(t) \leq 0$, $i \neq j$). In other words Q is a positive operator and his results reduce to the classical maximum principle. On the other hand we assume the existence of an admissible family satisfying (iii), which is essentially a disconjugacy condition. Hence our result can be looked upon as an extension of the generalized maximum principle. Moreover we can work with cones that do not have an interior such as the positive cone in $\mathbb{R}^p$ spaces.

The requirement that $P(t)$ be a diagonal matrix is rather natural for it represents weakly coupled systems and such systems appear in applications. This condition can be weakened only by requiring rather stringent requirements on the lower and upper solution (see [4], [8]). The difficulty in eliminating this condition on $P(t)$, which is also present in $\mathbb{R}^n$, $n > 1$, is that at a maximum point t_0 of a component $x_i(t)$ of a solution $x(t)$ (where $x(t) \in \mathbb{R}^n$) we have $x_i'(t_0) = 0$; we do not however have any control on $x_j'(t_0)$ $j \neq i$. One way to get around this is to assume that the lower and upper solutions satisfy stronger inequalities. We will not elaborate on this further; the interested reader may see sections 1.11 and 1.12 in [4]. These added conditions are not compatible with the monotone method as we shall see.

3. Existence

Our aim in this section is to prove an existence result for the BVP (2.1) and (2.2) a set up suitable to monotone iterative methods. For existence in a different framework see [9]. We need the following results.

Theorem 3.1. Assume that

$$(i) \quad f \in C[I \times E \times E, E] \quad \text{and} \quad ||f(t, x, y)|| \leq M \quad \text{on} \quad I \times E \times E;$$

$$(ii) \quad \alpha(f(I \times A \times B)) \leq \beta \max(\alpha(A), \alpha(B)),$$

where $\alpha(A)$ denotes the measure of noncompactness of the set A ; A, B being bounded subsets of E .

Then the BVP (2.1) and (2.2) has a solution $x \in C^2[I, E]$ provided

$$2\rho\beta < 1 \quad \text{where} \quad \rho = \max_{I \times I} |G(t, S)|.$$

For a proof of Theorem 3.1 see [9].

Let us next state a Nagumo type theorem which is a tool to obtain bounds on the derivatives of solutions of equation (2.1). Its proof is very much similar to a corresponding result for finite systems given in [8] and hence is omitted.

Theorem 3.2. Assume that

$$(A_0) \quad V, W \in C^2[I, E] \quad \text{such that} \quad V(t) \leq W(t) \quad \text{on} \quad I;$$

$$(A_5) \quad h \in C[R_+, K] \quad \text{and for} \quad t \in I, \quad V(t) \leq x \leq W(t) \quad \text{and for} \quad x' \in E,$$

$$|\phi(f(t, x, x'))| \leq \phi(j(|\phi(x')|)),$$

for all $\phi \in S_u$;

(A₆) there exist $\lambda, N \in K$ such that for each $\phi \in S_u$,

$$\int_{\phi(\lambda)}^{\phi(N)} \frac{s \, ds}{\phi(j(s))} > \max_{t \in I} \phi(W(t)) - \min_{t \in I} \phi(V(t));$$

where $\phi(\lambda) = \max[|\phi(V(0) - W(1))|, |\phi(V(1) - W(0))|]$. Then for any solution $x \in C^2[I, E]$ of equation (2.1) with $V(t) \leq x(t) \leq W(t)$ on I , we have for each $\phi \in S_u$

$$|\phi(x'(t))| \leq |\phi(N)| \quad \text{on } I,$$

We are now in a position to prove our existence theorem. We observe that if the assumptions (A₁) to (A₄) hold, we have by Theorem 2.1, $V(t) \leq W(t)$ in I . Consequently, when (A₁) to (A₄) hold the condition (A₀) in Theorem 3.2 is superfluous. This observation will be useful later.

To prove this result we need to insist that an element x is uniquely determined by the action of all the linear functionals in S_u . That is we want that $\phi(x) = 0$ for all $\phi \in S_u$ implies $x = 0$. Since $\phi(x) = 0$ for all $\phi \in S_u$ implies $x \in K$ and since $\phi(-x) = 0$ for all $\phi \in S_u$ implies $-x \in K$ then $x \equiv 0$. Here we are using the fact that S_u generates K and that whenever $x, -x \in K$ then $x \equiv 0$.

Theorem 3.3. Assume that the hypotheses (A₁) to (A₆) hold. Suppose further that the condition (11) of Theorem 3.1 is satisfied. Then the BVP (2.1) and (2.2) has a solution $x \in C^2[I, E]$ such that $V(t) \leq x(t) \leq W(t)$ on I , provided $2\rho B < 1$. Moreover, there is an $N_0 \in K$ such that $|\phi(x'(t))| \leq \phi(N)$ on I for $\phi \in S_u$.

Proof: By (A₁) to (A₄), Theorem 2.1 shows that $V(t) \leq W(t)$ on I .

Hence by Theorem 3.2, there exists $N \in K$, depending only on V, W and j

such that for $\phi \in S_{\mathcal{U}}$, $|\phi(x'(t))| \leq \phi(N)$ on I , where $x(t)$ is any solution of (2.1). Let N_0 be chosen so that for $\phi \in S_{\mathcal{U}}$,

$$\phi(N_0) \geq \max(\phi(N), \max_I |\phi(V'(t))|, \max_I |\phi(W'(t))|)$$

and let $d > N_0$. Relative to the triple (V, W, d) we define a modified function $F(t, x, x')$ as follows:

We let

$$F^*(t, x, x') = f(t, x, \bar{x}')$$

where

$$\phi(\bar{x}') = \begin{cases} \phi(d), & \text{if } \phi(x') > \phi(d), \\ \phi(x'), & \text{if } -\phi(d) \leq \phi(x') \leq \phi(d), \\ \phi(-d), & \text{if } \phi(x') < \phi(-d), \end{cases}$$

and

$$\phi(F(t, x, x')) = \begin{cases} \phi(F^*(t, \bar{x}, x')) + \frac{\eta\phi(x - W(t))}{(1 + \phi(x)^2)}, & \text{if } \phi(x) > \phi(W(t)), \\ \phi(F^*(t, x, x')), & \text{if } \phi(V(t)) \leq \phi(x) \leq \phi(W(t)), \\ \phi(F^*(t, \bar{x}, x')) + \frac{\eta\phi(x - V(t))}{(1 + \phi(x)^2)}, & \text{if } \phi(x) < \phi(V(t)), \end{cases}$$

where $\eta > 0$ is sufficiently small such that $(\beta + \eta)2\rho < 1$ and

$$\phi(\bar{x}) = \begin{cases} \phi(W(t)), & \text{if } \phi(x) > \phi(W(t)), \\ \phi(x), & \text{if } \phi(V(t)) \leq \phi(x) \leq \phi(W(t)), \\ \phi(V(t)), & \text{if } \phi(x) < \phi(V(t)), \end{cases}$$

for $\phi \in S_u$.

From the definition of F , it is clear that F is well defined (see discussion preceding the theorem) and bounded on $I \times E \times E$. Furthermore in view of (ii) of Theorem 3.1,

$$\alpha(F(I \times A \times B)) \leq (\beta + \eta) \max(\alpha(A), \alpha(B)).$$

We also note that $|\phi(\bar{x}')| \leq \phi(N)$ and $v(t) \leq \bar{x} \leq w(t)$ on I . Hence by Theorem 3.1, the modified BVP

$$x'' = F(t, x, x'), \quad B^i x = b_i, \quad i = 0, 1,$$

has a solution $x \in C^2[I, E]$ if $2\rho(\beta + \eta) < 1$. We shall show that $x(t)$ is actually a solution of the BVP (2.1) and (2.2). For this purpose we need to show that $V(t) \leq x(t) \leq W(t)$ on I . We will only prove that $V(t) \leq x(t)$ on I . Similar arguments may be used to prove $x(t) \leq W(t)$ on I .

Suppose that the inequality $V(t) \leq x(t)$ on I is false. Then by Lemma 2.1 there exists a $\psi \in \bar{S}_u$ and $t_0 \in I$ such that $m(t_0) = \min_I m(t) < 0$ where $m(t) = \psi(x(t) - V(t))$. Let $\varepsilon > 0$ be such that $m(t_0) + \varepsilon = 0$. If $t_0 \in (0, 1)$, we then have $m'(t_0) = 0$ and $m''(t_0) \geq 0$. This means that $\psi(x(t_0) - V(t_0)) < 0$, $\psi(x'(t_0) - V'(t_0)) = 0$ and $\psi(x''(t_0) - V''(t_0)) \geq 0$. Also, we have $|\psi(x'(t_0))| < |\psi(d)|$ and consequently,

$$\begin{aligned} \psi(V''(t_0) - x''(t_0)) &\geq \psi[f(t_0, V(t_0), V'(t_0)) - F(t_0, x(t_0), x'(t_0))] \\ &= \psi \left[f(t_0, V(t_0), V'(t_0)) - f(t_0, x(t_0), x'(t_0)) + \frac{\eta(V(t_0) - x(t_0))}{1 + \psi(x(t_0))^2} \right]. \end{aligned}$$

we note that $V(t_0) \leq x(t_0)$, $\psi(\bar{x}(t_0)) = \psi(V(t_0))$ and $\psi(x'(t_0)) = \psi(V'(t_0))$. Hence quasimonotonicity of f gives

$$\psi(f(t_0, \bar{x}(t_0), \bar{x}'(t_0))) \leq \psi(f(t_0, V(t_0), V'(t_0))).$$

It therefore follows that

$$\psi(V''(t_0) - x''(t_0)) \geq \psi \left[\frac{\eta(V(t_0) - x(t_0))}{1 + \psi(x(t_0))^2} \right] > 0,$$

which is a contradiction to the fact $\psi(x''(t_0) - V''(t_0)) \geq 0$.

If $t_0 = 0$, we have $\psi(x(0) - V(0)) + \varepsilon = 0$. Since $\psi(x(t) - V(t)) + \varepsilon \geq 0$, it follows that $\psi(x'(0) - V'(0)) \geq 0$. By the boundary conditions $B^0 V(0) \leq b_0$ and $B^0 x(0) = b_0$, we have $\psi(B_0(x'(0) - V'(0))) \leq 0$. Since $\beta_0 > 0$, this implies that $\psi(x'(0)) = \psi(V'(0))$. We get, as before, $\psi(V''(t_0) - x''(t_0)) \geq 0$ which is a contradiction again. We can argue similarly the case $t_0 = 1$. We have thus established $V(t) \leq x(t) \leq W(t)$ on I . It therefore follows that

$$x'' = F(t, x, x') = f(t, x, \bar{x}').$$

Consequently, by (A_5) , we obtain

$$|\phi(f(t, x, \bar{x}'))| \leq \phi(j(|\phi(\bar{x}')|)), \quad \phi \in S_u,$$

whenever $V(t) \leq x \leq W(t)$ on I . By Theorem 3.2 and by the choice of $d > N_0$, it follows that $|\phi(x'(t))| \leq |\phi(N_0)|$ on I for $\phi \in S_u$. This then implies that $F(t, x, x') = f(t, x, x')$ and hence $x(t)$ is also a solution of the BVP (2.1) and (2.2). The proof of theorem is therefore complete.

Remark: A rather complete discussion concerning the compactness conditions on f for infinite dimensional boundary value problems appears in

[9]. In particular, the assumption of complete continuity of f is a special case of our result.

4. Monotone Method

In order to develop the monotone method for (2.1), (2.2), we need some preliminary results. Suppose $V(t) \leq W(t)$ are two functions on I with range in E . Assume that

(A₇) $f(t, x, y)$ is continuously differentiable in x and y and for $t \in I$, $V(t) \leq x \leq W(t)$, and $-N \leq y \leq N$, the Frechet derivative of f with respect to x , $D_x f$, satisfies for each $z \in B$

$$D_x f(t, x, y)(z) \leq Q(z)$$

where $Q(z)$ (notice Q depends on N) is quasimonotone nonincreasing in z with respect to K and maps bounded sets into bounded sets.

For any element $n \in C[I, E]$ such that $V(t) \leq n(t) \leq W(t)$ define

$$\begin{aligned} (4.1) \quad G(t, u, u') &\equiv G(t, u, u'; n) \\ &= F^*(t, n(t), u') + Q(u - n(t)), \end{aligned}$$

where F^* is the modified function of f with respect to u' (see Theorem 3.3) and N is the Nagumo constant given in (A₆). Notice that $G(t, u, u')$ is bounded for $t \in I$, $u \in \langle V, W \rangle$ and $u' \in E$

Since $F^*(t, n(t), u') = f(t, n(t), u')$ whenever $-N \leq u' \leq N$; then because of the quasimonotonicity of f we have for $t \in I$, $V(t) \leq n(t) \leq W(t)$, $-N \leq u' \leq N$

$$\frac{\partial G}{\partial u}(t, u, u') = P(t, n(t), u'), \quad \frac{\partial G}{\partial u'}(t, u, u') = Q,$$

where $P(t, n(t), u') = \frac{\partial f}{\partial u'}(t, n(t), u'(t))$ satisfies $\phi(P(t, n(t), u')) =$

$\phi(P(t, n(t), \phi(u')))$ (for example if $E = R^n$ P is a diagonal matrix).

The following lemmas are needed to show that for each $n(t)$ with $V(t) \leq n(t) \leq W(t)$ the BVP

$$(4.2) \quad u'' = G(t, u, u')$$

$$(4.3) \quad B^i u = b_i \quad i = 0, 1$$

has a unique solution $r(t)$ such that $V(t) \leq r(t) \leq W(t)$ and $-N \leq r(t) \leq N$. Notice then $r(t)$ satisfies

$$(4.4) \quad r'' = f(t, n(t), r'(t)) + Q(r - n(t))$$

$$B^i r = b_i$$

Lemma 4.1. Let the assumptions of Theorem 2.1 and (A_7) hold. Then

V and W are lower and upper solutions respectively of the BVP (4.2),

(4.3) whenever $-N \leq V'(t) \leq N$ and $-N \leq W'(t) \leq N$ for $t \in I$.

Proof: Consider the case of a lower solution. We need only to show $f(t, V, V') \geq G(t, V, V')$ for $t \in I$ since $(F^* \equiv f)$. To see this, note that

$$\begin{aligned} & f(t, V, V') - G(t, V, V') \\ &= f(t, V, V') - f(t, n, V') - Q(V - n) \\ &= \int_0^1 \frac{\partial f(t, sn + (1 - S)V, V')}{\partial x} (V - n) dt - Q(V - n) \\ &= \int_0^1 \left(\frac{\partial f}{\partial x}(t, sn + (1 - S)V, V') - Q \right) (V - n) dt \\ &\geq 0. \end{aligned}$$

Here we have used (A_7) and the fact that $V \leq \eta$. A similar argument shows that W is an upper solution.

The following lemmas will prove useful in obtaining a Nagumo condition for (4.2).

Lemma 4.2. Let $j(s)$ be a positive continuous function on $[0, \infty)$ for which $\int_0^\infty \frac{s \, ds}{j(s)} = \infty$. Let $\psi : [0, \infty) \rightarrow (0, \infty)$ such that $\psi(s) \leq K(1 + s)$ for some K . Then

$$\int_0^\infty \frac{s \, ds}{j(s) + \psi(s)} = \infty$$

if $j(s)$ has one of the two following properties

$$(4.5) \quad \text{if we let } S(C) = \{s : \frac{j(s)}{s} \leq C\}, \text{ then}$$

$m(S(C)) = \infty$ for some $C > 0$, where $m(S(C))$ represents the Lebesgue measure of $S(C)$

or

$$(4.6) \quad \int_{\sim S(C)} \frac{s \, ds}{j(s)} = \infty \text{ for some } C > 0,$$

where $\sim S(C)$ is the complement of $S(C)$.

Proof: The proof follows immediately from Corollary 4.2 in [2] by letting $j(s) = s h(s)$ and $\psi(s) = s \phi(s)$.

Remark: We note that if $j(s) \leq K(1 + s^2)$ then (4.6) is satisfied (this is precisely the quadratic growth condition on $f(t, u, u')$ in u' imposed in the previous papers (see [3], [1])). Of course we do not necessarily require quadratic growth condition under the conditions of Lemma 4.2.

Lemma 4.3. Let the assumptions of Theorem 2.1 together with (A_5) , (A_6) and (A_7) hold. Then the function G defined in (4.1) satisfies a Nagumo condition relative to V and W .

Proof: We shall first prove the result for the scalar case $E = R$.

The general result will follow from this due to the fact that the Nagumo condition in (A_5) is stated in terms of linear functionals and thus for each $\phi \in S_u$ we have essentially a scalar problem. Recalling that $E = R$, first observe that there exists a M such that $|F_u^*(t, n(t), u')| \leq M$ for all $t \in I$, $u \in \langle V, W \rangle$ and $u' \in R$ (see definition of F^* in Theorem 3.3). Thus we may assume that $|Q(u - n(t))| \leq M\gamma$ where $\gamma = \sup_{t \in I} W(t) - \inf_{t \in I} V(t)$, since $F^*(t, n(t), u') = f(t, n(t), \bar{u}')$. Hence

$$|G(t, n(t), u')| \leq j(|u'|) + M\gamma.$$

An application of Lemma 4.2 with $\psi(S) \equiv M\gamma$ yields $\int_0^\infty \frac{s \, ds}{j(s) + M\gamma} = \infty$.

Thus there exists some N such that

$$\int_\lambda^N \frac{s \, ds}{j(s) + M\gamma} > \max W(t) - \min V(t)$$

concluding the proof.

Remark: In [3], [8] the function $f(t, n(t), u')$ was used instead of the modified function $F^*(t, n(t), u')$ in (4.1). This allows us to not require the technical condition that $\frac{Q(N)}{N^2} \rightarrow 0$ as $N \rightarrow \infty$ as was assumed in [3] and [8]. The idea to use $F^*(t, n(t), u')$ is due to Amann and Crandall [1].

Lemma 4.4. Let the assumptions of Lemma 4.3 hold. Suppose further that there exist $\gamma, \beta > 0$ such that

$$(1) \quad \alpha(f(I \times A \times B)) \leq \beta(\max(\alpha(A), \alpha(B))) \quad \text{and}$$

$$\alpha(Q(C)) \leq \gamma(\alpha(C)),$$

where A, B, C are arbitrary bounded subsets of E and

$$N \leq \max(N, \sup_{t \in I} \sup_{\phi \in S_u} |\phi(V'(t))|, \sup_{t \in I} \sup_{\phi \in S_u} |\phi(W'(t))|).$$

Then if $2\rho(\beta + \gamma) < 1$ (where ρ is defined in Theorem 3.1) there exists a unique solution of the BVP (4.2), (4.3) such that $V(t) \leq x(t) \leq W(t)$ and $|\phi(x'(t))| \leq |\phi(N_0)|$ for each $t \in I$ and $\phi \in S_u$.

Proof: By Theorem 2.1 we know that $V(t) \leq W(t)$ on I . By Lemma 4.1 and 4.3 V, W are lower and upper solutions of the BVP (4.2), (4.3) and G satisfies a Nagumo condition. Because of (1) and the requirement $2\rho(\beta + \gamma) < 1$ we can use Theorem 3.1 to obtain the existence of a solution $x \in C^2[I, E]$ of the BVP (4.2), (4.3) such that $V(t) \leq x(t) \leq W(t)$ and $|\phi(x'(t))| \leq |\phi(N_0)|$ for all $\phi \in S_u$ and $t \in I$.

We now show the uniqueness of $x(t)$. If not, there exists another solution $y \in C^2[I, E]$ such that $V(t) \leq y(t) \leq W(t)$ with $|\phi(y'(t))| \leq |\phi(N_0)|$, $t \in I$. Setting $\psi(t) = x(t) - y(t)$, we obtain $B^i \psi(i) = 0$, $i = 0, 1$ and

$$\begin{aligned} \psi'' &= F^*(t, n, x') - F^*(t, n, y') + Q(x - y) \\ &= \int_0^1 \frac{\partial F^*(t, n, sx' + (1-s)y')}{\partial x'} dt \cdot \psi' + Q\psi \\ &= \int_0^1 P(t, n, sx'(t) + (1-s)y'(t)) ds \cdot \psi'(t) + Q\psi \end{aligned}$$

since $|\phi(sx' + (1-s)y')| \leq |\phi(N_0)|$ for all $\phi \in S_u$. Since

$$\begin{aligned} & \phi \left[\int_0^1 P(t, n, s x'(t) + (1-s)y'(t)) dt \cdot \psi'(t) \right] \\ &= \phi \left[\int_0^1 P(t, n, \phi(s x'(t) + (1-s)y'(t))) dt \cdot \psi'(t) \right], \end{aligned}$$

an application of Corollary 2.1 shows that $\phi \equiv 0$. This completes the proof.

For each $\eta \in C[I, E]$ such that $V(t) \leq \eta(t) \leq W(t)$ on I define the mapping A by

$$(4.7) \quad A\eta = z$$

where z is the unique solution of the BVP (4.2), (4.3). Notice that fixed points of A are solutions of the BVP (2.1) (2.2). We will show that A is a monotone mapping and is used to obtain monotone sequences that converge to the minimal and maximal solution of the BVP (2.1) (2.2).

For this we need the following:

Lemma 4.5. Under the assumptions of Lemma 4.4, the mapping A defined by (4.7) satisfies

$$(i) \quad V \leq AV, \quad W \geq AW;$$

$$(ii) \quad A \text{ is monotone on the segment } \langle V, W \rangle.$$

Proof: Suppose $AV = y$ where y is the unique solution of BVP (4.2) (4.3) corresponding to $\eta = V$. Setting $\psi = y - V$ and proceeding as in Lemma 4.4, we obtain $B^i \psi(i) \geq 0$ $i = 0, 1$, and

$$\psi'' \leq \int_0^1 P(t, V, s y' + (1-s)V') ds \cdot \psi'(t) + Q\psi.$$

Since $|\phi(s y' + (1-s)V')| \leq |\phi(N_0)|$ for all $\phi \in K_u$ we may apply Corollary 2.1 and conclude that $y \geq V$. Similarly we can show $AW \leq W$.

To prove (ii) let $\eta_1, \eta_2 \in C(I, E)$ such that $\eta_1, \eta_2 \in \langle V, W \rangle$ and

$\eta_1 \leq \eta_2$. Let $A\eta_1 = x_1$ and $A\eta_2 = x_2$. Letting $\psi = x_2 - x_1$, we obtain $B_i\psi(i) = 0$ $i = 0, 1$, and

$$\begin{aligned}
 \psi'' &= F^*(t, \eta_2, x_2') - F^*(t, \eta_1, x_1') + Q\psi - Q(\eta_2 - \eta_1) \\
 &= F^*(t, \eta_2, x_2') - F^*(t, \eta_2, x_1') \\
 &\quad + F^*(t, \eta_2, x_1') - F^*(t, \eta_1, x_1') + Q\psi - Q(\eta_2 - \eta_1) \\
 &\leq \int_0^1 P(t, \eta_2, sx_2' + (1-s)x_1') ds \cdot \psi'(t) \\
 &\quad + \int_0^1 \frac{\partial f}{\partial x}(t, s\eta_2 + (1-s)\eta_1, x_1') ds \cdot (\eta_2 - \eta_1) \\
 &\quad + Q\psi + Q(\eta_2 - \eta_1) \\
 &= \int_0^1 P(t, \eta_2, sx_2' + (1-s)x_1') ds \cdot \psi'(t) \\
 &\quad + Q\psi + \left[\int_0^1 \left[\frac{\partial f}{\partial x}(t, s\eta_2 + (1-s)\eta_1, x_1') - Q \right] ds \cdot (\eta_2 - \eta_1) \right] \\
 &\leq \int_0^1 P(t, \eta_2, sx_2' + (1-s)x_1') ds \cdot \psi'(t) + Q\psi
 \end{aligned}$$

since $\eta_1 \leq \eta_2$, $s\eta_2 + (1-s)\eta_1 \in \langle V, W \rangle$ for $s \in [0, 1]$ and $|\phi(sx_2' + (1-s)x_1')| \leq |\phi(N_0)|$ for all $\phi \in S_u$. An application of Corollary 2.1 yields $x_2 \geq x_1$, completing the proof.

If we define the sequences

$$(4.8) \quad V_n = AV_{n-1}, \quad W_n = AW_{n-1}$$

where $V_0 = V$ and $W_0 = W$ then Lemma 4.5 tells us that $V_{n+1} \geq V_n$ and $W_{n+1} \leq W_n$ for all n . We are interested in showing the convergence of these sequences to solutions of the BVP (1.1) (1.2). In order to do

so we must assume that K is a regular cone, that is if $\{x_n\}$ is a monotone sequence of elements of K bounded above by some element z ; namely $x_n \leq x_{n+1} \leq z$ for all n then x_n converges to some element $y \in E$. Notice that the positive cone in R^n , $1 \leq n \leq \infty$ is regular.

We can now state the main result of this section.

Theorem 4.1. Assume K is regular and let the condition of Lemma 4.4 hold. Then the sequence $\{V_n\}$, $\{W_n\}$ defined by (4.8) converge uniformly and monotonically to the minimal and maximal solutions V_{min}, W_{max} respectively of the BVP (2.1) (2.2) on $\langle V, W \rangle$; that is, if x is any solution of the BVP (2.1) (2.2) on $\langle V, W \rangle$ then

$$V \leq V_1 \leq V_2 \leq \dots, V_{min} \leq x \leq W_{max} \leq \dots W_2 \leq W_1 \leq W$$

Proof: In view of Lemma 4.5 the proof follows essentially along the same lines as the proof given of theorem 1 in [6] (See also [3]). We omit the details.

5. Concluding Remarks

As mentioned previously, we have addressed ourselves to weakly coupled systems which may be characterized by assuming

$$(5.1) \quad \phi(f(t, u, u')) = \phi(f(t, u, \phi(u')))$$

for each $\phi \in S_u$. Recall when $E = \mathbb{R}^n$ and K is the set of positive functionals (5.1) reduces to $f_i(t, u, u') = f_i(t, u, u'_i)$. This necessitated us to include the condition $\phi(x') = \phi(y')$ in the definition of the quasimonotonicity of $f(t, x, y)$ with respect to x . The only known ways (to the authors) to weaken the restriction (5.1) requires modifying the definition of lower and upper solutions, causing a breakdown in the development of the monotone method. The authors have been attempting to construct two cones K_1, K_2 in the development of the comparison theorem. As before K_1 would provide an ordering for $V(t)$ and $W(t)$ whereas K_2 would provide an ordering for $V'(t)$ and $W'(t)$. We would like to conclude that for important classes of BVP that $V(t) \leq W(t)$ with respect to K_1 implies $V'(t) \leq W'(t)$ with respect to K_2 . This would allow us to consider certain types of strongly coupled systems.

Our results have natural consequences in the development of monotone methods for systems of partial differential equations. It will be necessary to extend the monotone method for scalar elliptic equations [1] to systems by developing a comparison theorem analogous to Theorem 2.1. Moreover, in the same vein, we feel our work can be applied

to the case of systems of parabolic equations. Recently Chandra and Davis [7] have developed monotone methods for scalar parabolic equations.

Finally we see that the disconjugacy condition ((iii) in Corollary 2.1 or more generally A_4 in Theorem 2.1) allows us to conclude the results on an interval $[a,b]$ in which $b < a^*$ where a^* is the conjugate point of a (we are now thinking in terms of Corollary 2.1). Suppose we want to solve the boundary value problem on $[c,d]$ and condition (iii) doesn't hold because $d > c^*$, the conjugate point of c . Then we can divide $[c,d]$ into subintervals $[c_i, d_i]$ where $[c,d] = \bigcup_{i=1}^n [c_i, d_i]$, $c_0 = c$, $d_n = d$ such that $d_i < c_i^*$, the conjugate point of c_i and $c_{i+1} < d_i^*$, the conjugate point of d_i . We then apply our results on the subintervals (c_i, d_i) obtaining monotone approximations that are C^2 on the open intervals (c_i, d_i) . By properly assigning the values of the approximations at the boundary points c_i, d_i , we obtain an approximation of the solution on the interval $[c,d]$. The details will appear elsewhere.

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