PERIODIC SOLUTIONS OF NONLINEAR
BOUNDARY VALUE PROBLEMS*

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Technical Report No. 129
April, 1980

*Research partially supported by U.S. Army Grant DAAG29-80-C-0060.

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Introduction

One of the well known techniques in the theory of nonlinear boundary value problems (BVP) is the method of differential inequalities or the method of upper and lower solutions. The method of "Alternative Problems", a global variant of the Lyapunov-Schmidt method, has been used in the study of problems at resonance. The investigation of periodic BVP's forms an important subclass of problems at resonance. Our aim is to combine the two approaches to discuss nonlinear problems at resonance. We restrict ourselves in this paper to the discussion of periodic boundary value problems.

In Section 1, we shall indicate the method of upper and lower solutions. The results for systems are given in a general way so as to include known results and also offer new directions. Section 2 deals with the abstract existence results at resonance in the desired framework. In Section 3, we consider the scalar periodic BVP's by combining the two techniques namely alternative methods and the method of differential inequalities.

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1. Existence Results for Systems not at Resonance

Let us consider the BVP

$$(1.1) \quad x'' = f(t, x, x'),$$

$$(1.2) \quad x(a) = x_1, \quad x(b) = x_2.$$

More general boundary conditions than (1.2) are

$$(1.3) \quad B(a)x = c_1 x(a) - d_1 x'(a) = x_1, \quad B(b)x = c_2 x(b) + d_2 x'(b) = x_2,$$

where $c_1, c_2, d_1, d_2 \in R_+$. The following result concerning existence of solutions of (1.1), (1.2) or (1.1), (1.3) is due to Scorza-Dragoni [14] which is fundamental in subsequent discussions.

Theorem 1.1. Assume that

(1) $f \in C[J \times R^n \times R^n, R^n]$ and $\|f(t, x, y)\| \leq L$ for $(t, x, y) \in J \times R^n \times R^n$. Then the BVP (1.1), (1.2) or (1.1), (1.3) has a solution.

Hereafter we shall concentrate on the BVP (1.1), (1.3) where $E = R^n$, and $f \in C[J \times R^n \times R, R^n]$. This means that the system $x'' = f(t, x, x')$ is coupled in x but not in x' . When needed we write for each i , $f_i(t, x, x'_i)$ to denote this weak coupling of f . Without further mention we assume that $i = 1, 2, \dots, n$ and all inequalities between vectors are componentwise.

In order to treat unbounded f we need a priori estimates on x and x' . Let us first consider a priori bounds on x' . We shall give two types of results.

Lemma 1.1. Let $\alpha, \beta \in C[J, R^n]$ with $\alpha(t) \leq \beta(t)$ in J . Suppose that $x \in C^2[J, R^n]$ is such that $\alpha(t) \leq x(t) \leq \beta(t)$ and $|x''_i| \leq h_i(|x'_i|)$ in J , where $h_i \in C[R_+, (0, \infty)]$ satisfies $\int_0^\infty \frac{s ds}{h_i(s)} = \infty$. Then there exists a constant vector $N > 0$ depending only on α, β, h such that $|x'(t)| \leq N$ in J .

For a proof see [2]. When $n = 1$, Lemma 1.1 is the well-known Nagumo condition.

Lemma 1.2. Let $x \in C^2[J, R^n]$ be such that $\|x\| \leq R$ and $\|x''\| \leq \phi(\|x'\|)$ in J . Suppose that $\phi \in C[R_+, (0, \infty)]$ is increasing and $\lim_{s \rightarrow \infty} \frac{s^2}{\phi(s)} = \infty$. Then there exists a constant $N > 0$ depending only on R and ϕ such that $\|x'(t)\| \leq N$ in J .

We shall indicate the proof of Lemma 1.2 which is given in [11].

Proof. Let $N = \|x'(t_0)\| = \max_J \|x'(t)\|$. If μ is a real number such that $|\mu| \leq \frac{1}{2(b-a)}$ and $t_0 + \mu \in J$, then it follows from

$$x(t_0 + \mu) = x(t_0) + \mu x'(t_0) + \int_0^1 x''(t_0 + s\mu) \frac{\mu^2}{2} ds$$

that $|\mu|N \leq 2R + \frac{\mu^2}{2}\phi(N)$. If μ is such that $|\mu|N = 3R$ and this choice is possible for $N \geq 6R(b-a)$, then $R \leq \frac{1}{2}\mu^2\phi(N) = \frac{9}{2}R^2 \frac{\phi(N)}{N^2}$ which leads to a contradiction for large N . This proves the lemma.

Let $\alpha, \beta \in C^2[J, R^n]$ with $\alpha(t) \leq \beta(t)$ on J and let a constant vector $c > 0$ be such that $|\alpha'(t)|, |\beta'(t)| < c$ in J . We define the functions

$$q_i(x') = \max[-c_i, \min(x'_i, c_i)],$$

$$p_i(t, x) = \max[\alpha_i(t), \min(x_i, \beta_i(t))],$$

and

$$Q_i(x) = \begin{cases} \frac{x_i - \beta_i(t)}{1+x_i^2} & \text{if } x_i > \beta_i(t), \\ 0 & \text{if } \alpha_i(t) \leq x_i \leq \beta_i(t), \\ \frac{x_i - \alpha_i(t)}{1+x_i^2} & \text{if } x_i < \alpha_i(t), \end{cases}$$

and consider the modified function $F(t, x, x') \equiv f(t, p(t, x), q(x')) + Q(x)$ relative to (α, β, c) . It is clear that F is continuous and bounded on $J \times R^n \times R$. Also we note that $\alpha(t) \leq p(t, x) \leq \beta(t)$ and $|q(x')| \leq c$ on J .

We then have the following existence theorem for unbounded f .

Theorem 1.2. Suppose that

(a) $\alpha, \beta \in C^2[J, R^n]$ with $\alpha(t) \leq \beta(t)$ on J and

$\alpha_i'' \geq f_i(t, \sigma, \alpha_i')$ for all σ such that $\alpha(t) \leq \sigma \leq \beta(t)$, $\sigma_i = \alpha_i(t)$,

$\beta_i'' \leq f_i(t, \sigma, \beta_i')$ for all σ such that $\alpha(t) \leq \sigma \leq \beta(t)$, $\sigma_i = \beta_i(t)$,

for $a < t < b$;

(b) $x \in C^2[J, R^n]$ is such that $\alpha(t) \leq x \leq \beta(t)$ and

$|x_i''| \leq h_i(|x_i'|)$ on J where $h_i \in C[R_+, (0, \infty))$ satisfies $\int_0^\infty \frac{s ds}{h_i(s)} = \infty$;

(c) $B(a)\alpha \leq x_1 \leq B(a)\beta$ and $B(b)\alpha \leq x_2 \leq B(b)\beta$ with $d_1, d_2 > 0$.

Then the BVP (1.1), (1.3) has a solution $x \in C^2[J, R^n]$ satisfying

$\alpha(t) \leq x(t) \leq \beta(t)$ on J . Moreover there exists an $N_0 > 0$ such that

$|x'(t)| \leq N_0$ on J .

Proof. By hypothesis (b) we can choose an $N > 0$ depending only on α, β, h as in Lemma 1.1. Let N_0 be such that

$$N_0 > \max[N, \max_J |\alpha'(t)|, \max_J |\beta'(t)|].$$

choose $c > N_0$. Let F be the modified function relative to (α, β, c) . By Theorem 1.1 the BVP

$$(1.4) \quad x'' = F(t, x, x'), \quad B(a)x = x_1, \quad B(b)x = x_2,$$

has a solution $x \in C^2[J, R^n]$. Let us first show that $\alpha(t) \leq x(t) \leq \beta(t)$ on J . We will show that $\alpha(t) \leq x(t)$ on J since the proof of $x(t) \leq \beta(t)$ on J is similar.

Suppose that $\alpha(t) \leq x(t)$ on J is not true. Then there is an index k , $1 \leq k \leq n$ and a smallest $\epsilon > 0$ such that

$$\alpha_k(t) \leq x_k(t) + \epsilon, \quad t \in J$$

and at least one $t_0 \in J$ satisfying

$$\alpha_k(t_0) = x_k(t_0) + \epsilon.$$

If $t_0 \in (a, b)$, then we have $\alpha_k(t_0) > x_k(t_0)$ and $\alpha'_k(t_0) = x'_k(t_0)$. Hence $|x'_k(t_0)| < c_k$. Consequently choosing $\sigma = p(t_0, x(t_0))$, we obtain

$$\begin{aligned} \alpha''_k(t_0) - x''_k(t_0) &\geq f_k(t_0, p(t_0, x(t_0)), \alpha'_k(t_0)) - F_k(t_0, x(t_0), x'_k(t_0)) \\ &= f_k(t_0, p(t_0, x(t_0)), \alpha'_k(t_0)) - f_k(t_0, p(t_0, x(t_0)), \alpha'_k(t_0)) \\ &\quad - \frac{(x_k(t_0) - \alpha_k(t_0))}{1 + x_k^2(t_0)} = \frac{\alpha_k(t_0) - x_k(t_0)}{1 + x_k^2(t_0)} > 0 \end{aligned}$$

in view of the definition of F and α . This contradicts the relation

$$x_k(t) + \epsilon \geq \alpha_k(t).$$

If $t_0 = a$, we have $x_k(a) + \varepsilon = \alpha_k(a)$. Since $x_k(t) + \varepsilon \geq \alpha_k(t)$, it follows that $x'_k(a) \geq \alpha'_k(a)$. By the boundary conditions $B(a)\alpha \leq x_1$ and $B(a)x = x_1$, we get $d_1(x'_k(a) - \alpha'_k(a)) \leq 0$. Since $d_1 > 0$ we must have $x'_k(a) = \alpha'_k(a)$. Hence as before we arrive at $\alpha''_k(a) - x''_k(a) > 0$ which contradicts the assumption $x_k(t) + \varepsilon \geq \alpha_k(t)$. A similar contradiction holds in case $t_0 = b$. This establishes $\alpha(t) \leq x(t) \leq \beta(t)$ on J . Hence we have

$$x'' = F(t, x, x') = f(t, x, q(x')).$$

As a result, we obtain by (b)

$$|f_i(t, x, q_i(x'))| \leq h_i(|q_i(x')|)$$

whenever $\alpha(t) \leq x \leq \beta(t)$ on J . Recalling that $|q_i(x')| \leq c_i$, Lemma 1.1 shows that $|x'(t)| \leq N_0$ on J . This implies $F(t, x, x') = f(t, x, x')$ and hence $x(t)$ is actually a solution of (1.1), (1.3). The proof is complete.

Remark 1. Let us define lower and upper solution α, β in a natural way, namely

$$\alpha''_i \geq f_i(t, \alpha, \alpha'_i), \quad \beta''_i \leq f_i(t, \beta, \beta'_i).$$

Then to prove Theorem 1.2 we need that f possesses a quasi-monotone non-increasing property in x . See [2]. In this case, it is easily seen that α, β also satisfy the inequalities demanded in the assumption (a) of Theorem 1.2. Hence Theorem 1.2 contains this special case.

Remark 2. For each i , let p_i, q_i be two nonnegative numbers such that $p_i + q_i = n - 1$ so that we can split $x \in R^n$ into $x = (x_i, [x]_{p_i}, [x]_{q_i})$. Then (1.1) becomes

$$x_i'' = f_i(t, x_i, [x]_{p_i}, [x]_{q_i}, x_i').$$

Let $\alpha, \beta \in C^2[J, R^n]$ with $\alpha(t) \leq \beta(t)$ be coupled lower and upper quasi-solutions that is,

$$\alpha_i'' \geq f_i(t, \alpha_i, [\alpha]_{p_i}, [\beta]_{q_i}, \alpha_i'), \quad \beta_i'' \leq f_i(t, \beta_i, [\beta]_{p_i}, [\alpha]_{q_i}, \beta_i').$$

This situation requires a mixed quasi-monotone property (*mqmp*) for f in order to prove Theorem 1.2. See for details [9]. The function f is said to possess a *mqmp* if for each i , $f_i(t, x_i, [x]_{p_i}, [x]_{q_i}, x_i')$ is monotone nonincreasing in $[x]_{p_i}$ and nondecreasing in $[x]_{q_i}$. When f has *mqmp*, it is also easy to see that α, β again satisfy the assumption (a) of Theorem 1.2, showing that this situation is also a special case of Theorem 1.2.

Remark 3. For convenience we have considered weakly coupled systems (1.1) in x' . One could consider the systems (1.1) where f is also coupled in x' and this demands that the lower and upper solutions α, β have to be defined more stringently to prove Theorem 1.2. See for details [2]. Lemma 1.2 can also be employed in Theorem 1.2 instead of Lemma 1.1 in which case we need to choose R of Lemma 1.2 such that $R \geq |\alpha(t)|, |\beta(t)|$ and make appropriate modifications. The advantage of Lemma 1.2 is in considering abstract BVP's.

Remark 4. For a related treatment of the method of upper and lower solutions for generalized two-point boundary value problems see [13].

2. An Abstract Existence Theorem at Resonance

Consider the equation

$$(2.1) \quad Ex = Nx,$$

where $E : D(E) \rightarrow Y$ is a linear operator, and $N : D(N) \rightarrow Y$ is a nonlinear operator, both E and N have domains $D(E)$, $D(N)$ in a Banach space X and ranges $R(E)$, $R(N)$ in a Banach space Y with $D(E) \cap D(N) \neq \emptyset$.

Let us suppose that there exist projection operators $P : X \rightarrow X$, $Q : Y \rightarrow Y$ (that is, linear, bounded and idempotent), and a linear operator H which is called partial inverse of E such that

$$(i) \quad H(I-Q)Ex = (I-P)x, \quad x \in D(E),$$

$$(ii) \quad QEx = EPx, \quad x \in D(E),$$

$$(iii) \quad EH(I-Q)Nx = (I-Q)Nx, \quad x \in D(E) \cap D(N).$$

By applying $H(I-Q)$ to (2.1) we obtain

$$H(I-Q)Ex = H(I-Q)Nx,$$

which, in view of (i), yields

$$(2.2) \quad x = Px + H(I-Q)Nx,$$

showing that every solution of (2.1) is a solution of (2.2). If $x \in X$ is a solution of (2.2), by applying E to (2.2) and using (ii) and (iii) we get

$$Ex - Nx = Q(Ex - Nx).$$

Thus a solution of (2.2) is a solution of (2.1) if and only if

$$(2.3) \quad Q(Ex - Nx) = 0.$$

Equations (2.2) and (2.3) are usually called as the auxiliary and the bifurcation equations respectively. We have therefore shown that (2.1) is equivalent to the system of equations (2.2) and (2.3).

We shall assume that the spaces X and Y are such that the operator $\langle y, x \rangle : Y \times X \rightarrow R$ is defined, linear in both x and y and has the following properties:

(a) $|\langle y, x \rangle| \leq K \|y\| \|x\|$ for all $x \in X$, $x, y \in Y$ and some constant $K > 0$;

(b) for $y_0 \in Y_0$ we have $y_0 = 0$ if and only if $\langle y_0, x_0 \rangle = 0$ for all $x_0 \in X_0$.

Let $X_0 = PX$, $X_1 = (I-P)X$, $Y_0 = QY$ and $Y_1 = (I-Q)Y$. We assume that the direct sum decompositions $X = X_0 + X_1$ and $Y = Y_0 + Y_1$ hold and that the operator $H : Y_1 \rightarrow X_1$ is compact. Further let $\dim X_0 = \dim Y_0 < \infty$ and $J : Y_0 \rightarrow X_0$ be one-to-one, continuous, and such that $Jy_0 = 0$ if and only if $y_0 = 0$. Moreover let $\langle Jy_0, y_0 \rangle \geq 0$ for all $y_0 \in Y_0$ (and thus $\langle J^{-1}x_0, x_0 \rangle \geq 0$ for all $x_0 \in X_0$), with equality holding only if $y_0 = 0$ (or $x_0 = 0$) and $\langle Jy_0, J^{-1}x_0 \rangle = \langle y_0, x_0 \rangle$ for all $x_0 \in X_0$ and $y_0 \in Y_0$. Also suppose that $X_0 = \ker E$ so that $QEx = EPx = 0$ for $x \in X_0$. This assumption reduces (2.3) to

$$(2.4) \quad QNx = 0.$$

when the $\ker E$ is nontrivial, we say that we have a resonance case.

If $x = x_0 + x_1$, $x_0 \in X_0$, $x_1 \in X_1$, then (2.2), (2.4) can be solved in the following manner. Let $T : X \rightarrow X$ be a nonlinear operator defined by

$$(2.5) \quad Tx = Px + H(I-Q)Nx + JQNx.$$

Then if $T_\lambda : X \rightarrow Y$ is denoted by

$$T_\lambda x = \lambda Px + H(I-Q)Nx + \lambda JQNx,$$

we see that the assumption $X_0 = \ker E$ is finite dimensional implies that $T_\lambda, \lambda \in [0, 1]$ is compact provided N is continuous. Let us suppose that

$$(2.6) \quad \|Nx\| \leq J_0, \quad x \in D(N),$$

and

$$(2.7) \quad \langle QNx, x_0 \rangle \geq 0 \quad (\text{or } \langle QNx, x_0 \rangle \leq 0)$$

for all $\|x_0\| = R_0$ and $x = x_1 + x_0$ where $x_1 = H(I-Q)N(x_1 + x_0)$.

We wish to apply the Leray-Schauder principle to prove the existence of solutions of (2.5). We let

$$B = \{x_0 \in X_0, x_1 \in X_1 : \|x_0\| \leq R_0, \|x_1\| \leq r_0\},$$

where R_0 and r_0 will be specified later. It suffices to show that $x = T_\lambda x$ does not have a solution on the boundary ∂B of B so that (2.5) has a solution in B . That is, there exist $x_0 \in X_0, x_1 \in X_1$ such that

$$x_1 = H(I-Q)Nx \quad \text{and} \quad QNx = 0,$$

that is, (2.2) and (2.4) are satisfied.

If possible, let $x = T_\lambda x$ on ∂B . Then either

$$(1) \quad x \in \partial B, \quad x = x_0 + x_1, \quad \|x_1\| = r_0 \quad \text{and} \quad \|x_0\| < R_0$$

or

(ii) $x \in \partial B$, $x = x_0 + x_1$, $\|x_1\| < r_0$ and $\|x_0\| = R_0$.

If (i) holds, the relation $x_1 = H(I-Q)N(x_1+x_0)$ implies

$$r_0 = \|x_1\| \leq \|H(I-Q)\| \|N(x_1+x_0)\| \leq LJ_0$$

by (2.6) where $L = \|H(I-Q)\|$. Hence if r_0 is chosen such that $r_0 > LJ_0$ we have a contradiction. If (ii) is true, then $x_0 = Px = \lambda Px + \lambda JQNx$ shows that

$$0 = (1-\lambda)x_0 + \lambda JQNx$$

or

$$\begin{aligned} 0 &= \langle (1-\lambda)x_0, J^{-1}x_0 \rangle + \langle \lambda JQNx, J^{-1}x_0 \rangle \\ &> \langle QNx, x_0 \rangle \end{aligned}$$

which is a contradiction. Thus we have proved the following abstract existence result at resonance.

Theorem 2.1. Under the conditions (2.6) and (2.7) the compactness of H implies that the equation (2.1) has at least one solution.

In case one wishes to avoid the knowledge of solutions x_1 given x_0 , the assumption (2.7) of Theorem 2.1 can be strengthened to:

$$(2.7^*) \quad \langle QNx, x_0 \rangle \geq 0 \quad (\text{or } \langle QNx, x_0 \rangle \leq 0)$$

for all $\|x_0\| = R_0$ and $\|x - x_0\| \leq r_0$, where r_0, R_0 are constants such that $R_0 > 0$ and $r_0 > LJ_0$, $L = \|H(I-Q)\|$.

Theorem 2.1 can be viewed as an extension of Scorza-Dragoni's Theorem 1.1 to the case of resonance which is essentially given in [4, 5]. See also [3].

The proof of Theorem 2.1 shows that the following version is also true which is interesting in itself.

Theorem 2.2. Suppose that for every $\|x_0\| \leq R_0$, the equation $x_1 = H(I-Q)N(x_1+x_0)$ has solutions which are bounded. Then the assumption (2.7) (or (2.7*)) implies that the equation (2.1) has at least one solution.

3. Nonlinear Periodic Boundary Value Problem

We consider the nonlinear boundary value problem

$$(3.1) \quad x'' = f(t, x, x'), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi),$$

where $f \in C[[0, 2\pi] \times R \times R, R]$. Let X denote the space of all 2π -periodic functions that are continuous in $(-\infty, \infty)$, absolutely continuous in $[0, 2\pi]$ with derivative $x' \in L_2[0, 2\pi]$ and such that $x(0) = x(2\pi)$. Thus X is a Hilbert space with the usual inner product and norm. Let $Y = L_2[0, 2\pi]$ and $Ex = x''$ with the domain $D(E) = [x \in X, x, x' \text{ are absolutely continuous, } x'(0) = x'(2\pi) \text{ and } x'' \in Y]$. Suppose that $N : D(N) \subset X \rightarrow Y$ is the nonlinear Nemytskii operator generated by $f(t, x, x')$ and we assume that $D(N) \cap D(E) \neq \emptyset$. The system (3.1) can then be written as an equivalent operator equation

$$(3.2) \quad Ex = Nx.$$

We now derive sufficient analytical conditions that imply the operator theoretic inequality (2.7). Let X_0 be the subspace of X generated by all constant functions and let X_1 be the complementary subspace. Thus we can define the projection operator $P : X \rightarrow X_0$ by $Px = x(0) = x(2\pi)$ and then the functions in X_1 can be characterized by $x_1(t) \in X_1$ implies $x_1(0) = x_1(2\pi) = 0$. Let Y be the space $L_2[0, 2\pi]$ and let Y_0 be the subspace of all constant functions so that $Q : Y \rightarrow Y_0$ is defined by $Qy = \frac{1}{2\pi} \int_0^{2\pi} y(t) dt$. This defines the complementary subspace Y_1 . The operator $H : Y_1 \rightarrow X_1$ can now be defined to be the unique solution $x(t)$ of the equation

$$x_1''(t) = y_1(t)$$

$$x_1(0) = x_1(2\pi) = 0.$$

It can now be easily seen that the operators H, P, Q, E and N satisfy the hypotheses of the previous section. The operator $\langle y, x \rangle$ is chosen to be the L^2 -inner product and finally $J : Y_0 \rightarrow X_0$ can be chosen to be the identity map.

The inequality $\langle QNx, x_0 \rangle \geq 0$ for all $|x_0| = R_0$ and x_1 satisfying $x_1 = H(I-Q)N(x_1+x_0)$ translates into

$$\int_0^{2\pi} f(s, x_1 + x_0, x_1') x_0 ds \geq 0$$

for $x_0 = \pm R_0$ which in turn yields the two inequalities

$$(3.3) \quad (a) \quad \int_0^{2\pi} f(s, x_{1+} + R_0, x_{1+}') ds \geq 0,$$

$$(b) \quad \int_0^{2\pi} f(s, x_{1-} - R_0, x_{1-}') ds \leq 0,$$

where $x_{1\pm}$ are solutions of the equation $x_1 = H(I-Q)N(x_1+x_0)$ with $Px = \pm R_0$. Thus the assumptions (3.3) and (2.7) are equivalent.

Let us next investigate situations where sufficient conditions can be obtained that verify (3.3).

Suppose that f is bounded on $[0, 2\pi] \times R \times R$ which implies that Nx is bounded by some J_0 . Since assuming (3.3) is equivalent to assuming (2.7), an application of Theorem 2.1 shows that the BVP (3.1) has a 2π -periodic solution.

Consider next the class of functions f that are not bounded but however satisfy the following conditions:

(3.4) $\alpha, \beta \in C^2[[0, 2\pi], R]$ such that $\alpha(t) \leq \beta(t)$ on $0 \leq t \leq 2\pi$ and

$$(i) \quad \alpha'' \geq f(t, \alpha, \alpha'), \quad \alpha(0) = \alpha(2\pi),$$

$$\beta'' \leq f(t, \beta, \beta'), \quad \beta(0) = \beta(2\pi),$$

(ii) f satisfies a Nagumo condition with respect to α, β .

Without loss of generality, we also suppose that $Pa = -P\beta = R_0$.

Then by the methods of Section 1, we know that for any x_0 such that $\alpha(0) = -R_0 \leq x_0 \leq R_0 = \beta(0)$, the BVP

$$(3.5) \quad x'' = f(t, x, x'), \quad x(0) = x(2\pi) = x_0$$

has a solution $x(t)$ such that $\alpha(t) \leq x(t) \leq \beta(t)$, $|x'(t)| \leq N$ in $[0, 2\pi]$.

Now to verify the inequalities (3.3) for all x_1 such that

$$(3.6) \quad x = x_1 + x_0 = Px + H(I-Q)N(x_1 + x_0),$$

it is clear from the definitions of P , $H(I-Q)$ and N that (3.6) is equivalent to (3.5). Consequently when $x_0 = \pm R_0$, every x such that $x = x_1 + x_0$ satisfies (3.5) and therefore the relations (3.3) hold. Hence whenever f satisfies (3.3) and (3.4), the BVP (3.1) has a 2π -periodic solution. Thus we have proved the following result.

Theorem 3.1. Assume that either f is bounded or f satisfies the conditions (3.4). Suppose further that the assumptions (3.3) hold. Then the BVP (3.1) has a 2π -periodic solution.

We shall now give sufficient conditions which guarantee the conditions (3.3). Assume that α, β of (3.4) satisfy

$$(3.7) \quad \alpha'(0) \geq \alpha'(2\pi) \quad \text{and} \quad \beta'(0) \leq \beta'(2\pi).$$

Let x_+, x_- be any solutions of (3.6) such that $x_+ = x_{1+} + R_0$, $x_- = x_{1-} - R_0$. Since (3.6) and (3.5) are equivalent, it is clear that

$$(3.8) \quad \alpha(t) \leq x_+(t), \quad x_-(t) \leq \beta(t)$$

on $[0, 2\pi]$. Also the relations (3.3) can in this case be rewritten as

$$(3.9) \quad \int_0^{2\pi} x_+'' ds \geq 0 \quad \text{and} \quad \int_0^{2\pi} x_-'' ds < 0.$$

It is now easy to show that (3.7) implies (3.9) because of (3.8) and the fact $x_0 = \pm R_0$.

We remark that the sufficient conditions (3.7) which are assumed in [1, 6, 7, 8, 10] are used only to verify the abstract inequality $\langle QNx, x_0 \rangle \geq 0$.

To show that the conditions (3.7) are not the best possible conditions, let us give an example [15]. Consider the BVP

$$x'' = -\frac{d}{dt}F(t, x) - g(t, x), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi)$$

where g is such that $x \leq y$ implies $g(t, x) \leq g(t, y)$ and $g(t, x) = g(t, y)$, $x \leq y$ implies $x = y$. Then if α, β are lower and upper solutions, we get on integrating from 0 to 2π

$$\begin{aligned} 0 &\leq \int_0^{2\pi} [g(s, \beta(s)) - g(s, \alpha(s))] ds \\ &\leq -\beta'(2\pi) + \beta'(0) + \alpha'(2\pi) - \alpha'(0) \leq 0 \end{aligned}$$

in view of (3.7). The properties of g and $\alpha(t) \leq \beta(t)$ now imply that $\alpha(t) \equiv \beta(t)$ for $[0, 2\pi]$ which amounts to assuming the existence of a periodic solution. Consequently the existing results for periodic solutions as given in [1, 6, 7, 8, 10] are not applicable. However our condition (3.3) is verified. Consider a simple example

$$x'' = e(t) - 2Dx' - g(x)$$

where g is increasing in x and $g(A) \leq e(t) \leq g(B)$ with $A \leq B$, A, B being constants. We then have

$$\int_0^{2\pi} f(s, A, 0) ds \geq 0 \quad \text{and} \quad \int_0^{2\pi} f(s, B, 0) ds \leq 0$$

verifying (3.3). Here A, B are lower and upper solutions.

Finally we remark that the ideas of this section and the preceding can be applied to obtain results analogous to Scorza-Dragoni's theorem [24] as results on "existence in the small." Similar ideas for the periodic boundary value problem may be seen in [12]. Thus we consider the nonlinear problem

$$x'' = f(t, x, x')$$

$$x(0) = x(\omega), \quad x'(0) = x'(\omega)$$

and we obtain sufficient conditions on f and ω in order that the above problem may have ω -periodic solutions. Because of the abstract nature of our approach, these ideas can be applied to other boundary conditions as well, e.g., the Neumann problem.

We use the same notations as before. Note that the inequality $\langle QNx, x_0 \rangle \geq 0$ can still be satisfied by sufficient conditions of the type

$\alpha'(0) \geq \alpha'(2\pi)$ and $\beta'(0) \leq \beta'(2\pi)$. Thus the only steps to guarantee are the existence of a solution to the auxiliary equation $x = Px + H(I-Q)Nx$ with $Px = x_0$ and the inequality $\alpha(t) \leq x(t) \leq \beta(t)$ to be satisfied by such solutions. The latter follows from the theory of upper and lower solutions. The existence of solutions to $x = Px + H(I-Q)Nx$ with $Px = x_0$ is equivalent to the existence of solutions of $x'' = f(t, x, x')$, $x(0) = x(\omega) = x_0$ and this follows by an application of Scorza-Dragoni's theorem in the small. Note that $\|H(I-Q)\|$ is small if ω is small.

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