VARIATION OF CONSTANTS, VECTOR LYAPUNOV
FUNCTIONS AND COMPARISON THEOREM

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Technical Report No. 131

April, 1980

* Research supported by Organized Research Grant at the University of Texas at Arlington, Arlington, Texas 76019

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1. INTRODUCTION

As is well known [4] the method of variation of constants is a very useful tool in studying the properties of solutions of perturbed differential equations. Two different versions of nonlinear variation of constant formulas are available for use [4,6]. In investigating the effect of perturbations of differential equations, the two different techniques employed are the method of variation of constants and the method of Lyapunov functions [4]. In [5], a new comparison theorem which connects the solutions of perturbed and unperturbed differential systems in a manner useful in the theory of perturbations is developed. It is also well-known that the method of vector Lyapunov functions provides an effective mechanism to deal with the study of large scale systems compared to employing a single Lyapunov function [2,3,7,8]. Furthermore, in the study of comparison theorems and extremal solutions for systems we need to impose a condition called quasi-monotone condition [4]. In systems which represent physical situations we find that such a property is not satisfied. To deal with such situations, a notion of quasi-solutions was introduced in [1].

*Research supported by Organized Research Grant at the University of Texas at Arlington, Arlington, Texas 76019.

In this paper we combine the above ideas to obtain a new comparison result and discuss its relation to known results. A simple application to stability theory is also given to indicate the usefulness of the comparison result. Our result is expected to play a prominent role in the qualitative study of differential equations.

2. COMPARISON THEOREM

Consider the three differential systems

$$x' = X(t, x), \quad x(t_0) = x_0, \quad (2.1)$$

$$y' = Y(t, y), \quad y(t_0) = x_0, \quad (2.2)$$

$$z' = Z(t, z), \quad z(t_0) = x_0, \quad t_0 \geq 0, \quad (2.3)$$

where $X, Y, Z \in C[R_+ \times S(\rho), R^n]$. Here R_+ denotes the nonnegative real line, R^n the Euclidian n -space, $C[R_+ \times S(\rho), R^n]$ the class of continuous functions from $R_+ \times S(\rho)$ to R^n and $S(\rho) = \{x \in R^n : \|x\| < \rho\}$ where $\|\cdot\|$ is any convenient norm in R^n . Relative to system (2.1) we assume that

- (H) the solutions $x(t, t_0, x_0)$ of (2.1) exist for all $t \geq t_0$, unique, continuous with respect to the initial data and $x(t, t_0, x_0)$ is locally Lipschitzian in x_0 .

For any $V \in C[R_+ \times S(\rho), R^N]$ and for any $t \in (t_0, \infty)$, we define

$$D_- V(s, x(t, t_0, z)) \equiv \liminf_{h \rightarrow 0} \frac{1}{h} [V(s+h, x(t, t_0, z+hZ(t, z))) - V(s, x(t, t_0, z))] \quad (2.4)$$

and

$$D_-V(s, x(t, s, y)) \equiv \liminf_{h \rightarrow 0} \frac{1}{h} [V(s+h, x(t, s+h, y+hY(s, y))) - V(s, x(t, s, y))] \quad (2.5)$$

where $s \in [t_0, t]$ and $z, y \in S(\rho)$.

We also consider the comparison system

$$u' = g(t, u), \quad u(t_0) = u_0, \quad (2.6)$$

where $g \in C[R_+ \times R^N, R^N]$. We need the following notions before we proceed further. For each $i \in \{1, 2, 3, \dots, N\}$, we fix two nonnegative integers p_i, q_i such that $p_i + q_i = N-1$ and we split $u \in R^N$ into $u = (u_i, [u]p_i, [u]q_i)$. Then (2.6) takes the form

$$u_i' = g_i(t, u_i, [u]p_i, [u]q_i), \quad u(t_0) = u_0; \quad 1 \leq i \leq N. \quad (2.7)$$

Since we also need inequalities between vectors, let us note that these inequalities will always be understood to hold componentwise.

Let $a \in C[R_+, R^N]$. Then $u \in C^1[R_+, R^N]$ is said to be a quasi-solution of (2.6) with respect to a , if

$$u_i' = g_i(t, u_i, [u]p_i, [a]q_i), \quad u(t_0) = u_0, \quad 1 \leq i \leq N. \quad (2.8)$$

Clearly we can define maximal and minimal quasi-solutions of (2.8). In the special case $q_i = 0$ for all i , quasi-solutions are just solutions.

The function g is said to possess a mixed quasi-monotone property (mqmp for short) if for each $i, g_i(t, u_i, [u]p_i, [u]q_i)$ is monotone non-decreasing in $[u]p_i$ and monotone nonincreasing in $[u]q_i$.

We are now ready to prove the following comparison result.

Theorem 2.1. Assume that

- (i) the hypothesis (H) holds;
- (ii) $V \in C[R_+ \times S(\rho), R^N]$, $V(t, x)$ is locally Lipschitzian in x , $V(t, x) \geq a(t)$ for $(t, x) \in R_+ \times S(\rho)$ where $a \in C[R_+, R^N]$;
- (iii) $g \in C[R_+ \times R^N, R^N]$, $g(t, u)$ possesses mqmp in u and the maximal quasi-solution $r(t, t_0, u_0)$ of

$$u_1' = g_1(t, u_1, [u]_{p_1}, [a]_{q_1}), \quad u(t_0) = u_0,$$

exists on $[t_0, \infty)$;

- (iv) either

(a)

$$D_- V(s, x(t, t_0, z)) \leq g(s, V(s, x(t, t_0, z))), \quad s \in [t_0, t],$$

$$z \in S(\rho)$$

(2.9)

and

$$y(t, t_0, x_0) = x(t, t_0, z(t, t_0, x_0))$$

(2.10)

holds where $y(t, t_0, x_0)$, $z(t, t_0, x_0)$ are solutions of (2.2), (2.3) respectively existing on $[t_0, \infty)$ or

(b)

$$D_- V(s, x(t, s, y)) \leq g(s, v(s, x(t, s, y))), \quad s \in [t_0, t],$$

$$y \in S(\rho)$$

(2.11)

Then

$$V(t_0, x(t, t_0, x_0)) \leq u_0$$

implies

$$V(t, y(t, t_0, x_0)) \leq r(t, t_0, u_0), \quad t \geq t_0. \quad (2.12)$$

Proof: Let $x(t, t_0, x_0)$ be any solution of (2.1) satisfying (H) and suppose that (iv) (a) holds. We set $m(s) = V(s, x(t, t_0, z(s)))$ for $s \in [t_0, t]$ where $z(s) = z(s, t_0, x_0)$ is any solution of (2.3) existing on $[t_0, \infty)$ so that $m(t_0) = V(t_0, x(t, t_0, x_0))$. Then using the assumptions (H), (ii) and (iv) (a), it is easy to obtain the differential inequality

$$D_- m(s) \leq g(s, m(s)), \quad t_0 < s \leq t \quad (2.13)$$

which yields by Comparison Theorem [Th. 2.1 in [1]] the estimate

$$m(s) \leq r(s, t_0, u_0), \quad t_0 < s \leq t \quad (2.14)$$

provided $m(t_0) \leq u_0$. Since $m(t) = V(t, x(t, t_0, z(t))) = V(t, y(t, t_0, x_0))$, the desired estimate (2.12) follows from (2.14) by setting $s = t$.

If, on the other hand, (iv) (b) holds, we set

$$m(s) = V(s, x(t, s, y(s)))$$

where $y(s) = y(s, t_0, x_0)$ is any solution of (2.2) existing on $[t_0, \infty)$ and obtain the differential inequality (2.13) using the hypothesis (H), (2.11) and (ii). The relation (2.14) follows as before. Noting that $m(t) = V(t, x(t, t, y(t))) = V(t, y(t, t_0, x_0))$, the claim (2.12) is again true

provided $V(t_0, x(t, t_0, x_0)) \leq u_0$. The proof is complete.

Taking $u_0 = V(t_0, x(t, t_0, x_0))$, the inequality (2.12) becomes

$$V(t, y(t, t_0, x_0)) \leq r(t, t_0, V(t_0, x(t, t_0, x_0))), \quad t \in [t_0, \infty) \quad (2.15)$$

which shows the connection between solutions of system (2.1) and (2.2) in terms of maximal quasi-solution of (2.8).

Remarks. Several important comments are now in order concerning Theorem 2.1.

(I) Suppose that $\phi(t, t_0, x_0) = \frac{\partial x}{\partial x_0}(t, t_0, x_0)$ exists and is continuous on $[t_0, \infty)$. Assume that $\phi^{-1}(t, t_0, x_0)$ exists and is continuous on $[t_0, \infty)$. Then the assumption (2.10) yields

$$y'(t, t_0, x_0) = x'(t, t_0, z(t)) + \phi(t, t_0, z(t))z'(t),$$

which in turn leads to

$$z'(t) = \phi^{-1}(t, t_0, z(t)) [Y(t, y(t)) - X(t, x(t, t_0, z(t)))] \equiv Z(t, z(t)), \quad (2.16)$$

showing the relation between the systems (2.1), (2.2) and (2.3).

(II) Suppose that $X(t, x) \equiv 0$ so that $x(t, t_0, x_0) \equiv x_0$. The hypothesis

(H) is trivially satisfied. The assumption (2.10) shows that $y(t) \equiv z(t)$.

Consequently the relation (2.16) implies that $Y(t, y) \equiv Z(t, z)$. As a result, the definitions (2.4) and (2.5) reduce to

$$D_- V(s, y) \equiv \liminf_{h \rightarrow 0} \frac{1}{h} [V(s+h, y+hY(s, y)) - V(s, y)]$$

which is the usual definition of the generalized derivative of the vector

Lyapunov function relative to system (2.2) or (2.3). If we assume that $q_i = 0$ for each i , then Theorem 2.1 reduces, in this special case, to a well-known Comparison Theorem [Th. 4.1.1 in 4].

(III) Suppose that $X(t, x) = Ax$ where A is $n \times n$ matrix. The solutions $x(t, t_0, x_0)$ of (2.1) are then given by

$$x(t, t_0, x_0) = U(t, t_0)x_0$$

where $U(t, t_0)$ is the fundamental matrix solution of $x' = Ax$ and $U(t_0, t_0) = I = \text{identity matrix}$. Clearly (H) holds. If $g(t, u) \equiv 0$, then (2.15) gives

$$V(t, y(t, t_0, x_0)) \leq V(t_0, U(t, t_0)x_0), \quad t \geq t_0. \quad (2.17)$$

If on the other hand $g(t, u) = Bu$ where $b_{ii} \leq 0$, $b_{ij} \leq 0$ for $j < i$ and $b_{ij} \geq 0$ for $j > i$. Then we get

$$V(t, y(t, t_0, x_0)) \leq \exp[\tilde{B}(t - t_0)]V(t_0, U(t, t_0)x_0), \quad t \geq t_0, \quad (2.18)$$

where $\tilde{B}$ is such that $\tilde{b}_{ii} = b_{ii}$, $\tilde{b}_{ij} = 0$ for $j < i$ and $\tilde{b}_{ij} = b_{ij}$ for $j > i$. Here we have assumed $V(t, x) \geq 0$ so that we can take $a(t) \equiv 0$.

If further $V(t, x) = |x|$, then it follows that

$$|y(t, t_0, x_0)| \leq \exp[\tilde{B}(t - t_0)]|U(t, t_0)x_0|, \quad t \geq t_0. \quad (2.19)$$

It is easy to see that the relation (2.19) helps in improving the behavior of solutions of (2.2) relative to the behavior of solutions of (2.1).

(IV) Suppose that $X_x(t, x)$ exists and is continuous for $(t, x) \in R_+ \times S(\rho)$. Then it is known [4] that the solutions $x(t, t_0, x_0)$ of (2.1) are unique,

differentiable with respect to initial data and satisfy

$$\begin{aligned}\frac{\partial x(t, t_0, x_0)}{\partial x_0} &= \Phi(t, t_0, x_0) \quad , \\ \frac{\partial x(t, t_0, x_0)}{\partial t_0} &= -\Phi(t, t_0, x_0)X(t_0, x_0)\end{aligned}\tag{2.20}$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix solution of variational equation

$$z' = f_x(t, x(t, t_0, x_0))z, \quad \Phi(t_0, t_0, x_0) = I.$$

Noting that

$$x(t, t_0, z(s)) = x(t, s, y(s)), \quad s \in [t_0, t]\tag{2.21}$$

we have because of (2.20)

$$\Phi(t, t_0, z(s))z'(s) = \Phi(t, s, y(s))[Y(s, y(s)) - X(s, y(s))].$$

Consequently, we get by (2.16)

$$\begin{aligned}\Phi(t, t_0, z(s))\Phi^{-1}(s, t_0, z(s))[Y(s, y(s)) - X(s, y(s))] &= \\ \Phi(t, s, y(s))[Y(s, y(s)) - X(s, y(s))].\end{aligned}\tag{2.22}$$

If we now assume that $v(s, x)$ is differentiable then (2.4), (2.5) reduce respectively to

$$\begin{aligned}D_V(s, x(t, t_0, z)) &= V_s(s, x(t, t_0, z)) + V_x(s, x(t, t_0, z))\Phi(t, t_0, z) \cdot \\ &\quad \Phi^{-1}(s, t_0, z)[Y(s, y) - X(s, y)] \quad ,\end{aligned}\tag{2.23}$$

and

$$D_V(s, x(t, s, y)) = V_s(s, x(t, s, y)) + V_x(s, x(t, s, y))\phi(t, s, y) \cdot [Y(s, y) - X(s, y)] . \quad (2.24)$$

Thus in this case the two definitions (2.4), (2.5) are identical in view of (2.22).

(V) Taking $V(t, x) = x$ and $Y(t, y) = X(t, y) + R(t, y)$ so that $R(t, y)$ may be considered as perturbation, we see that the expressions (2.23), (2.24) reduce to

$$\frac{dx(t, t_0, z(s))}{ds} = \phi(t, t_0, z(s))\phi^{-1}(s, t_0, z(s))R(s, y(s)) ,$$

$$\frac{dx(t, s, y(s))}{ds} = \phi(t, s, y(s))R(s, y(s)) ,$$

integrating these relations from t_0 to t , we arrive at

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^t \phi(t, t_0, z(s))\phi^{-1}(s, t_0, z(s))R(s, y(s))ds , \quad (2.25)$$

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^t \phi(t, s, y(s))R(s, y(s))ds , \quad (2.26)$$

which are the two different versions of the nonlinear variation of constants formula. See [4, 6].

(IV) When the solutions of (2.1) are known explicitly, a possible Lyapunov function is

$$w(s,x) = V(s,x(t,t_0,x_0)) \quad \text{or} \quad w(s,x) = V(s,x(t,s,x_0))$$

according as we utilize the definition (2.4) or (2.5). In fact, in certain cases, one could take a convenient $V(t,x)$ such as $\|x\|^2$.

(VII) The scalar version of Theorem 2.1 corresponding to the conditions (2.11) appears in [5].

3. APPLICATION

As an application of the comparison Theorem 2.1, we shall consider in this section a result on stability and asymptotic behavior of solutions of (2.2).

For this purpose, we need a new notation of stability called connective stability. Let $x(t,t_0,x_0)$, $u(t,t_0,u_0)$ be solutions of (2.1), (2.6) respectively. Then we define

$$w(t,t_0,x_0) = u[t,t_0,V(t_0,x(t,t_0,x_0))] \quad (3.1)$$

and note that

$$w(t_0,t_0,x_0) = V(t_0,x_0) \quad .$$

Definition. The differential equations (2.1) and (2.6) are said to be

(1) connectively equi-stable if, for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0,\epsilon)$ that is continuous in t_0 for each ϵ such that the inequality

$$\sum_{i=1}^N V_i(t_0,x_0) \leq \delta$$

implies

$$\sum_{i=1}^N w_i(t, t_0, x_0) < \varepsilon, \quad t > t_0 ;$$

(ii) connectively quasi-equi-asymptotically stable if, for each $\varepsilon > 0$, $t_0 \in R_+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and

$T = T(t_0, \varepsilon)$ such that, $\sum_{i=1}^N V_i(t_0, x_0) \leq \delta_0$ implies

$$\sum_{i=1}^N w_i(t, t_0, x_0) < \varepsilon, \quad t > t_0 + T ;$$

(iii) connectively equi-asymptotically stable if, (i) and (ii) hold at the same time.

We shall be content in giving the following simple result as an application of Theorem 2.1.

Theorem 3.1. Suppose that the conditions (i), (ii), (iv) of Theorem 2.1 are satisfied. Assume further that $V(t, 0) \equiv 0$ and

$$b(\|x\|) < \sum_{i=1}^N V_i(t, x), \quad (t, x) \in R_+ \times S(\rho) \quad (3.2)$$

where $b \in C[[0, \rho), R_+]$, $b(u)$ is increasing in u and $b(0) = 0$. Then the connective equi-asymptotic stability of equations (2.1) and (2.6) implies that the system (2.2) is equi-asymptotically stable.

Proof: Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given. Since (2.1) and (2.6) are connectively equi-stable, given $b(\varepsilon) > 0$, $t_0 \in R_+$, there exists a positive function $\delta^* = \delta^*(t_0, \varepsilon)$ that is continuous in t_0 such that

$$\sum_{i=1}^N V_i(t_0, x_0) \leq \delta^*$$

implies

$$\sum_{i=1}^N w_i(t, t_0, x_0) < b(\varepsilon), \quad t \geq t_0.$$

Since $V(t, x)$ is continuous and $V(t, 0) \equiv 0$, it is possible to find a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_0 for each ε , satisfying the inequalities $\|x_0\| \leq \delta$, $\sum_{i=1}^N V_i(t_0, x_0) \leq \delta^*$ simultaneously.

We claim that if $\|x_0\| \leq \delta$, then $\|y(t, t_0, x_0)\| < \varepsilon$, $t \geq t_0$, where $y(t) = y(t, t_0, x_0)$ is any solution of (2.2). Suppose that this is not true. Then, there exists $y(t) = y(t, t_0, x_0)$ with $\|x_0\| \leq \delta$ and a $t_1 > t_0$ such that

$$\|y(t_1)\| = \varepsilon \quad \text{and} \quad \|y(t)\| \leq \varepsilon \quad \text{for} \quad t_0 \leq t \leq t_1.$$

By Theorem 2.1 and (3.2), we have for $t_0 \leq t \leq t_1$

$$\begin{aligned} b(\|y(t, t_0, x_0)\|) &\leq \sum_{i=1}^N V_i(t, y(t, t_0, x_0)) \leq \sum_{i=1}^N r_i(t, t_0, V(t_0, x(t, t_0, x_0))) \\ &= \sum_{i=1}^N w_i(t, t_0, x_0) \end{aligned}$$

and consequently

$$b(\varepsilon) \leq b(\|y(t_1)\|) \leq \sum_{i=1}^N V_i(t_1, y(t_1, t_0, x_0)) \leq \sum_{i=1}^N w_i(t_1, t_0, x_0) < b(\varepsilon).$$

This contradiction proves that the system (2.2) is equi-stable.

To prove equi-asymptotic stability we only need to prove that the system (2.2) is quasi-equi-asymptotically stable. To do this, let $\epsilon > 0$, $t_0 \in R_+$ be given. Then since (2.1) and (2.6) are connectively quasi-equi-asymptotically stable, given $b(\epsilon) > 0$, $t_0 \in R_+$, there exist $\delta_0^* = \delta_0^*(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that

$$\sum_{i=1}^N V_i(t_0, x_0) \leq \delta_0^*$$

implies

$$\sum_{i=1}^N w_i(t, t_0, x_0) < b(\epsilon), \quad t \geq t_0 + T.$$

As before there exists a $\delta_0 = \delta_0(t_0) > 0$ such that $\|x_0\| \leq \delta_0$,

$\sum_{i=1}^N V_i(t_0, x_0) \leq \delta_0^*$ hold simultaneously. We claim that with these T ,

δ_0 the system (2.2) is quasi-equi-asymptotic stable. If it is not true, then there is a sequence $\{t_k\}$, $t_k \geq t_0 + T$ and $t_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$\|y(t_k, t_0, x_0)\| \geq \epsilon \quad \text{where} \quad \|x_0\| \leq \delta_0.$$

Since we now have for $t_0 \leq t < \infty$,

$$\begin{aligned} b(\|y(t, t_0, x_0)\|) &\leq \sum_{i=1}^N V_i(t, y(t, t_0, x_0)) \leq \sum_{i=1}^N r_i(t, t_0, V(t_0, x(t, t_0, x_0))) \\ &= \sum_{i=1}^N w_i(t, t_0, x_0), \end{aligned}$$

it easily follows that

$$b(\epsilon) \leq b(\|y(t_k, t_0, x_0)\|) \leq \sum_{i=1}^N V_i(t_k, y(t_k, t_0, x_0)) < b(\epsilon) \quad .$$

This contradiction, proves the theorem.

Remark. It is important to realize that the connective stability notion is a joint property of the systems (2.1) and (2.6) and consequently does not necessarily imply that each system (2.1) and (2.6) admits the same kind of stability. It is this important fact that offers flexibility in applications. Also it may be instructive to find whether $w(t, t_0, x_0)$ is infact a solution of a differential system. To do this, assume that $V(t, x)$ is differentiable. Then we can obtain easily the differential system satisfied by $w = w(t, t_0, x_0)$, in view of the relation (3.1) as

$$w' = g(t, w) + \sigma(t, t_0, x_0), \quad w(t_0) = V(t_0, x_0) \quad (3.2)$$

where $\sigma(t, t_0, x_0) = V_x(t, x(t, t_0, x_0))X(t, x(t, t_0, x_0))$. We note that if $x_0 = 0$, $X(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, then $w \equiv 0$ is a solution.

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