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AND MONOTONE METHOD

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1. INTRODUCTION

It is now well known that the method of vector Lyapunov functions provides an effective tool to investigate the properties of large scale interconnected dynamical and control systems. [3,4,5,11-15]. Several Lyapunov functions result in a natural way in the study of such systems by the decomposition and aggregation method [1-4,12-14]. However an unpleasant fact in this approach is the requirement of quasi-monotone property on the comparison system since comparison systems with a desired property like stability exist without satisfying quasi-monotone property. Also in the study of comparison theorems and extremal solutions for differential systems one usually imposes this quasi-monotone property. To avoid this difficulty two ideas are suggested recently, namely to use an appropriate cone other than R_+^n and to exploit the new notion of quasi-solutions [6,9,10]. In [8] the idea of quasi-solutions is developed to some extent. In this paper we obtain further results on quasi-solutions. We introduce the notion of coupled quasi-solutions in addition to quasi-solutions. We show how the idea of quasi-solutions leads

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to isolated subsystems, obtain error estimates between solutions and quasi-solutions and develop monotone iterative techniques to obtain coupled maximal and minimal quasi-solutions.

2. QUASI-SOLUTIONS

Consider the initial value problem

$$x' = f(t, x), \quad x(0) = x_0 \quad (2.1)$$

where $f \in C[I \times R^n, R^n]$ and $I = [0, T]$. To define quasi-solutions of (2.1), we fix for each $i \in \{1, 2, \dots, n\}$ two nonnegative integers p_i, q_i such that $p_i + q_i = n - 1$ and split $x \in R^n$ into $x = (x_i, [x]_{p_i}, [x]_{q_i})$. Then the system (2.1) becomes

$$x_i' = f_i(t, x_i, [x]_{p_i}, [x]_{q_i}), \quad x(0) = x_0. \quad (2.2)$$

Also for any $y, z \in R^n$, we let $[y, z]_i$ denote an element of R^n with the description $[y, z]_i = (y_i, [y]_{p_i}, [z]_{q_i})$. Without further mention we assume that $i = 1, 2, \dots, n$ and all inequalities between vectors are componentwise.

Definition 1. Let $a \in C[I, R^n]$. Then $y \in C^1[I, R^n]$ is said to be a quasi-solution of (2.1) relative to a if

$$y_i' = f_i(t, y_i, [y]_{p_i}, [a]_{q_i}), \quad y(0) = x_0 \quad (2.3)$$

The function y is called an upper quasi-solution of (2.1) if

$$y_i' \geq f_i(t, y_i, [y]_{pi}, [\alpha]_{qi}), \quad y(0) \geq x_0,$$

and a lower quasi-solution if all inequalities are reversed. The functions $v, w \in C^1[I, R^n]$ with $v \leq w$ on I are said to be coupled lower and upper quasi-solutions of (2.1) if

$$v_i' \leq f_i(t, v_i, [v]_{pi}, [w]_{qi}), \quad v(0) \leq x_0,$$

$$w_i' \geq f_i(t, w_i, [w]_{pi}, [v]_{qi}), \quad w(0) \geq x_0.$$

The functions $x, y \in C^1[I, R^n]$ are said to be coupled quasi-solutions of (2.1) if

$$\begin{cases} x_i' = f_i(t, x_i, [x]_{pi}, [y]_{qi}), & x(0) = x_0, \\ y_i' = f_i(t, y_i, [y]_{pi}, [x]_{qi}), & y(0) = x_0. \end{cases} \quad (2.4)$$

In the special case $q_i = 0$ for all i , quasi-solutions (2.3) are just solutions and in case $p_i = 0$ for all i , quasi-solutions that result are most useful since they can be determined most easily. It is clear from the foregoing definition that one can define maximal and minimal quasi-solutions relative to α as well as coupled maximal and minimal quasi-solutions.

Definition 2. The function f is said to possess a mixed quasimonotone property (mqmp for short) if for each i , $f_i(t, x_i, [x]_{pi}, [x]_{qi})$ is monotone nondecreasing in $[x]_{pi}$ and monotone nonincreasing in $[x]_{qi}$.

The following comparison theorem is known [8].

Theorem 2.1. Let $a \in C[I, R^n]$, $Q_a = \{(t, x) : t \in I \text{ and } x \geq a(t)\}$ and $f \in C[Q_a, R^n]$ have the mqmp. Suppose that $m \in C[I, R^n]$, $m(t) \in Q_a(t) = \{x : (t, x) \in Q_a\}$ for all t and $D_-m(t) \leq f(t, m(t))$, $t \in I$. Then $m(0) \leq x_0$ implies that $m(t) \leq r(t)$ on I where $r(t)$ is the maximal quasi-solution with respect to a of (2.1).

We need the following result concerning coupled quasi-solutions and quasi-solutions relative to a .

Theorem 2.2. Let v, w be coupled lower and upper quasi-solutions of (2.1) where $f \in C[D, R^n]$ and $D = \{(t, x) : v(t) \leq x \leq w(t), t \in I\}$. Let f possess mqmp. Then there exist coupled quasi-solutions (x, y) for (2.1) and for any $a \in C[I, R^n]$ such that $v(t) \leq a(t) \leq w(t)$ on I , there exists a quasi-solution relative to a of (2.1).

The proof of this result follows closely the proof of Müller's theorem as given in [7] with appropriate modifications. We omit the details.

3. THE METHOD OF VECTOR LYAPUNOV FUNCTIONS

We shall now indicate an application of Theorem 2.1 to large scale systems by the method of vector Lyapunov functions. Consider the overall large scale system of the form

$$x' = F(t, x), \quad x(0) = x_0. \quad (3.1)$$

Suppose that (3.1) consists of isolated subsystems

$$z_\eta' = g_\eta(t, z_\eta) + R_\eta(t, x), \quad z_\eta(0) = z_{\eta 0}, \quad (3.2)$$

where $x \in R^N$, $z_\eta \in R^{N_\eta}$ such that $\sum_{\eta=1}^n N_\eta = N$, and the R_η are the interaction terms, usually of the form $R_\eta(t, x) = \sum_{j=1}^n R_{\eta j}(t, z_j)$. If there exist functions V_η, d_η and w_η such that

$$D^+ V_\eta(t, z_\eta) \leq -d_\eta(t, z_\eta) + w_\eta(t, z_\eta, x) \quad (3.3)$$

and if the righthand side of (3.3) can be majorized so that (3.3) takes the form

$$D^+ V(t, z) \leq f(t, V(t, z)) \quad (3.4)$$

where $f_i(t, z)$ is nondecreasing in z_j , $j \neq i$, then the method of vector Lyapunov functions can effectively be utilized with appropriate assumptions on V and f to obtain the stability of the zero solution in (3.1); see [1-5, 9, 10, 12, 13]. In many situations that occur in science and engineering, $f(t, V) = -AV$, where the matrix A is assumed to have positive diagonal and nonpositive off-diagonal elements. It is then well known that the comparison system $u' = -Au$ is stable if and only if A is a Metzler matrix. This method fails if A has positive off-diagonal

elements. However, Theorem 2.1 is applicable in this situation. Since all $V_i \geq 0$, we can choose $\alpha \equiv 0$ and consequently we can get the desired estimate

$$V(t, z(t)) \leq r(t), \quad t \geq 0,$$

where $r(t)$ is now the maximal quasi-solution of the system

$$r_i' = f_i(t, r_i, [r]_{pi}, [0]_{qi}), \quad r(0) = V(0, z_0).$$

It is then a simple matter to obtain stability of the large scale system (3.1) from the stability of the quasi-solutions.

The following example demonstrates the usefulness of quasi-solutions. Consider $u' = Au$ where $a_{ii} < 0$, $a_{ij} \leq 0$ for $j < i$ and $a_{ij} \geq 0$ for $j > i$. The quasi-solutions relative to $\alpha \equiv 0$ are determined by $v' = \tilde{A}u$ where $\tilde{a}_{ii} = a_{ii}$, $\tilde{a}_{ij} = 0$ for $j < i$ and $\tilde{a}_{ij} = a_{ij}$ for $j > i$. $\tilde{A}$ is stable since it has eigenvalues $a_{11}, \dots, a_{nn} < 0$.

The comparison Theorem 2.1 can naturally be employed in place of the usual ones [5] in several situations to discuss qualitative behavior of a given system (3.1) under weaker hypotheses on the comparison system, that is, without demanding the quasimonotone nondecreasing property.

Recall that the decomposition of (3.1) in the form (3.2) is crucial for the method of vector Lyapunov functions to be effective. We shall now show that the idea of quasi-solutions leads automatically to isolated subsystems and consequently the decomposition needed is always possible. Let us rewrite (3.1) as

$$z_{\mu}' = F_{\mu}(t, z_{\mu_1}, z_{\mu_2}, \dots, z_{\mu_n})$$

where as before, $x = (z_{\mu_1}, \dots, z_{\mu_n})$, $z_{\mu} \in R^{N_{\mu}}$ and $\sum_{\mu=1}^n N_{\mu} = N$. Let $a \in C[I, R^N]$ and consider the subsystems

$$z_{\mu_i}' = F_{\mu_i}(t, a_{\mu_1}, \dots, z_{\mu_i}, \dots, a_{\mu_n}) = g_{\mu_i}(t, z_{\mu_i}) \quad (3.5)$$

so that the overall large scale system can be written as

$$z_{\mu}' = g_{\mu}(t, z_{\mu}) + R_{\mu}(t, x, z_{\mu})$$

where $R_{\mu_i}(t, x, z_{\mu_i}) \equiv F_{\mu_i}(t, x) - F_{\mu_i}(t, a_{\mu_1}, \dots, z_{\mu_i}, \dots, a_{\mu_n})$. Thus it is seen that the decomposition of (3.1) into the required isolated subsystems and interconnections, demanded by the method of vector Lyapunov functions, is always possible. This suggests that it would be nice to know the error estimates between quasi-solutions and the choice function a and solutions and quasi-solutions so that a judicial choice can be made whenever possible.

4. ERROR ESTIMATES

For convenience let us consider componentwise error estimates. Suppose that f satisfies a Lipschitz condition of the form

$$|f_i(t, x) - f_i(t, y)| \leq L \sum_{i=1}^n |x_i - y_i|.$$

Let $a \in C[I, R^N]$ and $y(t)$ be the quasi-solution given by (2.3). Then

setting $v_i(t) = |y_i(t) - a_i(t)|$, we get the differential inequality

$$D_- v_i(t) \leq L \sum_{j \in \{i, p_i\}} v_j(t) + \sigma_i(t), \quad v_i(0) = |y_i(0) - a_i(0)|,$$

where $\sigma_i(t) = |f_i(t, a) - D_- a_i|$. Hence we have the estimate

$$v(t) \leq v(0)e^{At} + \int_0^t \sigma(s)e^{A(t-s)} ds \equiv \delta(t), \quad t \in I, \quad (4.1)$$

where the matrix A has for each i , q_i zeros in the i th row and the rest L 's. We next let $m_i(t) = |x_i(t) - y_i(t)|$ where $x(t)$ is the solution of (2.1) and note $m(0) = 0$. Then we have

$$D_- m_i(t) \leq L \sum_{j=1}^n m_j(t) + L \sum_{j \in \{p_i\}} v_j(t).$$

Consequently, using (4.1), we arrive at

$$D_- m(t) \leq Bm(t) + \eta(t), \quad m(0) = 0,$$

where $b_{ij} = L$ and $\eta_i(t) = L \sum_{j \in \{q_i\}} \delta_j(t)$. It therefore follows that

$$m(t) \leq \int_0^t \eta(s)e^{B(t-s)} ds, \quad t \in I. \quad (4.2)$$

The estimates (4.1) and (4.2) are the desired error estimates.

5. MONOTONE METHOD

In order to develop monotone method for (2.1), we need the following assumptions.

(A₁) $v, w \in C^1[I, R^n]$ with $v(t) \leq w(t)$ on I are coupled lower and upper quasi-solutions of (2.1);

(A₂) for each i , $f_i(t, x_i, [x]_{pi}, [x]_{qi}) - f_i(t, y_i, [x]_{pi}, [x]_{qi}) \geq -M_i(x_i - y_i)$ whenever $v_i(t) \leq y_i \leq x_i \leq w_i(t)$ on I , where $M_i \geq 0$.

For any $\eta, \mu \in C[I, R^n]$ such that, $v \leq \eta$, $\mu \leq w$ on I , we define

$$F_i(t, x) = f_i(t, \eta_i, [\eta]_{pi}, [\mu]_{qi}) - M_i(x_i - \eta_i) \quad (5.1)$$

and consider the uncoupled linear differential system

$$x_i' = F_i(t, x), \quad x(0) = x_0. \quad (5.2)$$

clearly for a given η, μ , the system (5.2) possess a unique solution $x(t)$ defined on I .

For each $\eta, \mu \in C[I, R^n]$ such that $v \leq \eta \leq \mu \leq w$ on I we define the mapping A by

$$A[\eta, \mu] = x \quad (5.3)$$

where x is the unique solution of (5.2). We can then prove the following lemma concerning the mapping A .

Lemma 5.1. Suppose that f possesses $mqmp$ and that the assumptions (A_1) , (A_2) hold. Then (i) the unique solution x of (5.2) satisfies $v(t) \leq x(t) \leq w(t)$ on I , (ii) $v \leq A[v, w]$, $w \geq A[w, v]$, (iii) A possesses $mqmp$ on the segment $\langle v, w \rangle = \{x \in R^n : v(t) \leq x \leq w(t), t \in I\}$.

Proof: Let $\phi_i(t) = v_i(t) - x_i(t)$. Then $\phi_i(0) \leq 0$ and

$$\begin{aligned} \phi_i' &\leq f_i(t, v_i, [v]_{pi}, [w]_{qi}) - f_i(t, \eta_i, [\eta]_{pi}, [\mu]_{qi}) + M_i(x_i - \eta_i) \\ &\leq f_i(t, v_i, [\eta]_{pi}, [\mu]_{qi}) - f_i(t, \eta_i, [\eta]_{pi}, [\mu]_{qi}) + M_i(x_i - \eta_i) \\ &\leq M_i(\eta_i - v_i) + M_i(x_i - \eta_i) = M_i \phi_i. \end{aligned}$$

Here we have used (A_1) , (A_2) and the fact that f has $mqmp$. It thus follows that $\phi_i(t) \leq \phi_i(0)e^{M_i t} \leq 0$ on I and hence $v(t) \leq x(t)$ on I . Similarly we can show that $x(t) \leq w(t)$ on I . This proves (i).

Let $A[v, w] = y$ where y is the unique solution of (5.2) corresponding to the choice $\eta = v$, $\mu = w$. Setting $\phi_i = y_i - v_i$ we get easily $\phi_i' \geq -M_i \phi_i$ and $\phi_i(0) \geq 0$ which implies $\phi_i(t) \geq 0$ proving $v \leq A[v, w]$. Similarly we can prove $w \geq A[w, v]$ proving (ii).

To prove (iii), let $\eta \leq \mu$ be such that $\eta, \mu \in \langle v, w \rangle$. Suppose that $A[\eta, \mu] = x$ and $A[\mu, \eta] = y$. Then setting $\phi_i = x_i - y_i$, we find using $mqmp$ of f and (A_2) that

$$\begin{aligned} \phi_i' &\leq f_i(t, \eta_i, [\mu]_{pi}, [\eta]_{qi}) - f_i(t, \mu_i, [\mu]_{pi}, [\eta]_{qi}) - M_i \phi_i \\ &\quad + M_i(\eta_i - \mu_i) \leq -M_i \phi_i. \end{aligned}$$

Since $\phi_i(0) = 0$, this yields $x_i \leq y_i$. It therefore follows that the mapping A has *mcmp* proving (iii). The proof of lemma is complete.

In view of Lemma 5.1, we can define the sequences

$$v_n = A[v_{n-1}, w_{n-1}], \quad w_n = A[w_{n-1}, v_{n-1}] \quad (5.4)$$

with $v_0 = v$ and $w_0 = w$. It is then easy to prove that the sequences $\{v_n\}$, $\{w_n\}$ are monotonic and converge uniformly and monotonically to coupled quasi-solutions (α, β) of (2.1). Letting $\alpha = \lim_{n \rightarrow \infty} v_n$, $\beta = \lim_{n \rightarrow \infty} w_n$, we find

$$\alpha_i' = f_i(t, \alpha_i, [\alpha]_{pi}, [\beta]_{qi}), \quad \alpha(0) = x_0,$$

$$\beta_i' = f_i(t, \beta_i, [\beta]_{pi}, [\alpha]_{qi}), \quad \beta(0) = x_0.$$

We shall show that α, β are coupled minimal and maximal quasi-solutions respectively. Let (x, y) be any coupled quasi-solution of (2.1) such that $x, y \in (v, w)$. Let us assume that for some integer $k > 0$, $v_{k-1} \leq x, y \leq w_{k-1}$ on I . Then setting $\phi_i = v_{k,i} - x_i$ and employing *mcmp* of f and (A_2) we arrive at

$$\begin{aligned} \phi_i' &\leq f_i(t, v_{k-1}, [x]_{pi}, [y]_{qi}) - M_i(v_{k,i} - v_{k-1,i}) \\ &\quad - f_i(t, x_i, [x]_{pi}, [y]_{qi}) \\ &\leq M_i(x_i - v_{k-1,i}) - M_i(v_{k,i} - v_{k-1,i}) = -M_i \phi_i. \end{aligned}$$

Since $\phi_i(0) \leq 0$ this implies that $v_k \leq x$. Using similar arguments it is easy to conclude that $v_k \leq x, y \leq w_k$ on I . Since $v_0 \leq x, y \leq w_0$ on I , it follows by induction $v_n \leq x, y \leq w_n$ on I for all n . Hence we have $\alpha \leq x, y \leq \beta$ on I proving (α, β) are coupled minimal and maximal quasi-solutions of (2.1). Since any solution x of (2.1) such that $x \in \langle v, w \rangle$ can be considered as (x, x) coupled quasi-solution of (2.1), we also have $\alpha \leq x \leq \beta$ on I . We have thus proved the following result.

Theorem 5.1. Let f possess *mqmp* and the assumptions $(A_1), (A_2)$ hold. Then the sequences $\{v_n\}, \{w_n\}$ defined by (5.4) converge uniformly and monotonically to coupled minimal and maximal quasi-solutions (α, β) of (2.1), that is, if (x, y) are any coupled quasi-solutions of (2.1) such that $x, y \in \langle v, w \rangle$ then

$$v \leq v_1 \leq \dots \leq v_n \leq \alpha \leq x, y \leq \beta \leq w_n \leq \dots \leq w_1 \leq w \text{ on } I. \quad (5.5)$$

Furthermore any solution x of (2.1) such that $x \in \langle v, w \rangle$ also satisfies the inequality (5.5).

Remark. Let $\alpha \in C[I, R^n]$ with $v \leq \alpha \leq w$ on I . Then the coupled lower and upper quasi-solutions (v, w) yield

$$\begin{aligned} v_i' &\leq f_i(t, v_i, [v]_{pi}, [w]_{qi}) \leq f_i(t, v_i, [v]_{pi}, [\alpha]_{qi}) \\ w_i' &\geq f_i(t, w_i, [w]_{pi}, [v]_{qi}) \geq f_i(t, w_i, [w]_{pi}, [\alpha]_{qi}), \\ v(0) &\leq x_0 \leq w(0). \end{aligned}$$

This implies that v, w are lower and upper quasi-solutions of (2.1) with respect to $a \in \langle v, w \rangle$. Hence choosing $\mu = a$ in the definition of F given in (5.1) and proceeding with suitable modifications, it is easy to prove that α, β are minimal and maximal quasi-solutions of (2.1) with respect to a , that is, any quasi-solution of (2.1) relative to a satisfies

$$v \leq v_1 \leq \dots \leq v_n \leq \alpha \leq x \leq \beta \leq w_n \leq \dots \leq w \leq w_1 \text{ on } I.$$

If $q_i = 0$ for each i , then f is quasi-monotone nondecreasing consequently α, β reduce to minimal and maximal solutions of (2.1).

REFERENCES

- [1] Arakai, M., Stability of large scale nonlinear systems, *IEEE Trans. Auto. Control* 23, 129-142 (1978).
- [2] Sandell, N. R., Varaiya, P., Athans, M. & Safanov, M. G., Survey of decentralized control methods for large scale systems, *IEEE Trans. Auto. Control* 23, 108-128 (1978).
- [3] Siljak, D. D., *Large Scale Dynamical Systems*, North Holland, New York (1978).
- [4] Michel, A. N. & Miller, R. K., *Qualitative Analysis of Large Scale Dynamical Systems*. Academic Press, New York (1977).
- [5] Lakshmikantham, V. & Leela, S., *Differential and Integral Inequalities*. Vol. I. Academic Press, New York (1969).
- [6] Lakshmikantham, V., *Comparison Results for Reaction Diffusion Equations in a Banach Space*. Univ. of Texas at Arlington, Technical Rept. No. 94 (1978).
- [7] Deimling, K. & Lakshmikantham V., Existence and comparison theorems for differential equations in Banach spaces, *Nonlinear Analysis TMA* 3(1979), 569-575.

- [8] Deimling, K. & Lakshmikantham, V., Quasi-solutions and their role in the qualitative theory of differential equations. to appear in *Nonlinear Analysis*.
- [9] Lakshmikantham, V. & Leela, S., Cone valued Lyapunov functions, *Nonlinear Analysis*, 1(1977), 215-222.
- [10] Lakshmikantham, V. & Leela, S., Cone valued Lyapunov functions and large scale systems, *Proc. IEEE Trans. Auto. Control*, (1979).
- [11] Matrosov, V. M., On the theory of stability of motion, *Prikl. Mat. Meh.* 26(1962), 992-1002.
- [12] Lakshmikantham, V., Several Lyapunov functions, *Proc. of the Fifth Internat. Conference on Nonlinear Oscillation*, Kiev, U.S.S.R., (1969), 268-275.
- [13] _____, On the method of vector Lyapunov functions, *Proc. Twelfth Annual Allerton Conference on Circuit and System Theory*, October, 1974, 71-76.
- [14] Bailey, F. N., The application of Lyapunov's second method to interconnected systems, *J. SIAM Control* 27, (1971), 543-548.
- [15] Bellman, R., Vector Lyapunov functions, *J. SIAM Control* 1, (1962), 32-34.