EXISTENCE AND COMPARISON RESULTS FOR NONLINEAR
VOLTERRA INTEGRAL EQUATIONS IN A BANACH SPACE

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1. Introduction

In this paper we prove the existence of solutions to nonlinear Volterra integral equations in a Banach Space. A comparison theorem and the existence of maximal solutions are also obtained using the notion of ordering with respect to a cone. As is known, in infinite dimensional Banach spaces compactness-type conditions are needed to prove existence, whereas in finite dimensional cases these assumptions are not necessary. The results of the paper generalize the corresponding results of Nohel [6]. See also Miller [5] and Lakshmikantham and Leela [2].

Throughout this paper, E will denote a Banach space, and $[t_0, t_0 + a] = J \subset R$. We also use $B_\epsilon(x_0)$ to denote the ball of radius ϵ centered at x_0 , i.e., $B_\epsilon(x_0) = \{y \mid \|y - x_0\| < \epsilon\}$. The equation under consideration is

$$(1.1) \quad x(t) = x_0(t) + \int_{t_0}^t K(t, s, x(s)) ds$$

where $\Omega \subseteq E$ is open, $x_0 : J \rightarrow \Omega$ and $K : J \times J \times \Omega \rightarrow E$.

2. Preliminaries

The compactness type conditions which we impose on the function K will be in terms of the Kuratowski measure of noncompactness, α . For $A \subseteq E$ bounded, the Kuratowski measure of noncompactness for A is defined by $\alpha(A) = \inf\{\epsilon > 0 \mid A \text{ can be covered by a finite number of sets each with diameter } \leq \epsilon\}$.

We list below some known properties of α which we need in our discussion.

Lemma 2.1. Let E be a Banach space, and let A, B be bounded subsets of E . Then

- a) $\alpha(A) = \alpha(\overline{A}) = \alpha(\text{co } A)$;
- b) $A \subseteq B$ implies $\alpha(A) \leq \alpha(B)$;
- c) $\alpha(A) = 0 \Leftrightarrow A$ is precompact;
- d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
- e) If $\lambda \in \mathbb{R}$, $\alpha(\lambda A) = |\lambda| \alpha(A)$;
- f) If $A \subseteq C(J, E)$ is bounded, then $\sup_{t \in J} \alpha(A(t)) \leq \alpha(A(J)) \leq \alpha(A)$

where $A(t) = \{f(t) \mid f \in A\}$. If A is also equicontinuous, then $\alpha(A) = \sup_{t \in J} \alpha(A(t)) = \alpha(A(J))$.

- g) If $\{x_n\} \subseteq C(J, E)$ is an equicontinuous family then:

$$i) \quad \alpha \left\{ \left\{ \int_{t_0}^t x_n(s) ds \right\} \right\} \leq \int_{t_0}^t \alpha(\{x_n(s)\}) ds,$$

- ii) If $\{\varepsilon_n\} \subseteq \mathbb{R}$ with $\varepsilon_n \downarrow 0$, then $\alpha(\{x_n(t - \varepsilon_n)\}) = \alpha(\{x_n(t)\})$, $\forall t > t_0$.

- h) If $f(t, s, x(s))$ is integrable for each $x \in A \subseteq C(J, E)$, then

$$\alpha \left\{ \left\{ \int_{t_0}^t f(t, s, x(s)) ds \mid x \in A \right\} \right\} \leq |t - t_0| \cdot \alpha(\overline{\text{co}}(f(t, [t_0, t], A[t_0, t]))).$$

For a proof of the properties given in Lemma 2.1 see Martin [3].

We also need the following fixed point theorem of Darbo [1].

THEOREM 2.1. Let E be a Banach space and A a closed, bounded, convex subset of E . If $T \in C[A, A]$ is such that $\alpha(T(B)) \leq K\alpha(B)$ for $K < 1$ for each bounded subset B of A then T has a fixed point.

The following theorem due to Mazur [4], will be used in proving the existence of maximal solutions.

THEOREM 2.2. Let E be a Banach space and let $H \subseteq E$ be a cone. Let $x \in \partial H$, the boundary of H , there $\exists$ a continuous linear functional $c : E \rightarrow R$ with $c(x) = 0$, further $c(y) > 0$ for every y in the interior of H .

3. Existence Criteria

We are now in a position to prove our existence results. Let us start with the following:

THEOREM 3.1. Let $K \in C[J \times J \times \Omega, \Omega]$, $x_0 \in C[J, \Omega]$ and suppose that the following conditions are satisfied:

- (A1) $||K(t, s, x)|| \leq M$ for all $(t, s, x) \in J \times J \times \Omega$;
- (A2) $\lim_{t \rightarrow \tau} (\sup_I \int_I ||K(t, s, x(s)) - K(\tau, s, x(s))|| ds : x \in C[I, B]) = 0$
for every bounded set $B \subseteq \Omega$ and every interval $I \subseteq J$;
- (A3) $\alpha(K(J \times J \times B)) \leq \beta \alpha(B)$, where $\beta > 0$, for each bounded set $B \subseteq \Omega$.

Then $\exists$ a solution $x(t)$ for the problem (1.1) on $[t_0, t_0 + \gamma]$ for some $\gamma > 0$.

Proof: Set $\eta = \sup\{\epsilon > 0 | B_\epsilon(x_0(t_0)) \subseteq \Omega\}$. Since x_0 is uniformly continuous on the interval J , $\exists \delta_1 > 0$ so that

$$|t - t_0| < \delta_1 \Rightarrow ||x_0(t) - x_0(t_0)|| < \frac{\eta}{2}. \text{ Let } \gamma = \min\left[a, \delta_1, \frac{\eta}{2M}, \frac{1}{2\beta}\right],$$

and let $J_0 = [t_0, t_0 + \gamma]$.

Define $A \subseteq C(J_0, \Omega)$ by

$$A = \left\{ \phi \in C[J_0, \Omega] : \sup_{t \in J_0} ||\phi(t) - x_0(t)|| \leq \frac{\eta}{2} \right\}.$$

Clearly A is closed, bounded and convex.

Define $T : C(J_0, \Omega) \rightarrow C(J_0, \Omega)$ by

$$(T\phi)(t) = x_0(t) + \int_{t_0}^t K(t, s, \phi(s)) ds .$$

If $\phi \in A$, then

$$|| (T\phi)(t) - x_0(t) || \leq \int_{t_0}^t || K(t, s, \phi(s)) || ds \leq M(t - t_0) \leq \frac{\eta}{2} .$$

Thus $TA \subseteq A$ and T is bounded.

To see that T is continuous, let $\{x_n\} \subseteq A$ be a sequence converging to x . By the continuity of K , we have

$$K(t, s, x_n(s)) \rightarrow K(t, s, x(s)) .$$

Further $|| K(t, s, x_n(s)) || \leq M$. By applying the Bounded convergence theorem, it follows that

$$\int_{t_0}^t K(t, s, x(s)) ds = \lim_{n \rightarrow \infty} \int_{t_0}^t K(t, s, x_n(s)) ds .$$

Hence $Tx_n \rightarrow Tx$.

If $t, \tau \in J_0$, and $\phi \in A$, we get

$$\begin{aligned} || (T\phi)(t) - (T\phi)(\tau) || &\leq || x_0(t) - x_0(\tau) || + \int_{\tau}^t || K(t, s, \phi(s)) || ds \\ &\quad + \int_{t_0}^{\tau} || K(t, s, \phi(s)) - K(\tau, s, \phi(s)) || ds \\ &\leq || x_0(t) - x_0(\tau) || + M(t - \tau) + \int_{\tau}^t || K(t, s, \phi(s)) - K(\tau, s, \phi(s)) || ds \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned}$$

Since x_0 is uniformly continuous on J_0 , given $\varepsilon > 0$ $\exists \delta_1^* > 0$ so that $|t - \tau| < \delta_1^* \Rightarrow I_1 = ||x_0(t) - x_0(\tau)|| < \frac{\varepsilon}{3}$. Let $\delta_2^* < \frac{\varepsilon}{3M}$, then $|t - \tau| < \delta_2^* \Rightarrow M(t - \tau) < \frac{\varepsilon}{3}$. By assumption (A2), $\exists \delta_3^* > 0$ so that

$$|t - \tau| < \delta_3^* \Rightarrow \int_{t_0}^{\tau} ||K(t, s, \phi(s)) - K(\tau, s, \phi(s))|| ds < \frac{\varepsilon}{3}.$$

Consequently, if $|t - \tau| < \min(\delta_1^*, \delta_2^*, \delta_3^*)$, $|(T\phi)(t) - (T\phi)(\tau)| < \varepsilon$.

Thus TA is equicontinuous at each point $t \in J_0$.

Now let $B \subseteq A$, then, using the properties of α listed in Lemma 2.1, we get

$$\begin{aligned} \alpha(TB(t)) &= \alpha\left\{\left\{x_0(t) + \int_{t_0}^t K(t, s, \phi(s))ds : \phi \in B\right\}\right\} \\ &\leq \alpha\left\{\left\{\int_{t_0}^t K(t, s, \phi(s))ds : \phi \in B\right\}\right\} \\ &\leq |t - t_0| \cdot \alpha(\overline{co} K(t, [t_0, t], B[t_0, t])) \\ &\leq \gamma \cdot \alpha(K(J_0 \times J_0 \times B(J_0))) \leq \gamma \cdot \alpha(K(J \times J \times B(J_0))) \\ &\leq \gamma \cdot \beta \cdot \alpha(B(J_0)) \leq \frac{1}{2} \alpha(B). \end{aligned}$$

Thus $\alpha(TB) = \sup_{t \in J_0} (TB(t)) \leq \frac{1}{2} \alpha(B)$. By the Darbo fixed point theorem $\exists$ a fixed point ϕ of T . Clearly such a fixed point is a solution of (1.1).

By requiring that K be uniformly continuous we may weaken the compactness-type condition. We use the following result of Nohel [6].

THEOREM: Let $G \in C[J \times J \times R^+, R^+]$, $G(t, s, u)$ monotone nondecreasing in u for each $(t, s) \in J \times J$, and

$$m(t) \leq \int_{t_0}^t G(t, s, m(s)) ds, \text{ for } t \geq t_0,$$

where $m \in C(J, R^+)$. Then $m(t) \leq r(t)$, where $r(t)$ is the maximal solution of

$$v(t) = \int_{t_0}^t G(t, s, v(s)) ds.$$

The proof of the following theorem is slightly different than that of Theorem 3.1, we indicate the proof as approximate solutions to (1.1) are constructed.

THEOREM 3.2. Let $x_0 \in C[J, \Omega]$ and let $K : J \times J \times \Omega \rightarrow \Omega$ be uniformly continuous. Assume (A1) and (A2) of Theorem 3.1 hold and that the following compactness condition is satisfied:

(A3) $\alpha(K(t, s, B)) \leq g(t, s, \alpha(B))$ for every bounded $B \subseteq \Omega$, where $g \in C(J \times J \times R^+, R^+)$, $g(t, s, u)$ is monotone nondecreasing in u for each $(t, s) \in J \times J$, and the equation

$$u(t) = \int_{t_0}^t g(t, s, u(s)) ds$$

has unique solution $u(t) \equiv 0$.

Then $\exists$ a solution $x(t)$ to (1.1) on $[t_0, t_0 + \gamma]$ for some $\gamma > 0$.

Proof: Let η and δ_1 be as in Theorem 3.1. Let $\gamma = \min\left(\alpha, \delta_1, \frac{\eta}{2M}\right)$ and let $J_0 = [t_0, t_0 + \gamma]$.

Define $\{x_n\}_{n=1}^{\infty} \subseteq C[J_0, \Omega]$ by

$$x_n(t) = x_0(t_0) ; \quad t_0 - \gamma < t < t_0 ,$$

$$x_n(t) = x_0(t) + \int_{t_0}^t K\left(t, s, x_n\left(s - \frac{\gamma}{n}\right)\right) ds ; \quad t_0 \leq t \leq t_0 + \gamma.$$

The equicontinuity and uniform boundedness of $\{x_n\}$ follows as in Theorem 3.1.

As K is uniformly continuous, and $\{x_n\}$ is equicontinuous the sequence $\left\{K\left(t, s, x_n\left(s - \frac{\gamma}{n}\right)\right)\right\}$ is equicontinuous, using Lemma 2.1 and assumption ($\hat{A}3$), we get

$$\begin{aligned} \alpha(\{x_n(t)\}) &= \alpha\left\{\int_{t_0}^t K\left(t, s, x_n\left(s - \frac{\gamma}{n}\right)\right) ds\right\} \\ &\leq \int_{t_0}^t \alpha\left\{K\left(t, s, x_n\left(s - \frac{\gamma}{n}\right)\right)\right\} ds \\ &= \int_{t_0}^t \alpha\left\{K\left(t, s, \left\{x_n\left(s - \frac{\gamma}{n}\right)\right\}\right)\right\} ds \\ &\leq \int_{t_0}^t g\left(t, s, \alpha\left\{x_n\left(s - \frac{\gamma}{n}\right)\right\}\right) ds = \int_{t_0}^t g\left(t, s, \alpha(\{x_n(s)\})\right) ds . \end{aligned}$$

Consequently $\alpha(\{x_n(t)\}) \leq r(t)$ where $r(t)$ is the maximal solution of

$$v(t) = \int_{t_0}^t g(t, s, v(s)) ds, \quad \text{hence } \alpha(\{x_n(t)\}) \equiv 0, \quad \text{in view of } (\hat{A}3).$$

Thus $\{x_n\}$ contains a uniformly convergent subsequence $\{\psi_n\}$.

If $\psi_n \rightarrow \psi$, the bounded convergence theorem implies

$$\int_{t_0}^t K(t, s, \psi_n(s)) ds \rightarrow \int_{t_0}^t K(t, s, \psi(s)) ds.$$

Thus ψ is a solution of (1.1).

4. Maximal Solutions

We now prove that, under suitable conditions, the maximal solution to (1.1) exists. To do so we define an order relation on the Banach space E .

Let $H \subseteq E$ be a proper cone, and let H^0 denote the interior of H ($H^0 \neq \emptyset$). For $u, v \in E$ we define $u \leq v$ if $v - u \in H$ and $u < v$ if $v - u \in H^0$. We also denote

$$H_0^* = \{c \in L(E, R) \mid x \in H^0 \Rightarrow c(x) > 0\}$$

and

$$H^* = \{c \in L(E, R) \mid x \in H \Rightarrow c(x) \geq 0\},$$

where $L(E, R)$ is the space of all continuous linear functionals from E to R .

The following integral inequality will be of use in the remainder of this paper.

THEOREM 4.1. Let $g \in C[J \times J \times E, E]$, $x_0, u, v \in C[J, E]$ with $g(t, s, u)$ monotone nondecreasing in u for each $(t, s) \in J \times J$. Then

$$u(t) \leq x_0(t) + \int_{t_0}^t g(t, s, u(s)) ds,$$

$$v(t) \geq x_0(t) + \int_{t_0}^t g(t, s, v(s)) ds,$$

with one of the inequalities strict, and $u(t_0) < v(t_0)$ implies $u(t) < v(t) \forall t \geq t_0$.

Proof: Suppose the theorem is false. Then $\exists t_1 > t_0$ so that $v(t_1) - u(t_1) \in \partial H$, the boundary of H , and $v(t) - u(t) \in H^0$ for $t \in [t_0, t_1]$. By Theorem 2.2 there exists a $c \in H_0^*$ with $c(u(t_1)) = c(v(t_1))$. By the monotone property of g

$$g(t_1, s, u(s)) \leq g(t_1, s, v(s))$$

for all $s \in [t_0, t_1]$. Thus

$$\begin{aligned} c(u(t_1)) &\leq c(x_0(t_1)) + \int_{t_0}^{t_1} g(t_1, s, u(s)) ds \\ &= c(x_0(t_1)) + \int_{t_0}^{t_1} c(g(t_1, s, u(s))) ds \\ &\leq c(x_0(t_1)) + \int_{t_0}^{t_1} c(g(t_1, s, v(s))) ds \\ &< c(v(t_1)) . \end{aligned}$$

which contradicts $c(u(t_1)) = c(v(t_1))$. Thus the theorem is true.

We now prove that maximal solutions to (1.1) exist if K is monotone and satisfies (A1)-(A3) of Theorem 3.1.

THEOREM 4.2. Let K and x_0 be as in Theorem 3.1 and suppose $K(t, s, u)$ is monotone nondecreasing in u for each $(t, s) \in J \times J$ then $\exists \gamma > 0$ so that the maximal solution to (1.1) exists on $[t_0, t_0 + \gamma]$.

Proof: Let η and δ_1 be as in Theorem 3.1. Let $\gamma = \min\left[a, \delta_1, \frac{\eta}{4M}, \frac{1}{2\beta}\right]$.

Let $A = \left\{ \phi \in C([t_0, t_0 + \gamma], \Omega) \mid \|\phi - x_0\| \leq \frac{\eta}{2} \right\}$.

Again define $T : C([t_0, t_0 + \gamma], \Omega) \rightarrow C([t_0, t_0 + \gamma], \Omega)$ by

$$(T\phi)(t) = x_0(t) + \int_{t_0}^t K(t, s, \phi(s)) ds .$$

Let $y_0 \in H^0$ with $||y_0|| \leq \frac{\eta}{4}$, and $y_n = \frac{1}{n} y_0$, $n = 1, 2, \dots$.

We define $T_n : C([t_0, t_0 + \gamma], \Omega) \rightarrow C([t_0, t_0 + \gamma], \Omega)$ by

$$T_n \phi = T\phi + y_n, \quad n = 1, 2, \dots$$

The continuity of T_n follows from the continuity of T .

Further

$$|| (T_n \phi)(t) - x_0(t) || = || y_n + \int_{t_0}^t K(t, s, \phi(s)) ds || \leq || y_n || + M(t - t_0) \leq \frac{\eta}{2}.$$

Thus $T_n A \subseteq A$ for each n .

As in Theorem 3.1 it can be shown that $T_n A$ is equicontinuous at each $t \in [t_0, t_0 + \gamma]$.

If $B \subseteq A$,

$$\alpha(\{T_n \phi | \phi \in B\}) = \alpha(\{T\phi + y_n | \phi \in B\}) = \alpha(\{T\phi | \phi \in B\}) = \alpha(TB) \leq \frac{1}{2} \alpha(B).$$

Thus each T_n has a fixed point ϕ_n .

Also,

$$\phi_1(t) = x_0(t) + y_1 + \int_{t_0}^t K(t, s, \phi_1(s)) ds > x_0(t) + y_2 + \int_{t_0}^t K(t, s, \phi_1(s)) ds = T_2 \phi_1(t),$$

and $\phi_1(t_0) = y_1 > y_2 = \phi_2(t_0)$, thus by Theorem 4.1 $\phi_1(t) > \phi_2(t) \forall t \geq t_0$.

Similarly $\phi_n(t) > \phi_m(t) \forall t \in [t_0, t_0 + \gamma]$ if $m > n$.

Now consider $\{\phi_n\}$. Since

$$\begin{aligned} || \phi_n(t) - x_0(t_0) || &= || (T_n \phi_n)(t) - x_0(t_0) || = || (T_n \phi_n)(t) + y_n - x_0(t_0) || \\ &\leq || (T\phi_n)(t) - x_0(t) || + || x_0(t) - x_0(t_0) || + || y_n || \\ &< \frac{\eta}{4} + \frac{\eta}{2} + \frac{\eta}{4} = \eta. \end{aligned}$$

We see that the sequence $\{\phi_n\}$ is uniformly bounded.

Further

$$||\phi_n(t) - \phi_n(\tau)|| = ||(T\phi_n)(t) - (T\phi_n)(\tau)|| = ||(T\phi_n)(t) - (T\phi_n)(\tau)||$$

shows that $\{\phi_n\}$ is equicontinuous at each point $t \in [t_0, t_0 + \gamma]$.

Suppose $\{\phi_n\}$ is not compact. Then $\alpha(\{\phi_n\}) > 0$, and

$$\begin{aligned}\alpha(\{\phi_n(t)\}) &= \alpha(\{(T\phi_n)(t) + y_n\}) \\ &\leq \alpha(\{(T\phi_n)(t)\}) = \alpha(T\{\phi_n(t)\}) \\ &\leq \frac{1}{2} \alpha(\{\phi_n(t)\})\end{aligned}$$

thus $\alpha(\{\phi_n\}) \leq \frac{1}{2} \alpha(\{\phi_n\})$ which is impossible. Thus $\alpha(\{\phi_n\}) = 0$ and consequently $\{\phi_n\}$ is precompact. Thus $\exists$ a uniformly convergent subsequence $\{\phi_{nk}\}$ of $\{\phi_n\}$. Suppose $\phi_{nk} \rightarrow \psi$. By assumption (A1), $||K(t, s, \phi_{nk}(s))|| \leq M$, and thus the bounded convergence theorem implies

$$\int_{t_0}^t K(t, s, \phi_{nk}(s)) ds \rightarrow \int_{t_0}^t K(t, s, \psi(s)) ds,$$

and that ψ is a fixed point of T .

If x is any other fixed point of T ,

$$\begin{aligned}x(t) &= x_0(t) + \int_{t_0}^t K(t, s, x(s)) ds \quad \text{and} \\ \phi_{nk}(t) &= x_0(t) + y_{nk} + \int_{t_0}^t K(t, s, \phi_{nk}(s)) ds \\ &> x_0(t) + \int_{t_0}^t K(t, s, \phi_{nk}(s)) ds\end{aligned}$$

and $x(t_0) = x_0 < x_0 + y_{nk} = \phi_{nk}(t_0)$, thus by Theorem 4.1 $x(t) < \phi_{nk}(t)$ for $t \in [t_0, t_0 + \gamma]$. This implies that $x(t) \leq \lim_{k \rightarrow \infty} \phi_{nk}(t) = \psi(t)$ for $t \in [t_0, t_0 + \gamma]$. And thus ψ is the maximal solution to (1.1) on $[t_0, t_0 + \gamma]$.

An analogous theorem may be stated under the conditions of Theorem 3.2.

THEOREM 4.3. Let K and x_0 be as in Theorem 3.2 and let $K(t, s, u)$ be monotone nondecreasing in u for each $(t, s) \in J \times J$, then $\exists \gamma > 0$ so that the maximal solution to (1.1) exists on $[t_0, t_0 + \gamma]$.

Having proved the existence of maximal solutions of (1.1) we can now prove a comparison result for integral inequalities.

THEOREM 4.4. Suppose the hypotheses of Theorem 4.2 are satisfied. Let

$m \in C[J, \Omega]$ and $m(t) \leq x_0(t) + \int_{t_0}^t K(t, s, m(s)) ds$ on J . Then $m(t) \leq r(t)$

$t \in [t_0, t_0 + \gamma]$, where $r(t)$ is the maximal solution of (1.1).

Proof: As in the proof of Theorem 4.2, let $y_0 \in H^0$ with

$\|y_0\| = \frac{1}{4}$, $y_n = \frac{1}{n} y_0$, $n = 1, 2, \dots$, and let $x_n(t)$ be a solution of

$$x(t) = x_0(t) + \int_{t_0}^t K(t, s, x(s)) ds + y_n$$

on $[t_0, t_0 + \gamma]$.

Since, $\lim_{n \rightarrow \infty} x_n(t) = r(t)$ for $t \in [t_0, t_0 + \gamma]$, it suffices to prove

that $m(t) < x_n(t)$ for $t \in [t_0, t_0 + \gamma]$.

Since $m(t_0) < x_n(t_0) = x_0(t_0) + y_n$ and

$$x_n(t) > x_0(t) + \int_{t_0}^t K(t,s,x_n(s))ds ,$$

Theorem 4.1 yields $m(t) < x_n(t)$ for $t \in [t_0, t_0 + \gamma]$ for each n .

Thus the proof is complete.

REFERENCES

- [1] Darbo, G. Punti uniti in Trasformazioni a codominio noncompatto, *Rundsch. Sem. Mat. Univ. Padova*. 24 (1955).
- [2] Lakshmikantham, V. and Leela, S. *Differential and Integral Inequalities*, Vol 1. Academic Press. 1969.
- [3] Martin, R. H. *Nonlinear Operators and Differential Equations in Banach Space*. J. Wiley and Sons, 1976.
- [4] Mazur, S. Über konvexe Mengen in linearen normierten Räumen, *Stud. Math.* 4 (1933), 70-84.
- [5] Miller, R. K. *Nonlinear Volterra Integral Equations*, W. A. Benjamin, Inc., 1971.
- [6] Nohel, J. A. Some problems in nonlinear Volterra Integral Equations, *Bull. Amer. Math. Soc.* 68 (1962), 373-329.