**GENERAL UNIQUENESS CRITERIA FOR ORDINARY
DIFFERENTIAL EQUATIONS ***

by

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Abstract

This paper generalizes various uniqueness results and offers very general uniqueness criteria which include different types of results considered so far.

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1. INTRODUCTION

Let E be a finite or infinite dimensional real Banach space with norm $\|\cdot\|$. Set $R_0 = \{(t, x) : 0 < t \leq a \text{ and } \|x - x_0\| \leq b\}$ where $a, b > 0$. We consider the initial value problem (IVP for short);

$$(1.1) \quad x' = f(t, x), \quad x(0) = x_0,$$

where $f \in C[R_0, E]$.

Several uniqueness criteria are known concerning the IVP (1.1) [1,3,4,7,10,13,15,16,17,24,29]. One of the general uniqueness results which is well known is that of Kamke's [15,19]. However, if f is assumed continuous (which is the case in many uniqueness results) for $0 \leq t \leq a$ and $\|x - x_0\| \leq b$, then Kamke's theorem is no more general than that of Perron's theorem, see [19,25]. When f is not assumed continuous at $(0, x_0)$, then a necessary condition is the requirement

$$\lim_{t \rightarrow 0^+} \frac{\|x(t) - y(t)\|}{t} = 0,$$

where $x(t)$ and $y(t)$ are any two solutions of (1.1). This condition can be generalized as follows:

$$(1.2) \quad \lim_{t \rightarrow 0^+} \frac{\|x(t) - y(t)\|}{B(t)} = 0,$$

where $B \in C[[0, a], \mathbb{R}^+]$ with $B(t) > 0$ for $0 < t \leq a$. Clearly (1.2) is not sufficient for guaranteeing uniqueness and a general uniqueness theorem in this direction which appears in [17,19] is the following.

Theorem 1.1

Suppose that;

- (i) $x(t), y(t)$ are any two solutions of (1.1) satisfying (1.2);
- (ii) $g(t,u)$ is continuous for $0 < t \leq a, 0 \leq u \leq 2b$, and the only solution $u(t)$ of $u' = g(t,u)$ on $0 \leq t \leq a$ such that $\lim_{t \rightarrow 0^+} \frac{u(t)}{B(t)} = 0$ is the trivial solution;
- (iii) $\|f(t,x) - f(t,y)\| \leq g(t, \|x-y\|)$ for $(t,x), (t,y) \in R_0$. Then the IVP (1.1) has at most one solution.

For a complete discussion of the foregoing remarks see [19]. Furthermore, it is known that using a Lyapunov-like function as a candidate to impose uniqueness conditions rather than the norm is more flexible, see [18,19,22]. In [3], an extension of known uniqueness results is given in terms of Lyapunov-like function under much weaker requirements so as to cover the pathological example $x' = x^{1/2} + t, x(0) = 0$, which can not be included in the well known uniqueness results.

If we examine various uniqueness results, we see that there are three different assumptions, namely (i), (ii) and (iii) or their variants of Theorem 1.1. For a special choice of $B(t)$ such as $B(t) = t, B(t) = t^n$ and $B(t) = e^{-1/t}$ or a general $B(t)$, one could prescribe conditions on f such that (1.2) holds. This is what was mostly done in several recent results [1,11,12,14,27]. One can also offer sufficient conditions on g to guarantee the hypothesis (ii) as is done in [21] for $B(t) = 1$. Finally condition (iii) can be made more flexible by utilizing the concept of a Lyapunov function which is the case in [3,18,19,22].

In this paper, we generalize all the three different assumptions in uniqueness criteria and offer very general uniqueness theorems which include various known results for different kinds of problems.

2. MAIN RESULTS

Let us begin by listing the following assumptions for convenience.

(A₀) $B \in C[[0, a], \mathbb{R}^+]$ such that $B(t) > 0$ for $0 < t \leq a$,

(A₁) (i) $g \in C[(0, a] \times \mathbb{R}^+, \mathbb{R}]$, $g(t, 0) \equiv 0$, $g(t, u)$ is nondecreasing in u for each t ,

(ii) For each (t_0, u_0) such that $0 < t_0 \leq a$, $u_0 > 0$, the left minimal solution $p(t, t_0, u_0)$ of $u' = g(t, u)$, $u(t_0) = u_0$, exists on $(0, t_0]$;

(iii) For each $m \in C[(0, a], \mathbb{R}^+]$ satisfying

$$D_- m(t) \leq g(t, m(t)) \quad 0 < t \leq a,$$

and $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$, there exists a function $H \in C[[0, a] \times \mathbb{R}^+, \mathbb{R}^+]$

and a number δ , $0 < \delta \leq a$ such that $H(0, u) \equiv 0$ and

$$B(t)D_- m(t) < D_- B(t)m(t) + g(t, p(t, t_0, H(t, m(t)))) B^2(t) \quad 0 < t \leq \delta;$$

(A₂) (i) $g_0 \in C[(0, a] \times \mathbb{R}^+, \mathbb{R}]$, $g_0(t, 0) \equiv 0$, $A \in C[[0, a], \mathbb{R}^+]$ and

$$\lim_{t \rightarrow 0^+} \frac{A(t)}{B(t)} = 0;$$

(ii) All solutions $v(t)$ of $v' = g_0(t, v)$, $v(0) = 0$ defined on $(0, a]$ satisfy the relation $v(t) \leq A(t)$ for $0 < t \leq a$.

(A₃) $V \in C[(0, a] \times E \times E, \mathbb{R}^+]$, $V(t, x, y)$ is locally Lipschitzian in x, y , $V(t, x, y) \equiv 0$ iff $x = y$ and

$$D_- V(t, x, y) \leq g(t, V(t, x, y)) \quad \text{for } (t, x), (t, y) \in D,$$

where $D \subset \mathbb{R}_0$ is such that for any solution $x(t)$ of (1.1),

$$(t, x(t)) \in D \quad \text{for } 0 < t \leq a \quad \text{and}$$

$$D_- V(t, x, y) = \liminf_{h \rightarrow 0} \frac{1}{h} (V(t+h, x+hf(t, x), y+hf(t, y)) - V(t, x, y)).$$

(A₄) (A₃) holds with $g_0(t, u)$ instead of $g(t, u)$.

We shall first prove the following lemmas.

Lemma 2.1

Let the assumptions (A_0) and (A_1) hold. Then $m(t) \neq 0$ for $0 < t \leq a$ implies that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} \neq 0$.

Proof:

Suppose on the contrary that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$. Define $m(0) = 0$. Then since $m(t) \neq 0$, there exists a $\sigma, 0 < \sigma \leq a$ such that $m(\sigma) > 0$. Let $\rho(t, \sigma, m(\sigma))$ be the left minimal solution of $u' = g(t, u)$, $u(\sigma) = m(\sigma)$ on $(0, a]$. Then by Theorem 1.4.6 in [19], we get

$$(2.1) \quad 0 \leq \rho(t, \sigma, m(\sigma)) \leq m(t) \quad \text{for } 0 < t \leq \sigma.$$

Now for each $\eta \in (0, m(\sigma))$ there exists a $t_1 \in (0, \sigma)$ such that

$$0 < H(t, m(t)) \leq \eta \quad 0 < t \leq t_1,$$

because of the property of H . Since $\rho(t, \sigma, \xi)$ is nondecreasing in ξ , we have

$$(2.2) \quad \rho(t, \sigma, H(t, m(t))) \leq \rho(t, \sigma, \eta) \leq \rho(t, \sigma, m(\sigma)) \quad 0 < t \leq t_1.$$

We shall show that there exists a $t_2 \in (0, t_1]$ such that

$$(2.3) \quad \frac{m(t_2)}{B(t_2)} - \rho(t_2, \sigma, \eta) = \eta_0 > 0.$$

Without loss of generality, we can suppose that $0 \leq B(t) \leq 1$. Then it follows from (2.1) that

$$\frac{m(t)}{B(t)} \geq \rho(t, \sigma, \eta) \quad 0 < t \leq t_1.$$

If (2.3) is not true for any $t_2 \in (0, t_1]$, we would have

$$\frac{m(t)}{B(t)} \equiv \rho(t, \sigma, \eta) \quad 0 < t \leq t_1.$$

As a result, we have

$$\frac{D_- m(t) B(t) - m(t) D_- B(t)}{B^2(t)} = \rho'(t, \sigma, \eta) \quad 0 < t \leq t_1$$

which implies, for $0 < t \leq t_1$,

$$\begin{aligned} D_- m(t) B(t) &= m(t) D_- B(t) + g(t, \rho(t, \sigma, \eta)) B^2(t) \\ &\geq m(t) D_- B(t) + g(t, \rho(t, \sigma, H(t, m(t)))) B^2(t). \end{aligned}$$

This is a contradiction to (iii) of (A_1) . It therefore follows that there exists a $t_2 \in (0, t_1]$ such that (2.3) holds.

Because of (iii) of A_1 , we have, by setting $p(t) = \frac{m(t)}{B(t)}$,

$$(2.4) \quad D_- p(t) < g(t, \rho(t, \sigma, \eta)) \quad 0 < t \leq t_2 < t_1.$$

We now wish to show that $p(t) \geq q(t)$ for $0 \leq t \leq t_2$, where

$q(t) = \rho(t, \sigma, \eta) + \eta_0$. Since $p(t_2) = q(t_2)$, using the standard arguments (see the proof of Theorem 1.2.3 in [19]), we get

$$p(t) \geq q(t) \quad \text{for } 0 < t \leq t_2,$$

which implies, using the fact $\rho(t, \sigma, \eta) \geq 0$, that

$$\frac{m(t)}{B(t)} \geq \eta_0 > 0 \quad 0 < t \leq t_2.$$

Hence $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = \eta_0 > 0$ contradicting our supposition that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$.

It therefore follows that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} \neq 0$ and the proof is complete.

Corollary 2.1

If in Lemma 2.1, we suppose in addition that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$,

then $m(t) \equiv 0$ for $0 < t \leq a$.

Remark 2.1

The strict inequality assumed in (iii) of (A_1) becomes superfluous if left uniqueness of solutions $\rho(t, t_0, u_0)$ of $u' = g(t, u)$, $u(t_0) = u_0$ holds and $B, m \in C^1[(0, a], \mathbb{R}^+]$. For, in that case, we have directly

$$0 < \rho(t_2, \sigma, \eta) < \rho(t_2, \sigma, m(\sigma)) \leq m(t_2),$$

and hence

$$\frac{m(t_2)}{B(t_2)} - \rho(t_2, \sigma, \eta) = \eta_0 > 0 \text{ for each } t_2 \in (0, t_1].$$

Also (2.4) becomes $p'(t) \leq g(t, \rho(t, \sigma, \eta))$ $0 < t \leq t_2$.

Hence

$$\int_t^{t_2} p'(s) ds \leq \int_t^{t_2} \rho'(s, \sigma, \eta) ds \quad 0 < t \leq t_2,$$

which implies that

$$p(t) \geq \rho(t, \sigma, \eta) + p(t_2) - \rho(t_2, \sigma, \eta) = \rho(t, \sigma, \eta) + \eta_0 \geq \eta_0 > 0.$$

Lemma 2.2

Let the assumptions (A_0) and (A_2) hold. Suppose further that $m \in C[[0, a], \mathbb{R}^+]$, $m(0) = 0$ and $D_- m(t) \leq g_0(t, m(t))$ $0 < t \leq a$. Then

$$\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0.$$

Proof:

Suppose that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} \neq 0$. Then there exists a t_1 , $0 < t_1 \leq a$

such that $m(t_1) > A(t_1)$ because of (i) of (A_2) .

Let $\rho(t)$ be the left minimal solution of

$$\rho' = g_0(t, \rho), \quad \rho(t_1) = m(t_1).$$

Then by Theorem 1.4.6 in [19], we can show that

$$0 \leq \rho(t) \leq m(t) \quad 0 < t \leq t_1,$$

and $\rho(0) = 0$ since $m(0) = 0$. Consequently by (ii) of (A_2) we have

$$0 \leq \rho(t) \leq A(t) \quad 0 < t \leq a.$$

This is absurd because $m(t_1) = \rho(t_1) \leq A(t_1) < m(t_1)$. Hence we must have

$0 \leq m(t) \leq A(t)$, $0 < t \leq a$ and therefore by (i) of (A_2) , it follows that

$$\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0 \quad \text{and the proof of the lemma is complete.}$$

Now we are in a position to prove our main results on uniqueness of solutions of (1.1).

Theorem 2.1

Consider the IVP (1.1) and suppose that the assumptions (A_0) , (A_1) and (A_3) hold. If $x(t)$ and $y(t)$ are any two solutions of (1.1) existing on $(0, t^*]$, $0 < t^* \leq a$, such that $x(t) \neq y(t)$, then

$$\lim_{t \rightarrow 0^+} \frac{V(t, x(t), y(t))}{B(t)} \neq 0.$$

Proof:

Suppose that $x(t)$ and $y(t)$ are two solutions of (1.1) existing on $(0, t^*]$, $0 < t^* < a$ such that $x(t) \neq y(t)$. and $\lim_{t \rightarrow 0^+} \frac{V(t, x(t), y(t))}{B(t)} = 0$.

Define $m(t) = V(t, x(t), y(t))$ for $0 < t \leq t^*$. It is easy to obtain, using standard arguments and (A_3) , the differential inequality

$$(2.5) \quad D_- m(t) \leq g(t, m(t)) \quad 0 < t \leq t^*.$$

Since $m(t) \neq 0$ by assumption, Lemma 2.1 yields the stated result, completing the proof.

Theorem 2.2

Let the assumptions of theorem 2.1 hold. Suppose further that for any two solutions $x(t)$ and $y(t)$ of (1.1) existing on some interval $0 < t \leq t^* \leq a$, we have

$$(2.6) \quad \lim_{t \rightarrow 0^+} \frac{V(t, x(t), y(t))}{B(t)} = 0.$$

Then the IVP (1.1) has at most one solution.

The proof is immediate in view of corollary 2.1.

In order to employ Lemma 2.3, which offers a sufficient condition for the validity of (2.6), it is necessary to have somewhat smoother hypothesis on $V(t, x, y)$. The following uniqueness result addresses to this fact.

Theorem 2.3

Consider the IVP (1.1) and suppose that the assumptions (A_0) , (A_1) , (A_2) , (A_3) and (A_4) hold. Assume further that $V(t, x, y)$ in (A_3) is such that $V \in C[[0, a] \times E \times E, \mathbb{R}^+]$. Then the IVP (1.1) has at most one solution.

Proof:

Suppose, if possible, that $x(t)$ and $y(t)$ are two solutions of (1.1) existing on the interval $0 < t \leq t^* \leq a$ such that $x(t) \neq y(t)$. Defining $m(t) = V(t, x(t), y(t))$ for $0 < t \leq t^*$, we obtain (2.5) as well as

$$D_- m(t) \leq g_0(t, m(t)), \quad 0 < t \leq t^*.$$

Also $m(0) = 0$ and $m(t) \neq 0$ on $0 < t \leq t^*$. Hence by Lemma 2.2 it follows that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$. Then Theorem 2.2 gives us the stated result completing the proof.

Remark 2.2

Let $m(t)$ be a solution of

$$(2.7) \quad m' = g(t, m), \quad m(0) = 0,$$

with the property that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$. Then assumption (A_1) implies, in

view of corollary 2.1 that $m(t) \equiv 0$ is the only solution of (2.7) such

that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$. Consequently (A_1) may be replaced by

$(A_1)^*$: $g \in C[(0, a] \times \mathbb{R}^+, \mathbb{R}]$, $g(t, 0) \equiv 0$ and $u(t) \equiv 0$ is the only solution of $u' = g(t, u)$, $u(0) = 0$ satisfying $\lim_{t \rightarrow 0^+} \frac{u(t)}{B(t)} = 0$.

We can then state the following result which is essentially given in [3].

Theorem 2.4

Assume (A_0) , (A_1^*) and (A_3) hold. Suppose further that V in (A_3) is such that $V \in C[[0, a] \times E \times E, \mathbb{R}^+]$. Then if (2.6) holds for any two solutions $x(t)$ and $y(t)$ of (1.1) existing on some interval $0 \leq t \leq t^* \leq a$, the IVP (1.1) has at most one solution.

Let us now consider the special case where $V(t, x, y) = \|x - y\|^2$. We then have to impose conditions (A_3) and (A_4) in terms of generalized inner products $(x, y)_+$, see [18]. For example the inequality in (A_3) can now take one of the forms;

$$(2.8) \quad \begin{aligned} (i) \quad & (x - y, f(t, x) - f(t, y))_- \leq \frac{1}{2} g(t, \|x - y\|^2), \\ (ii) \quad & (x - y, f(t, x) - f(t, y))_- \leq g(t, \|x - y\|) \|x - y\|, \end{aligned}$$

for $(t, x), (t, y) \in D \subset R_0$, where D is the same set defined in (A_3) .

The form (2.8), (ii) is equivalent to choosing $V(t, x, y) = \|x - y\|$ so that

(11) becomes;

$$(2.9) \quad [x-y, f(t,x)-f(t,y)]_- \leq g(t, \|x-y\|) .$$

In view of the fact $\|x\| [x,y]_- = (x,y)_-$. We then have the following special cases.

Corollary 2.2

Let (A_0) , (A_1^*) and (2.9) hold. Assume that there exists some $u \in E$ such that

$$(2.10) \quad f(t,x)-u = o(B'(t)) \text{ as } (t,x) \rightarrow (0,0),$$

and $B'(t)$ exists and is continuous on $(0,a]$. Then the IVP;

$$(2.11) \quad x' = f(t,x), \quad x(0) = 0 ,$$

has at most one solution.

Proof:

Let $x(t)$ and $y(t)$ be any two solutions of (2.11) such that $x(t) \neq y(t)$ existing on $(0, t^*]$ and set $m(t) = \|x(t)-y(t)\|$. Then (2.9) implies $D_- m(t) \leq g(t, m(t))$, $0 < t \leq t^*$ and hence it is enough to show that $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$ so that we can conclude the claim by Theorem 2.4. We have

$$m(t) \leq \int_0^t \|f(s, x(s)) - f(s, y(s))\| ds, \quad 0 < t \leq t^* .$$

By assumption (2.10), given $\epsilon > 0$, there exists a $\delta > 0$ such that for $0 < s \leq \delta$,

$$\|f(s, x(s)) - u\| \leq \frac{\epsilon}{2} B'(s) \quad \text{and} \quad \|f(s, y(s)) - u\| \leq \frac{\epsilon}{2} B'(s) .$$

This implies

$$\|f(s, x(s)) - f(s, y(s))\| \leq \epsilon B'(s) \quad 0 < s \leq \delta$$

and therefore

$$0 \leq m(t) \leq \varepsilon \int_0^t B'(s) ds \leq \varepsilon B(t)$$

proving $\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0$.

Corollary 2.3

Assume (A_0) with $B'(t)$ exists and is continuous on $(0, a]$. Suppose further (2.9) holds with $g(t, u) = \frac{B'(t)}{B(t)} u$ and

$$\|f(t, x(t)) - f(t, y(t))\| = o(B'(t)) \text{ as } t \rightarrow 0,$$

for any two solutions $x(t)$ and $y(t)$ of (1.1) existing on $(0, t^*]$.

Then the IVP (1.1) has at most one solution.

Proof:

(A_1^*) holds with $c = 0$, because for any $c \geq 0$, $u' = g(t, u)$ has solutions $u(t) = cB(t)$. Furthermore, following the proof of corollary 2.2, we get easily

$$0 \leq m(t) \leq \int_0^t \|f(s, x(s)) - f(s, y(s))\| ds \leq \varepsilon \int_0^t B'(s) ds \quad 0 < t \leq \delta,$$

which implies

$$\lim_{t \rightarrow 0^+} \frac{m(t)}{B(t)} = 0.$$

Hence Theorem 2.4 proves the corollary.

Corollary 2.3 also contains as a special case, uniqueness results of singular Cauchy problem given in [13, 14] as was shown in [26].

If $D = R_0$, corollaries 2.2 and 2.3 are the results in [26] and [1] respectively which contain other special cases. The importance of requiring the condition (A_3) and its special cases to hold on a subset D of R_0 is stressed in [3] where the example $x' = x^\alpha + t^\beta$, $x(0) = 0$, is worked

out to show the advantages of the weaker requirement.

Lemma 2.1 and corollary 2.1 are generalizations of results in [21], while Lemma 2.2 utilizes the ideas of [8,16,19,28].

Consider the example $x' = x^{1/2} + (t+x)^{2/3}$, $x(0) = 0$. It is clear that the solutions $x(t)$ of this equation satisfy the relation $x(t) > \frac{t^2}{4}$ on $[0, a]$. Let $R_0 = \{(t, x) : 0 < t \leq a, 0 \leq x \leq b\}$ and $D \subset R_0$ be given by $D = \{(t, x) : 0 < t \leq a, x > \frac{t^2}{4}\}$. It is easy to compute and show that

$$|f(t, \xi) - f(t, \eta)| \leq \frac{|\xi - \eta|}{t} + \frac{2\sqrt[3]{4}}{3} \frac{|\xi - \eta|}{(t^2 + 4t)^{1/3}} \quad \text{in } D.$$

Consequently, $g(t, u) = \frac{u}{t} + \frac{2\sqrt[3]{4}}{3} \frac{u}{(t^2 + 4t)^{1/3}}$ and the solutions of $u' = g(t, u)$ are given by $u(t) = cte^{\phi(t)}$ where $\phi(t) = \frac{2\sqrt[3]{4}}{3} \int_0^t \frac{ds}{(s^2 + 4s)^{1/3}}$ and c is any

arbitrary constant. Choosing $B(t) = t$, we see that the only solution $u(t) \equiv 0$ satisfies $\lim_{t \rightarrow 0^+} \frac{u(t)}{t} = 0$. Also it is easy to see that

$$\lim_{t \rightarrow 0^+} \frac{\|x(t) - y(t)\|}{t} = 0.$$

Thus, uniqueness follows by Theorem 2.3. The results which are of different nature. Cover this example in [23].