**EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF REACTION-
DIFFUSION SYSTEMS VIA COUPLED QUASI-SOLUTIONS**

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1. INTRODUCTION

In the study of comparison theorems, existence of extremal solutions and monotone iterative techniques for differential systems a property called quasimonotone property is necessary [2,7,10]. However, there are several physical situations wherein such a property is not satisfied [1]. This difficulty has been overcome by introducing the notion of quasi-solutions [3,8,9].

In this paper we consider the reaction-diffusion system in which quasimonotone property is not satisfied but a mixed quasimonotone property holds. By utilizing fruitfully the notion of quasi-solutions we prove the existence of coupled maximal and minimal solutions. For this purpose we exploit the monotone iterative technique. We then offer methods of constructing explicit coupled upper and lower solutions from which we deduce the asymptotic behaviour of solutions. Our results are in the spirit of similar results given in [1,5,10] and shed much light on the whole situation.

2. NOTATIONS AND DEFINITIONS

We consider the following system of initial boundary value problem (I.B.V.P.) for short)

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$$\frac{\partial u^i}{\partial t} - L_i u^i \equiv \frac{\partial u^i}{\partial t} - \left(\sum_{j,k=1}^N a_{jk}^i(x) u_{x_j x_k}^i + \sum_{j=1}^N a_j^i(x) u_{x_j}^i \right) = f_i(t, x, u) \quad (2.1)$$

($t > 0, x \in \Omega$)

$$B_i[u_i] = b_1^i(x) \frac{\partial u_i}{\partial \tau} + b^i(x) u_i = 0 \quad (t > 0, x \in \partial\Omega) \quad (2.2)$$

$$u_i(0, x) = u_{0i}(x), \quad i = 1, 2, \dots, n \quad (2.3)$$

where for each i , L_i is a strongly elliptic operator in a bounded domain Ω in R^N , $\partial\Omega$ is the boundary of Ω , τ is the outward unit normal vector on $\partial\Omega$, and $b_1^i(x)$ and $b^i(x)$ are nonnegative continuous functions. Here $f \in C[R^+ \times \Omega \times R^n, R^n]$. We shall assume that the functions in question have outernormal-derivatives on $(0, \infty) \times \partial\Omega$.

To define quasi-solutions of (2.1), (2.2), (2.3), we fix for each $i \in \{1, 2, \dots, n\}$, two nonnegative integers p_i, q_i such that $p_i + q_i = n - 1$ and split $u \in R^n$ into $u = (u_i, [u]_{p_i}, \{u\}_{q_i})$. Then (2.1) can be written as

$$\frac{\partial u_i}{\partial t} - L_i u_i = f_i(t, x, u_i, [u]_{p_i}, \{u\}_{q_i}).$$

Also for any $y, z \in R^n$, we let $[y, z]_i$ denote an element of R^n with the description $[y, z]_i = (y_i, [y]_{p_i}, \{z\}_{q_i})$. Also the inequalities between vectors are to be understood componentwise.

If $u \in C[R^+ \times \bar{\Omega}, R^n]$ has partial derivatives u_t, u_x, u_{xx} which are continuous in $R^+ \times \Omega$, we shall say that u belongs to class $\mathcal{C}$ or $u \in \mathcal{C}$, where $\bar{\Omega}$ is the closure of Ω .

Definition 2.1. Let $\alpha \in C[R^+ \times \bar{\Omega}, R^n]$. Then

(1) $u \in \mathcal{C}$ is said to be a quasi-solution of (2.1), (2.2), (2.3) relative to α , if

$$\frac{\partial u_i}{\partial t} - L_i u_i = f_i(t, x, u_i, [u]_{p_i}, \{a\}_{q_i}) \quad (2.4)$$

$$B_i[u_i] = 0, \quad u_i(0, x) = u_{0i}(x);$$

(ii) $u \in \mathbb{C}$ is called an upper quasi-solution relative to a if

$$\frac{\partial u_i}{\partial t} - L_i u_i \geq f_i(t, x, u_i, [u]_{p_i}, \{a\}_{q_i}) \quad (2.5)$$

$$B_i[u_i] \geq 0, \quad u_i(0, x) \geq u_{0i}(x)$$

and a lower quasi-solution if both inequalities are reversed;

(iii) the functions $v, w \in \mathbb{C}$ with $v \leq w$ are said to be coupled lower and upper quasi-solutions respectively if

$$\left. \begin{aligned} \frac{\partial v_i}{\partial t} - L_i v_i &\leq f_i(t, x, v_i, [v]_{p_i}, \{w\}_{q_i}) \\ B_i[v_i] &\leq 0, \quad v_i(0, x) \leq u_{0i}(x) \\ \text{and} \\ \frac{\partial w_i}{\partial t} - L_i w_i &\geq f_i(t, x, w_i, [w]_{p_i}, \{v\}_{q_i}) \\ B_i[w_i] &\geq 0, \quad w_i(0, x) \geq u_{0i}(x); \end{aligned} \right\} \quad (2.6)$$

(iv) the functions $u, w \in \mathbb{C}$ are said to be coupled quasi-solutions of (2.1)-(2.2)-(2.3) if equality holds in the definition (iii).

Remark 2.1. In the special case $q_i = 0$ for all i , quasi-solutions are just solutions and in case $p_i = 0$ for all i , quasi-solutions become the

solutions of the uncoupled systems of equations which are easy to deal. It is clear from the above definition that, we can define maximal and minimal quasi-solutions relative to α as well as coupled maximal and minimal quasi-solutions.

Definition 2.2. The function f is said to possess a mixed quasimonotone property (mqmp, for short) if for each i , $f_i(t, x, u_i, [u]_{p_i}, \{u\}_{q_i})$ is monotone nondecreasing in $[u]_{p_i}$, and monotone nonincreasing in $\{u\}_{q_i}$. Further if f is monotone nondecreasing in u_i then f is said to possess mixed monotone property (mmp for short).

3. MONOTONE ITERATIVE TECHNIQUE

Throughout the paper we assume that for each $i = 1, 2, \dots, n$ the coefficients of L_i and the first-order derivatives a_{jk}^i are Hölder continuous in $R^+ \times \bar{\Omega}$ (of exponent α , $0 < \alpha < 1$), the matrix (a_{jk}^i) is symmetric positive definite in $\bar{\Omega}$, $f_i(t, x, u)$ are Hölder continuous in (t, x, u) in every bounded subset of $R^+ \times \Omega \times R^n$, the boundary $\partial\Omega$ is of class $C^{2+\alpha}$, $b_1^i, b^i \in H^{1+\alpha}(\partial\Omega)$ and finally $u_0^i \in H^{2+\alpha}(\bar{\Omega})$ and satisfies the boundary condition (2.3) at $t = 0$. $H^{2+\alpha}(\bar{\Omega})$, $H^{1+\alpha}(\partial\Omega)$ are function spaces in the sense of [6]. The above smoothness assumptions are required only for insuring the existence of a solution for the corresponding linear problem.

Let $D_T = (0, T] \times \Omega$, $S = (0, T] \times \partial\Omega$, where $T > 0$ is finite but can be arbitrarily large.

In order to prove the existence of coupled maximal and minimal solutions of (2.1), (2.2), (2.3) we make the following assumptions on $f(t, x, u)$:

(A₁) There exist coupled lower and upper solutions v, w such that $v \leq w$ on $\overline{D_T}$ and for some positive constants M_i ($i = 1, \dots, n$)

$$f_i(t, x, \eta_{1i}, [\eta]_{p_i}, \{\eta\}_{q_i}) - f_i(t, x, \eta_{2i}, [\eta]_{p_i}, \{\eta\}_{q_i}) \geq -M_i(\eta_{1i} - \eta_{2i}) \quad (3.1)$$

for every $(t, x) \in D_T$, $v_i(t, x) \leq \eta_{2i} \leq \eta_{1i} \leq w_i(t, x)$ and $v(t, x) \leq \eta \leq w(t, x)$.

(A₂) For any $\eta_1, \eta_2 \in C[\overline{D_T}, R^n]$ such that $v \leq \eta_1, \eta_2 \leq w$ we define

$$G_i(t, x, u) = f_i(t, x, \eta_{1i}, [\eta_1]_{p_i}, \{\eta_2\}_{q_i}) - M_i(u_i - \eta_{1i}) \quad (3.2)$$

where M_i is from assumption (A₁).

Consider now the problem

$$\frac{\partial u_i}{\partial t} - L_i u_i = G_i(t, x, u), \quad (3.3)$$

$$B_i[u_i] = 0, \quad u_i(0, x) = u_{0i}(x). \quad (3.4)$$

For a known η_1, η_2 the assumption on $f_i(t, x, u)$ ensures [4] the existence of a unique solution of (3.3)-(3.4).

For each $\eta_1, \eta_2 \in C[\overline{D_T}, R^n]$ such that $v \leq \eta_1, \eta_2 \leq w$ on D_T , define the mapping A by

$$A[\eta_1, \eta_2] = u \quad (3.5)$$

where $u \in \mathcal{C}$ is the unique solution of the I.B.V.P. of (3.3)-(3.4).

Lemma 3.1. Under the assumption (A₁), (A₂) the mapping A defined by (3.5) satisfies the following

$$(1) \quad v \leq A[v, w], \quad w \geq A[w, v];$$

(ii) A possesses mmp on the segment

$$\langle v, w \rangle = \{u \in R^n, v(t, x) \leq u \leq w(t, x), (t, x) \in \bar{D}_T\}.$$

Proof. Suppose $A[v, w] = y$ where $y \in C[D_T, R^n]$ and belongs to class $\mathcal{C}$ (on D_T) is the unique solution of the I.B.V.P. (3.3)-(3.4) corresponding to the choice $[\eta_1, \eta_2] = [v, w]$. Setting $\phi_i = y_i - v_i$, we get

$$\begin{aligned} \frac{\partial \phi_i}{\partial t} - L_i \phi_i &\geq G_i(t, x, y) - f_i(t, x, v_i, [v]_{p_i}, [w]_{q_i}) \\ &= f_i(t, x, v_i, [v]_{p_i}, [w]_{q_i}) - M_i \phi_i - f_i(t, x, v_i, [v]_{p_i}, [w]_{q_i}). \end{aligned}$$

This implies that

$$\frac{\partial \phi_i}{\partial t} - (L_i - M_i) \phi_i \geq 0$$

and also $B_i[\phi_i] \geq 0$, $\phi_i(0, x) \geq 0$ by (2.6), (3.4). Hence by the maximum principle $\phi_i \geq 0$ for each $i = 1, 2, \dots, n$. This proves $v \leq y = A[v, w]$. In a similar way we can prove $w \geq A[w, v]$.

To prove (ii) let $\eta_1, \eta_2 \in C[D_T, R^n]$ such that $\eta_1, \eta_2 \in \langle v, w \rangle$ and $\eta_1 \leq \eta_2$. Let $A[\eta_1, \eta_2] = y$ and $A[\eta_2, \eta_1] = z$. Then writing $\phi_i = y_i - z_i$, we see that

$$\begin{aligned} \frac{\partial \phi_i}{\partial t} - L_i \phi_i &= G_i(t, x, y) - G_i(t, x, z) \\ &= f_i(t, x, \eta_{1i}, [\eta_1]_{p_i}, [\eta_2]_{q_i}) - M_i(y_i - z_i) \\ &\quad - f_i(t, x, \eta_{2i}, [\eta_2]_{p_i}, [\eta_1]_{q_i}) + M_i(z_i - \eta_{2i}) \end{aligned}$$

This implies

$$\begin{aligned} \frac{\partial \phi_i}{\partial t} - (L_i - M_i) \phi_i &= f_i(t, x, \eta_{1i}, [\eta_1]_{p_i}, \{\eta_2\}_{q_i}) \\ &\quad - f_i(t, x, \eta_{2i}, [\eta_2]_{p_i}, \{\eta_1\}_{q_i}) + M(\eta_{1i} - \eta_{2i}) \end{aligned}$$

Using assumption (A_2) , (3.3) and (3.4) it follows that

$$\frac{\partial \phi_i}{\partial t} - (L_i - M_i) \phi_i \leq 0$$

and $B_i \in \mathbb{R}$, $\phi_i(0, x) = 0$. Hence by maximum principle $\phi_i \leq 0$, which proves (ii).

In view of Lemma 3.1, we can define the sequences

$$v_n = A[v_{n-1}, w_{n-1}], \quad w_n = A[w_{n-1}, v_{n-1}] \quad (3.6)$$

with $v_0 = v$, $w_0 = w$. It is easy to observe the sequences $\{v_n(t, x)\}$ and $\{w_n(t, x)\}$ are monotone nondecreasing and nonincreasing respectively and $v_k(t, x) \leq w_k(t, x)$ for every $k = 1, 2, \dots$. Hence the pointwise limits exist. Using the dominance convergence theorem, they converge uniformly and monotonically. Let $\lim_{n \rightarrow \infty} v_n(t, x) = \alpha(t, x)$, $\lim_{n \rightarrow \infty} w_n(t, x) = \beta(t, x)$. Then it is easily proved as in [1] that $\alpha(t, x)$, $\beta(t, x)$ satisfy

$$\frac{\partial \alpha_i}{\partial t} - L_i \alpha_i = f_i(t, x, \alpha_i, [\alpha]_{p_i}, \{\beta\}_{q_i}) \quad (3.7)$$

$$\frac{\partial \beta_i}{\partial t} - L_i \beta_i = f_i(t, x, \beta_i, [\beta]_{p_i}, \{\alpha\}_{q_i}) \quad (3.8)$$

and

$$B_i[\alpha_i] = 0 = B_i[\beta_i], \quad \alpha_i(0, x) = u_{0i}(x) = \beta_i(0, x).$$

We shall now prove that α, β are actually coupled minimal and maximal solutions of the problem (2.1), (2.2), (2.3).

Theorem 3.1. Assume that $(A_1), (A_2)$ holds. Then the sequences $\{v_n(t, x)\}, \{w_n(t, x)\}$ defined by (3.6) converge uniformly and monotonically to coupled minimal and maximal quasi-solutions α, β of (2.1), (2.2), (2.3), that is, if y, z is any coupled quasi-solution of (2.1), (2.2), (2.3) such that $v_0 \leq x, y \leq w_0$ on D_T , then

$$v_0 \leq v_1 \leq \dots \leq v_n \leq \alpha \leq y, z \leq \beta \leq w_n \leq \dots \leq w_1 \leq w_0 \text{ on } \overline{D_T}. \quad (3.9)$$

Furthermore if u is any solution of (2.1), (2.2), (2.3) such that $u \in \langle v, w \rangle$, then

$$v_0 \leq v_1 \leq \dots \leq v_n \leq \alpha \leq u \leq \beta \leq w_n \leq \dots \leq w_1 \leq w_0 \text{ on } \overline{D_T}. \quad (3.10)$$

Proof. Let y, z be any coupled quasi-solution of (2.1), (2.2), (2.3) such that $v = v_0 \leq y, z \leq w_0 = w$ on D_T . Assume that for some $k, v_{k-1} \leq y, z \leq w_{k-1}$; then setting $\phi_i = v_{k,i} - y_i$, we obtain because of assumptions $(A_1)-(A_2)$,

$$\begin{aligned} \frac{\partial \phi_i}{\partial t} - L_i \phi_i &= f_i(t, x, v_{k-1,i}, [v_{k-1}]_{p_i}, \{w_{k-1}\}_{q_i}) - M_i(v_{k,i} - v_{k-1,i}) \\ &\quad - f_i(t, x, y_i, [y]_{p_i}, \{z\}_{q_i}) \\ &\leq f_i(t, x, v_{k-1,i}, [y]_{p_i}, \{z\}_{q_i}) \\ &\quad - f_k(t, x, y_i, [y]_{p_i}, \{z\}_{q_i}) - M_i(v_{k,i} - v_{k-1,i}) \\ &\leq M_i(y_i - v_{k-1,i}) - M_i(v_{k,i} - v_{k-1,i}) \leq M_i(y_i - v_{k,i}) = -M_i \phi_i. \end{aligned}$$

Thus we have

$$\frac{\partial \phi_i}{\partial t} - (L_i - M_i) \phi_i \leq 0 \quad \text{and} \quad B_i[\phi_i] = 0, \quad \phi_i(0, x) = 0.$$

Hence by maximum principle $v_{k,i} \leq y_i$ for $i = 1, 2, \dots, n$. Using similar arguments we can conclude $v_k(t, x) \leq y(t, x)$, $z(t, x) \leq w_k(t, x)$. Since $v_0 \leq y$, $z \leq w_0$, it follows by induction that $v_n(t, x) \leq y(t, x)$, $z(t, x) \leq w_n(t, x)$ on $\bar{D}_T$ for all n . Hence we get $\alpha(t, x) \leq y(t, x)$, $z(t, x) \leq \beta(t, x)$ on D_T thus proving α, β are coupled minimal and maximal solutions of (2.1), (2.2), (2.3).

Since any solution $u(t, x)$ of (2.1), (2.2), (2.3) can be considered as (u, u) coupled quasi-solutions of (2.1), (2.2), (2.3), it is evident that $\alpha(t, x) \leq u(t, x) \leq \beta(t, x)$. Hence the proof is complete.

Remark 3.1. If f satisfies uniqueness condition of the form

$$|f_i(t, x, u) - f_i(t, x, \bar{u})| \leq K_i \sum_{i=1}^n |u_i - \bar{u}_i| \quad (3.11)$$

whenever $v(t, x) \leq u$, $\bar{u} \leq w(t, x)$, then it is easy to show [1,10] that $u(t, x) = \alpha(t, x) = \beta(t, x)$ which implies that there is a unique solution $u(t, x)$ of (2.1), (2.2), (2.3) such that $v(t, x) \leq u(t, x) \leq w(t, x)$ on D_T . Hence our results include those given in [1,10].

Remark 3.2. Let $a \in C[D_T, R^n]$ with $v \leq a \leq w$. Then the coupled lower and upper quasi-solution v, w yield

$$\begin{aligned} \frac{\partial v_i}{\partial t} - L_i v_i &\leq f_i(t, x, v_i, [v]_{p_i}, \{w\}_{q_i}) \leq f_i(t, x, v_i, [v]_{p_i}, \{a\}_{q_i}) \\ \frac{\partial w_i}{\partial t} - L_i w_i &\geq f_i(t, x, w_i, [w]_{p_i}, \{v\}_{q_i}) \geq f_i(t, x, w_i, [w]_{p_i}, \{a\}_{q_i}) \end{aligned}$$

This implies that v, w are lower and upper quasi-solutions of (2.1), (2.2), (2.3) with respect to α . By choosing $\eta_2 = \alpha$ in the definition of G , it is easy to prove that α, β are minimal and maximal quasi-solutions of (2.1), (2.2), (2.3) with respect to α , that is any quasi-solution $y(t, x)$ of (2.1), (2.2), (2.3) with respect to α satisfies

$$v = v_0 \leq v_1 \leq \dots \leq v_n \leq \alpha \leq y(y, x) \leq \beta \leq w_n \leq \dots \leq w_0 = w \text{ on } \bar{D}_T.$$

If $q_i = 0$, then f is quasimonotone nondecreasing and consequently (α, β) reduce to minimal and maximal solutions of (2.1), (2.2), (2.3). Thus our results include those given in [10].

4. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

In this section, we give a method of constructing explicit coupled upper and lower quasi-solutions by means of which we can deduce the asymptotic behaviour of coupled quasi-solutions of (2.1), (2.2), (2.3). Since from the point of view of physical applications, we need to consider nonnegative solutions only, we have to impose additional restrictions for that purpose; namely for each i ,

$$(A_3) \quad f_i(t, x, 0, [0]_{p_i}, \{\phi\}_{q_i}) \geq 0 \text{ for every } \phi \geq 0 \quad (4.1)$$

and

$$(A_4) \quad f_i(t, x, \psi_i, [\psi]_{p_i}, \{0\}_{q_i}) \leq 0 \text{ for every } \psi \geq 0 \quad (4.2)$$

Our construction of upper and lower quasi-solutions is based as usual on the eigenfunction ϕ_i corresponding to the least eigenvalue λ_i of the linear uncoupled eigenvalue problem

$$L_i \phi_i + \lambda_i \phi_i = 0, \quad x \in \Omega \quad (4.3)$$

$$B_i[\phi_i] = 0 \quad (4.4)$$

where B_i is as in (2.2). It is well known that the least eigenvalue λ_i of (4.3), (4.4) is positive and the corresponding ϕ_i is also positive on Ω [11,12]. In the case of $b_i > 0$ on $\partial\Omega$, the maximum principle implies that $\phi_i(x) > 0$ on $\bar{\Omega}$. We then have the following result.

Theorem 4.1. In addition to the assumptions of Theorem 3.1, let the hypothesis $(A_3), (A_4)$ hold and let $b_i > 0$ in (2.2). Then there exist coupled minimal and maximal solutions $\alpha(t, x)$ and $\beta(t, x)$ respectively of (2.1), (2.2), (2.3) such that

$$0 \leq \alpha(t, x) \leq \beta(t, x) \leq \rho e^{-\lambda t} \phi(x) \quad (4.5)$$

whenever $0 \leq u_0(x) \leq \rho \phi(x)$, $x \in \bar{\Omega}$ for some constant vector $\rho \geq 0$. In particular every coupled solutions of (2.1), (2.2), (2.3) such that $u_0(x) \geq 0$ tends to zero as $t \rightarrow \infty$.

Proof. Let us choose $v(t, x) \equiv 0$ and $w(t, x) = P(t)\phi(x)$ so that $0 \leq u_0(x) \leq \rho \phi(x)$ holds for some $\rho \geq 0$. Since $w(t, x) \geq 0$ the assumption (A_3) implies that $v(t, x)$ is a coupled lower solution. Hence it is enough to find $P(t)$ such that

$$P_i' \phi_i - L_i(P_i \phi_i) \geq f_i(t, x, P_i \phi_i, [P\phi]_{P_i}, \{0\}_{Q_i})$$

holds. The assumption (A_4) now shows that it suffices to choose $P(t)$ such that $P_i' + \lambda_i P_i = 0$, that is $P_i = \rho e^{-\lambda_i t}$. Consequently $v \equiv 0$ and

$w = \rho\phi(x)e^{-\lambda t}$ are lower and upper coupled quasi-solutions. The conclusion of the theorem is now a consequence of Theorem 3.1.

As an example consider a coupled reaction diffusion system of the special form

$$\frac{\partial u_i}{\partial t} - L_i u_i = \lambda u_i^m f_i(t, x, u_j), \quad (t > 0, x \in \Omega) \quad (4.6)$$

$i = 1, 2, j = 1, 2$ and $i \neq j$ which is a generalized form of gas liquid model, see [10]. Here L_i and the boundary conditions are as before. Assume that $m \geq 1$ and $\lambda < 0$. Suppose that

(A₅) for each $i = 1, 2, f_i(t, x, 0) \geq 0$ and for each $\gamma_i(t, x) > 0$, there exists a constant $K_i > 0$ such that for every $(t, x) \in \bar{D}_T$

$$0 \leq f_i(t, x, \eta_2) - f_i(t, x, \eta_1) \leq K_i(\eta_2 - \eta_1) \quad \text{when } 0 \leq \eta_1 \leq \eta_2 \leq \gamma_i(t, x). \quad (4.7)$$

Also, for each $i = 1, 2, f_i$ is monotone nondecreasing in u_j ($j \neq i$). By Theorems 3.1 and 4.1 and the Remark 3.1, we can conclude that

$$0 \leq u(t, x) \leq \rho e^{-\lambda t} \phi(x) \quad (t > 0, x \in \bar{\Omega})$$

whenever $0 \leq u(0, x) \leq \rho\phi(x)$, where $u(t, x)$ is the unique solution of (4.6), (2.2), (2.3). Here (2.2), (2.3) are taken into consideration only for $i = 1, 2$. Furthermore, the zero solution is exponentially asymptotically stable. Since the notion of quasi-solutions is not known and since the right hand side of (4.6) is not quasimonotone nondecreasing in u , the above result was obtained in a complicated way in [10].

We next consider the coupled system (2.1), (2.3) under Neumann boundary condition

$$\frac{\partial u_i}{\partial \tau} = 0, \quad t > 0, \quad x \in \partial\Omega$$

for each $i = 1, 2, \dots, n$.

In this case the eigenvalues λ_i ($i = 1, 2, \dots, n$) are zero and $\phi_i = 1$ for $i = 1, 2, \dots, n$. Hence Theorem 4.1 implies $0 \leq \alpha(t, x) \leq \beta(t, x) \leq \rho$ whenever $0 \leq u_0(x) \leq \rho$. This implies that the steady state solution is stable but not asymptotically stable. However, we can utilize in the situation, the properties of f and obtain v and w as follows. Let us assume that (A_3) holds. (A_4) is not needed now.

Define $\bar{f}(t, R) = \max\{f(t, x, R_i, [R]_{p_i}, \{0\}_{q_i}), x \in \bar{\Omega}\}$. Let $r(t)$ be the maximal quasi-solution of $r'_i = \bar{f}(t, r_i, [r]_{p_i}, \{0\}_{q_i})$, $r(0) \geq u_0(x)$ on $\bar{\Omega}$. Then the assumption (A_3) shows that $v(t, x) \equiv 0$, $w(t, x) \equiv r(t)$ are lower and upper quasi-solutions. Consequently we obtain

$$0 \leq \alpha(t, x) \leq \beta(t, x) \leq r(t).$$

If $\bar{f}$ is such that $\lim_{t \rightarrow \infty} r(t) = 0$, then one can derive asymptotic behaviour of solutions of (2.1), (2.2), (2.3).

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